Production, Manufacturing and Logistics

Incorporating one-way substitution policy into the newsboy problem with imprecise customer demand

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Article info

Article history:
Received 21 May 2007
Accepted 6 December 2008
Available online 24 December 2008

Keywords:
Inventory
Newsboy problem
Substitution
Fuzzy demand
Expected profit

ABSTRACT

This paper presents an approach for solving an inventory model for single-period products with maximizing its expected profit in a fuzzy environment, in which the retailer has the opportunity for substitution. Though various structures of substitution arise in real life, in this study we consider the fuzzy model for two-item with one-way substitution policy. This one-way substitutability is reasonable when the products can be stored according to certain attribute levels such as quality, brand or package size. Again, to describe uncertainty usually probability density functions are being used. However, there are many situations in real world that utilize knowledge-based information to describe the uncertainty. The objective of this study is to provide an analysis of single-period inventory model in a fuzzy environment that enables us to compute the expected resultant profit under substitution. An efficient numerical search procedure is provided to identify the optimal order quantities, in which the utilization of imprecise demand and the use of one-way substitution policy increase the average expected profit. The benefit of product substitution is illustrated through numerical example.

1. Introduction

Single-period inventory model is a mathematical description of a system in which only a single procurement is made. There is a wide application of this model in production management, like stocking of seasonal items, perishable goods, spare parts, etc. In the present day competitive business scenario, while dealing with single-period products the basic problem is how best one can manage its inventory. Earlier, several extensions to this model have been proposed and solved in the literature [1–4]. An excellent literature review and suggestion about the single-period newsvendor problem has been presented by Khouja [7]. While these models serve interesting extensions to the single-period inventory problem, but it assumes that in case of a shortage, unsatisfied demand is lost.

In practice, in multi-item phenomena, a product with a surplus can be used to substitute for a portion of the unmet demand of other out of stock products. That is, there is a chance to substitute a similar type product to satisfy the unmet demand. In literature, the effect of product substitution for a two-item newsboy problem with stochastic demands appear in Parlar and Goyal [6], Khouja et al. [8], Bassok et al. [11] and Rajaram and Tang [9]. Since substitutability involves a complicated profit function, most of the authors provided either a service rate heuristic or a simulation based approach to identify the optimal order quantities. Recently, Axster [10] used the concept of product substitution in a single-echelon inventory system consisting of a number of warehouses in which, in case of stockout at a warehouse, an emergency lateral transshipment from other warehouses may be possible. Inderfurth [12] presented a method for policy analysis in stochastic manufacturing/remanufacturing problems in single-period framework in which there is a downward substitution policy that allows the producer to offer a new item instead of a remanufactured one in case of a shortage of a remanufactured product. Bayindir et al. [5] investigated the benefits of remanufacturing option under one-way substitution policy in which the manufactured and remanufactured products are segmented to different markets and the product capacity is finite. So, in accordance with the applications and advancements of single-period inventory model there is a need to study the effect of product substitution in different context. However, all the above mentioned works develop the newsboy-type problem with substitution in a probabilistic framework. To the best of our knowledge, the newsboy problem with substitution in fuzzy environment has not yet been received attention. In this paper, an effort has been made to study the benefits of product's substitution for two-item newsboy problem with imprecise customer demand. Indeed, business decisions
are becoming complex in our day-to-day life due to incomplete and imprecise information. So the integration of fuzzy logic with inventory systems is important to take care of such imprecision. As compared to the existing literature, our paper differs not only from the kind of uncertainty but also in theoretical context in providing a new fuzzy optimization approach.

The analysis of single-period inventory model in fuzzy environment can be found in Petrovic et al. [13], Ishii and Konno [14], Li et al. [15] and Kao and Hsu [16]. Recently, Dutta et al. [17] extended the model in a mixed fuzzy-stochastic environment where the customer demand appears fuzzy randomly. This extension may be regarded as an amalgamation of fuzzy perception and random behavior into the demand level. Ji and Shao [19] studied the model with fuzzy demands and price discount policy in bi-level context, in which there is a manufacturer in the upper level and several retailers in the lower level. By considering credibility measure, they defuzzified the fuzzy profit function and then the optimal order quantities are computed using genetic algorithm. In [18], Dutta et al. presented a single-period model with reordering strategy along with fuzzy demand, in which the reorder is to be made during the mid-season after the early-season demand has been observed. Using possibilistic mean value method, an analytical solution procedure has been proposed to evaluate the optimality conditions and the expected resultant profit. But the issue of substitution in the fuzzy newsboy problem has not yet been taken care. In this paper, we make an attempt to address the following questions: (a) if substitution is taken place, then how does it affect the profit function when customer demands are characterized imprecisely? (b) how to develop the fuzzy model for computing the expected profit and the optimal order quantities? and (c) does substitution increase the expected resultant profit? To perform this analysis, though several ways of substitution arise in practice, for simplicity, here we study the model with one-way substitution only. That is, there is a possibility that a product with surplus inventory can be used as a substitute for other out of stock product, but not vice-versa. For example, suppose a retailer stores two grades of a product, a brand name product with higher rate and a low graded product with lower price. There is a chance for brand name customers to purchase the lower brand product if the brand name product is out of stock. In contrast, the customers for low brand product would prefer to satisfy their unmet demand at a different store because of their inability to purchase the brand name product.

The outline of the paper is as follows. In Section 2, we model a two-item newsboy problem with one-way substitution in fuzzy environment. Customer demands are taken as fuzzy in nature, which yields a fuzzy profit function. A concrete mathematical analysis of this specific version of the model is presented in Section 3. According to Dubois and Prade [20], the mean value of a fuzzy number is a closed interval bounded by the expectations calculated from its upper and lower distribution functions. In case of defuzzification, here we employ this principle to find the value of expected profit in the fuzzy sense. In Section 3.1, we present an approximation that gives a simpler way of understanding the insight of the solution procedure, and in Section 4, we provide an efficient numerical search procedure in computing the optimal order quantities and expected resultant profit. Computational study is performed in Section 5 to observe the benefits of product’s substitution and finally Section 6 contains some concluding remarks.

2. Fuzzy model for newsboy problem with one-way substitution

Consider a single-period ‘newsboy-type’ inventory problem. The so-called newsboy problem deals with situations where the demand of a product is uncertain and the items that remain unsold at the end of the period become obsolete. Here, we consider a two-item newsboy problem with one-way substitutability where the products demands are prescribed imprecisely.

The following notation is used to describe the model:

\( i = 1, 2 \) the product’s index,
\( c_i \) purchase cost for one unit of product \( i \),
\( p_i \) selling price for one unit of product \( i \),
\( Q_i \) order quantity for product \( i \),
\( D_i \) fuzzy demand for product \( i \),
\( \tau \) probability that unmet demand for product 1 is substitutable by product 2.

Let the demand of a product \( i \) be characterized by the triangular fuzzy number \( D_i = (\bar{D}_i, D_i, \overline{D}_i) \) with the membership function \( \mu_{D_i} \), where

\[
\mu_{D_i}(x) = \begin{cases} 
L_i(x) = \left( -\frac{x - \bar{D}_i}{\overline{D}_i - \bar{D}_i} \right), & \bar{D}_i \leq x \leq D_i, \\
R_i(x) = \left( \frac{x - D_i}{\overline{D}_i - D_i} \right), & D_i \leq x \leq \overline{D}_i, \\
0, & \text{otherwise.}
\end{cases}
\]

If \( d_i \) be the actual realization of fuzzy demand \( D_i \), for each \( Q_i \), the parameter \( d_i \) can assume any value between \( D_i \) and \( \overline{D}_i \). Of course, there may arise two situations, viz., over-stock (\( d_i \leq Q_i \)) or under-stock (\( d_i > Q_i \)). Let us assume that the substitution policy is upward i.e., unmet demand of product 1 (P-I say) can be supplied by the excess item of product 2 (P-II), if any, but not vice-versa. Therefore the profit function can be formulated as

\[
P(Q_1, Q_2) = \begin{cases} 
(p_1 d_1 - c_1 Q_1 + |p_2 \min(d_2, Q_2) - c_2 Q_2|) & \text{for } d_i \leq Q_i \\
(p_1 - c_1) Q_1 + |p_2 \min(d_2, Q_2) + p_2 \max(0, \min(\tau(d_i - Q_1), (Q_2 - d_i))) - c_2 Q_2| & \text{for } d_i > Q_i
\end{cases}
\]

In describing the profit function, (2) may be rewritten as an over-stocking profit and under-stocking profit in the following sense:

When P-I is excess, without regarding to product substitution, the profit function can be expressed as the over-stock profit with respect to P-I and is given by

\[
P_{oa}(Q_1, Q_2) = p_1 d_1 - c_1 Q_1 + |p_2 \min(d_2, Q_2) - c_2 Q_2|.
\]

On the other hand, when P-I is short and P-II is used to satisfy a portion \( \tau \) of unmet demand for P-I, then the profit function is expressed as

\[
P_{os}(Q_1, Q_2) = (p_1 - c_1) Q_1 + |p_2 \min(d_2, Q_2) + p_2 \max(0, \min(\tau(d_i - Q_1), (Q_2 - d_i))) - c_2 Q_2|.
\]
In this expression, since \( d_1 - Q_1 \) is the unsatisfied demand of P-I and \( Q_2 - d_2 \) is the excess items of P-II, \( p_2 \max \{0, \min\{\tau(d_1 - Q_1),(Q_2 - d_2)\}\} \) represents the extra revenue from P-II.

Since demands are fuzzy, the associated profit function defined above will also becomes fuzzy in nature. Let \( D_{1,OS} \) and \( D_{1,US} \) be the respective over-stocking demand and under-stocking demand for product \( i \) (see Figs. 1 and 2). Therefore, the said profit functions are transformed to

\[
P_{OS}(Q_1, Q_2) = p_1 D_{1,OS} - c_1 Q_1 + [(p_2 D_{2,OS} - c_2 Q_2)] \cup [(p_2 - c_2) Q_2] \tag{3}
\]

and

\[
P_{US}(Q_1, Q_2) = (p_1 - c_1) Q_1 + [(p_2 D_{2,OS} - c_2 Q_2) + p_2 \min\{\tau(D_{1,US} - Q_1),(Q_2 - D_{2,OS})\}] \cup [(p_2 - c_2) Q_2]. \tag{4}
\]

where the restrictions of \( D_{1,OS} \) and \( D_{1,US} \) are \([D_{1,OS}Q_1] \) and \([Q_2, D_{1,US}]\), respectively.

Naturally, a general fuzzy output comprises the logical combination of these two over-stocking and under-stocking profits. Consequently, the resultant profit function becomes a fuzzy quantity \( \tilde{P}(Q_1, Q_2) \) [21] and is given by

\[
\tilde{P}(Q_1, Q_2) = \tilde{P}_{OS}(Q_1, Q_2) \cup \tilde{P}_{US}(Q_1, Q_2) \tag{5}
\]

with its general membership function

\[
\mu_{\tilde{P}}(x) = \max\{\mu_{\tilde{P}_{OS}}(x), \mu_{\tilde{P}_{US}}(x)\}.
\]

In single-period inventory models, the objective functions are the expected values of the resultant profits. Hence the retailer’s problem is to determine the optimal order quantities \( Q_i \) for each product \( i \) that maximizes the total expected profit, \( E(P(Q_1, Q_2)) \). The expected value \( E(\tilde{P}(Q_1, Q_2)) \) may be computed using the interval-valued expectation of a fuzzy number proposed by Dubois and Prade [20]. In this case, we only need to know the \( \alpha \)-cut of \( \tilde{P}(Q_1, Q_2) \forall \alpha \in [0, 1].

3. Mathematical analysis

In order to analyze the effects of product substitution, let us assume that the excess item of P-II is being used to satisfy a portion \( \tau \) of unmet demand for P-I, when it is out of stock.

From (2)–(4), we observe that when P-I is surplus, there is no chance for substitution using excess item of P-II. In this case, it is rather rational to compute the expected value of \([p_2 D_{2,OS} - c_2 Q_2] \cup [(p_2 - c_2) Q_2]\) individually. On the other hand, when P-I is demanded more than its order quantity, to calculate the expected profit \( E([p_2 D_{2,OS} - c_2 Q_2] + p_2 \min\{\tau(D_{1,US} - Q_1),(Q_2 - D_{2,OS})\}] \cup [(p_2 - c_2) Q_2], \) we need to analyze the term \( \min\{\tau(D_{1,US} - Q_1),(Q_2 - D_{2,OS})\}\).

In this context, it may be useful to note that if the products are not substitutable (i.e., \( \tau = 0 \)), then Eq. (2) reduces to the sum of two independent newsboy problems with maximizing their individual expected profit. Consequently, the expected resultant profit can be written as \( E(P(Q_1, Q_2)) = E(P_i(Q_i)) + E(P_2(Q_2)), \) where,

\[
P_i(Q_i) = p_i \min\{d_i, Q_i\} - c_i Q_i, i = 1, 2.
\]

Now for any feasible order quantity \( Q_2 \), when the customer demand \( D_i \) is characterized imprecisely, it procures a fuzzy profit function \( \tilde{P}_i(Q_i) \). If \( P_i(Q_2, \alpha) \) be the \( \alpha \)-level set of \( P_i(Q_i) \), using the interval-valued expectation method [20] the expected value of fuzzy profit function \( \tilde{P}_i(Q_i) \) is computed as

\[
E(\tilde{P}_i(Q_i)) = \int_0^1 0.5(P_{i,l}(Q_i, x) + P_{i,r}(Q_i, x))dx,
\]

where \( P_{i,l}(Q_i, x) \) and \( P_{i,r}(Q_i, x) \) are the left and right endpoints of \( P_i(Q_2, \alpha) \), respectively.

Let \( E(P_i) \equiv E(\tilde{P}_i(Q_i)) \), then we review the single item fuzzy newsboy problem in the following proposition.

**Proposition 1.** For any order quantity \( Q_2 \), the expected value of the profit function \( \tilde{P}_i(Q_i) \) can be expressed as

\[
\]
(i) For $Q_i$ lying in the range of $D_i$ and $D_i$

$$E(\hat{P}_i) = p_i \int_0^{L_i(Q_i)} 0.5D_i(x) dx + p_i Q_i (1 - 0.5L_i(Q_i)) - c_i Q_i$$ (6)

(ii) For $Q_i$ lies in the range of $D_i$ and $D_i$

$$E(\tilde{P}_i) = p_i \int_0^{L_i(Q_i)} 0.5D_i(x) dx + p_i \int_{L_i(Q_i)}^{1} 0.5D_i(x) dx + 0.5p_i R_i(Q_i) - c_i Q_i$$ (7)

Moreover, the optimal order quantity $Q_i^0$ (say) can be obtained by $L_i(Q_i^0) = 2(p_i - c_i)/p_i$ or $R_i(Q_i^0) = 2c_i/p_i$ subject to the condition $2c_i - p_i \geq 0$ or $0 < p_i$ according to the position of $Q_i$ in $[D_i, D_i]$ or $[D_i, D_i]$, respectively.

Now, the objective function $\tilde{P}(Q_1, Q_2)$ as defined in (3)-(5) can be simplified in the form as

$$\tilde{P}(Q_1, Q_2) = \begin{cases} 
\tilde{P}_{OS}(Q_1, Q_2) = p_1 \tilde{D}_1(Q_1) - c_1 Q_1 + \tilde{E}(\tilde{P}_2) & \text{for } d_1 \leq Q_1 \\
\tilde{P}_{OS}(Q_1, Q_2) = (p_1 - c_1) Q_1 + \tilde{E}(\tilde{P}_2) & \text{for } d_1 > Q_1 
\end{cases}$$

where, $E(\tilde{P}_2)$ and $E(\tilde{P}_2)$ represent the respective expected profit earned from P-II individually and when it is being used for substitution for a portion ($\tau$) of unmet demand of P-I in the interval-valued expectation method.

Therefore, to obtain the expected resultant profit $E(\tilde{P}(Q_1, Q_2))$, the next step is to compute the value of $\tilde{E}(\tilde{P}_2)$.

3.1. Expected profit earned from product 2 after substitution

Since P-II is being used for substitution, the extra revenue $p_2 \min\{\tau(\tilde{D}_1 - Q_1), (Q_2 - \tilde{D}_2)\}$ earned after substitution is to be added to the over-stock profit function of P-II. We then write

$$\tilde{P}_2^2 = [(p_2 \tilde{D}_2 - c_2 Q_2) + p_2 \min\{\tau(\tilde{D}_1 - Q_1), (Q_2 - \tilde{D}_2)\}] = [(p_2 - c_2) Q_2].$$ (8)

In order to calculate the expected value of $\tilde{P}_2^2$, let us rewrite (8) as

$$\tilde{P}_2^2 = \tilde{P}_{OS}^2 + \tilde{P}_{2, OS}^2$$

where,

$$\tilde{P}_{OS}^2 = (p_2 \tilde{D}_2 - c_2 Q_2) + p_2 \min\{\tau(\tilde{D}_1 - Q_1), (Q_2 - \tilde{D}_2)\}$$ (9)

and

$$\tilde{P}_{2, OS}^2 = (p_2 - c_2) Q_2.$$ (10)

The term $\tilde{P}_{OS}^2$ indicates the event that demand for P-I is above the order quantity $Q_1$, whereas demand for P-II is below its order quantity $Q_2$ and therefore there is a chance for extra revenue by substitution. In the following, we will calculate the possible over-stock gain from P-II after substitution.

If $\tau(\tilde{D}_1 - Q_1) \leq (Q_2 - \tilde{D}_2)$ (i.e., unlikely unmet demand of P-I less than the excess item of P-II) then (9) reduces to

$$\tilde{P}_{OS}^2 = (p_2 \tilde{D}_2 - c_2 Q_2) + p_2 \tau(\tilde{D}_1 - Q_1).$$ (11)

Again, if $\tau(\tilde{D}_1 - Q_1) > (Q_2 - \tilde{D}_2)$ (i.e., all the excess quantity of P-II is demanded for unmet demand of P-I) then (9) reduces to

$$\tilde{P}_{OS}^2 = (p_2 - c_2) Q_2.$$ (12)

It is assumed that the demands for the products are independent, so the exact term $\min\{\tau(\tilde{D}_1 - Q_1), (Q_2 - \tilde{D}_2)\}$ can be realized only at the spot. In order to incorporate it into the optimization setting, the relation $\tau(\tilde{D}_1 - Q_1) \leq (Q_2 - \tilde{D}_2)$ can be rewritten as

$$\tilde{D}_2 - Q_2 \leq \tau(\tilde{D}_1 - Q_1).$$ (13)

For a given $x$-level set of $\tilde{D}_1$ in the right hand side of (13) reduces to either

$$Q_2 - \tau \times \begin{cases} 
0, D_1(x) - Q_1 & 0 \leq x \leq L_1(Q_1) \\
\tau(\tilde{D}_1 - Q_1, D_1(x) - Q_1) & L_1(Q_1) \leq x \leq 1
\end{cases}$$

or

$$Q_2 - \tau \times \begin{cases} 
\tau(D_1(x) - Q_1) & 0 \leq x \leq R_1(Q_1)
\end{cases}$$

according to the position of $Q_1$ in $[D_1, D_1]$ or $[D_1, D_1]$, respectively (see Fig. 3).

Thus, for any realization of the demand $\tilde{D}_1$, we may approximate $Q_2 - \tau(\tilde{D}_1 - Q_1)$ by defining

$$\text{d}_{2,0} = Q_2 - \tau E(\tilde{D}_1 - Q_1),$$ (14)

where $E(\tilde{D}_1 - Q_1)$ represents the expected number of shortages for P-I and is given by either

$$E(\tilde{D}_1 - Q_1) = \int_0^{L_1(Q_1)} 0.5(D_1(x) - Q_1) dx + \int_{L_1(Q_1)}^{1} 0.5(D_1(x) + D_1(x) - 2Q_1) dx$$

or

$$\int_0^{L_1(Q_1)} 0.5D_1(x) dx + \int_{L_1(Q_1)}^{1} 0.5D_1(x) dx - Q_1(1 - 0.5L_1(Q_1))$$ (15)
or

\[
E(\tilde{D}_{1.05} - Q_1) = \int_0^{R_1(Q_1)} 0.5(D_{1.05}(x) - Q_1) \, dx = \int_0^{R_1(Q_1)} 0.5D_{1.05}(x) \, dx - 0.5Q_1R_1(Q_1)
\]

(16)

according to the position of \(Q_1\) in \([D_1, D_1]\) or \([D_1, \tilde{D}_1]\), respectively.

Therefore, it is easy to identify that \(\tilde{P}^2_{2.05}\) will take the form of (11) or (12) if and only if \(d_2,0\) of \(e_{D_{2.05},OS}\) satisfies the relation 

\[
d_2,0 > d_{2,0},
\]

respectively. That is, when \(P-I\) is short, profit from \(P-II\) can be computed by (11) or (12) or (10), according to the position of \(d_2\) in \([D_2, d_{2,0}]\) or \([d_{2,0}, Q_2]\) or \([Q_2, \tilde{D}_2]\), respectively (see Figs. 4 and 5).

In view of the above results we may now propose the following:

**Proposition 2.** If \(d_2\) be the actual realization of \(\tilde{D}_2\), then

(i) For \(d_2,0\) lying in the range of \(D_2\) and \(d_{2,0}\) (i.e., \(D_2 < d_2,0 < d_{2,0}\)) the profit function \(\tilde{P}^2_{2.05}\) takes the form

\[
\tilde{P}^2_{2.05} = (p_2\tilde{D}_{2.05} - c_2Q_2) + p_2T(\tilde{D}_{1.05} - Q_1).
\]

(ii) Again, if \(d_2\) lies between \(d_{2,0}\) and \(Q_2\) (i.e., \(d_{2,0} < d_2 < Q_2\)), then \(\tilde{P}^2_{2.05}\) reduces to \((p_2 - c_2)Q_2\).

(iii) On the other hand, if \(d_2 > Q_2\), then the under-stock profit \(\tilde{P}^2_{2.05}\) is computed as \(\tilde{P}^2_{2.05} = (p_2 - c_2)Q_2\).

Now, it is easy to determine the expected value of \(\tilde{P}^2_{2}\). The complete derivation of \(E(\tilde{P}^2_2)\) is shown in the appendix. Below we present the summary of the results as follows:

**Case 1.** When \(Q_2\) lies between \(D_2\) and \(\tilde{D}_2\), we have

(i) For \(d_{2,0}\) lying out side the left endpoint of \(\tilde{D}_2\) (i.e., \(d_{2,0} < D_2\)), the expected resultant profit from \(P-II\) after substitution is given by

\[
E(\tilde{P}^2_2) = (p_2 - c_2)Q_2.
\]

(17)

(ii) For \(d_{2,0}\) lying between \(D_2\) and \(Q_2\) (i.e., \(D_2 < d_{2,0} < Q_2\)), the expected resultant profit from \(P-II\) after substitution is given by

\[
E(\tilde{P}^2_2) = p_2 \int_0^{1/d_{2,0}} 0.5D_{2.05}(x) \, dx + 0.5p_2T(d_{2,0})(d_1 - Q_1) + (1 - 0.5L(d_{2,0}))(p_2Q_2 - c_2Q_2).
\]

(18)

\[
\tilde{P}^2_{2.05} = (p_2 - c_2)Q_2
\]

\[
\tilde{P}^2_{2.05} = (p_2 - c_2)Q_2
\]
Case 2. When $Q_2$ lies between $D_2$ and $D_1$, we have

(i) For $d_{2,0}$ lying out side the left endpoint of $D_2$ (i.e., $d_{2,0} < D_2$), the expected resultant profit from $P_{II}$ after substitution is the same as in (17).

(ii) For $d_{2,0}$ lying between $D_2$ and $D_1$ (i.e., $D_2 < d_{2,0} < D_1$), the expected resultant profit from $P_{II}$ after substitution is the same as in (18).

(iii) For $d_{2,0}$ lying between $D_2$ and $Q_2$ (i.e., $D_2 < d_{2,0} < Q_2$), the expected resultant profit from $P_{II}$ after substitution is given by

$$E(P_2) = p_2 \int_{0}^{1} 0.5D_{2,osr}(\beta)d\beta + p_2 \int_{R_{1}(d_{2,0})}^{1} 0.5D_{2,osr}(\beta)d\beta + 0.5p_2 \tau(1 - 0.5R_2(d_{2,0}))(d_1 - Q_1) + 0.5R_2(d_{2,0})p_2 Q_2 - c_2 Q_2. \quad (19)$$

We may thus utilize $E(P_2)$ to evaluate the expected resultant profit.

3.2. Expected resultant profit

Since the resultant profit function $\tilde{P}(Q_1, Q_2)$ (as defined in (3)–(5)) is constructed as an over-stocking profit and under-stocking profit with respect to $P_I$, the $\alpha$-level set of the resultant profit is computed as $[P_L(Q_1, Q_2), P_R(Q_1, Q_2, \alpha)]$, where

$$P_L(Q_1, Q_2, \alpha) = \min\{P_{osl}(Q_1, Q_2, \alpha), P_{usr}(Q_1, Q_2, \alpha)\} \quad (20)$$

and

$$P_R(Q_1, Q_2, \alpha) = \max\{P_{osr}(Q_1, Q_2, \alpha), P_{usr}(Q_1, Q_2, \alpha)\}. \quad (21)$$

In order to calculate the expected resultant profit $E(P(Q_1, Q_2))$ two cases need to be considered according to the position of $Q_1$ in $[D_1, D_2]$ as follows:

Case 1. When $Q_1$ lies between $D_1$ and $D_1$, we have

$$P_{osl}(Q_1, Q_2, \alpha) = \begin{cases} p_1D_{1,1}(x) - c_1Q_1 + E(P_2) & \text{for } 0 \leq \alpha \leq L_1(Q_1) \\ \phi & \text{for } L_1(Q_1) < \alpha \leq 1 \end{cases}$$

$$P_{osr}(Q_1, Q_2, \alpha) = \begin{cases} (p_1 - c_1)Q_1 + E(P_2)_{stq_1} & \text{for } 0 \leq \alpha \leq L_1(Q_1) \\ (p_1 - c_1)Q_1 + E(P_2)_{std_1}(x) & \text{for } L_1(Q_1) < \alpha \leq 1 \end{cases}$$

and

$$P_{osr}(Q_1, Q_2, \alpha) = \begin{cases} (p_1 - c_1)Q_1 + E(P_2)_{stq_1} & \text{for } 0 \leq \alpha \leq L_1(Q_1) \\ \phi & \text{for } L_1(Q_1) < \alpha \leq 1 \end{cases}$$

$$P_{usr}(Q_1, Q_2, \alpha) = (p_1 - c_1)Q_1 + E(P_2)_{atd_1}(x) \quad \text{for } 0 \leq \alpha \leq L_1(Q_1)$$

The $\alpha$-level set of resultant profit function according as (20) and (21) can thus be expressed as

$$P(Q_1, Q_2, \alpha) = \begin{cases} [p_1D_{1,1}(x) - c_1Q_1 + E(P_2), (p_1 - c_1)Q_1 + E(P_2)_{atd_1}(x)] & \text{for } 0 \leq \alpha \leq L_1(Q_1) \\ [(p_1 - c_1)Q_1 + E(P_2)_{stq_1}, (p_1 - c_1)Q_1 + E(P_2)_{std_1}(x)] & \text{for } L_1(Q_1) < \alpha \leq 1 \end{cases}$$

Hence, the expected resultant profit using the interval-valued expectation is obtained as

$$E(P(Q_1, Q_2)) = \int_{0}^{L_1(Q_1)} 0.5(p_1D_{1,1}(x) - c_1Q_1 + E(P_2) + (p_1 - c_1)Q_1 + E(P_2)_{atd_1}(x)) dx$$

$$+ \int_{L_1(Q_1)}^{1} 0.5((p_1 - c_1)Q_1 + E(P_2)_{atd_1}(x), (p_1 - c_1)Q_1 + E(P_2)_{atd_1}(x)) dx$$

$$= p_1 \int_{0}^{L_1(Q_1)} 0.5D_{1,1}(x) dx + \int_{0}^{1} 0.5E(P_2)_{atd_1}(x) dx + \int_{L_1(Q_1)}^{1} 0.5E(P_2)_{atd_1}(x) dx + 0.5L_1(Q_1)E(P_2)$$

$$+ (1 - 0.5L_1(Q_1))p_1 Q_1 - c_1 Q_1, \quad (22)$$

where $E(P_2)$ and $E(P_2)$ are given by either (6) and (17.18) or (7) and (17.18.19) according to the position of $Q_2$ in $[D_2, D_2]$ or $[D_2, D_2]$, respectively.

Case 2. When $Q_1$ lies between $D_1$ and $D_1$, we have

$$P_{osl}(Q_1, Q_2, \alpha) = p_1D_{1,1}(x) - c_1Q_1 + E(P_2) \quad \text{for } 0 \leq \alpha \leq 1,$$

$$P_{osl}(Q_1, Q_2, \alpha) = \begin{cases} (p_1 - c_1)Q_1 + E(P_2)_{std_1}(x) & \text{for } 0 \leq \alpha \leq R_1(Q_1) \\ \phi & \text{for } R_1(Q_1) < \alpha \leq 1 \end{cases}.$$
and

\[
P_{\text{opt}}(Q_1, Q_2, x) = \begin{cases} 
(p_1 - c_1)Q_1 + \mathbb{E}(\bar{P}_2) & \text{for } 0 \leq x \leq R_1(Q_1), \\
 p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2) & \text{for } R_1(Q_1) \leq x \leq 1 
\end{cases}
\]

\[
P_{\text{opt}}(Q_1, Q_2, x) = \begin{cases} 
(p_1 - c_1)Q_1 + \mathbb{E}(\bar{P}_2) & \text{for } 0 \leq x \leq R_1(Q_1), \\
 \phi & \text{for } R_1(Q_1) \leq x \leq 1.
\end{cases}
\]

The \(x\)-level set of resultant profit function according as (20) and (21) can thus be expressed as

\[
P(Q_1, Q_2, x) = \begin{cases} 
(p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2), (p_1 - c_1)Q_1 + \mathbb{E}(\bar{P}_2)) & \text{for } 0 \leq x \leq R_1(Q_1), \\
p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2), p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2) & \text{for } R_1(Q_1) \leq x \leq 1.
\end{cases}
\]

Hence, the expected resultant profit using the interval-valued expectation is obtained as

\[
\mathbb{E}(\tilde{P}(Q_1, Q_2)) = \int_0^{R_1(Q_1)} 0.5(p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2)) + (p_1 - c_1)Q_1 + \mathbb{E}(\bar{P}_2)) dx + \int_{R_1(Q_1)}^1 0.5(p_1D_{1,2}(x) - c_1Q_1 + \mathbb{E}(\bar{P}_2)) + (p_1 - c_1)Q_1 + \mathbb{E}(\bar{P}_2)) dx
\]

\[
= p_1 \int_0^{R_1(Q_1)} 0.5D_{1,2}(x) dx + p_1 \int_{R_1(Q_1)}^1 0.5D_{1,2}(x) dx + \int_0^{R_1(Q_1)} 0.5\mathbb{E}(\bar{P}_2)) + (1 - 0.5R_1(Q_1))\mathbb{E}(\bar{P}_2)) dx + 0.5R_1(Q_1)p_1Q_1 - c_1Q_1
\]

(23)

where \(\mathbb{E}(\tilde{P}_2)\) and \(\mathbb{E}(\tilde{P}_2)\) are given by either (6) and (17, 18) or (7) and (17, 18, 19) according to the position of \(Q_2\) in \([D_2, D_3]\) or \([D_2, D_3]\), respectively.

Therefore, after analyzing the fuzzy model for a two-item newsvendor problem with upward substitution as described above, an equivalent deterministic optimization problem can be restated as:

Find the optimal \((Q_1', Q_2')\) pair that maximizes the expected profit \(\mathbb{E}(\tilde{P}(Q_1', Q_2'))\).

For the notational convenience, let \(\mathbb{E}(P(Q_1, Q_2)) = \mathbb{E}(Q_1, Q_2)\). We then derive the following proposition.

**Proposition 3.** Optimal solution \((Q_1^*, Q_2^*)\) for the non-substitutable case satisfies the relation \(\mathbb{E}(Q_1, Q_2) \geq \mathbb{E}(P(Q_1^*, Q_2^*))\).

Proof. We recall \(\mathbb{E}(\tilde{P}(Q_1^*, Q_2^*))\) as the expected resultant profit earned from product \(i(j = 1, 2)\) for the non-substitutable case and it is to be mentioned that by substituting the value \(\lambda = 0\) into Eq. (2), the resultant profit function reduces to the sum of two independent newsvendor problems. Again, the pair \((Q_1', Q_2')\) produces the maximum profit \(\mathbb{E}(Q_1', Q_2')\) for the model with substitution and hence it is obvious that \(\mathbb{E}(Q_1, Q_2) \geq \mathbb{E}(Q_1^*, Q_2^*)\). Now it would be sufficient to show that, by ordering \(Q_2^*\) units, the retailer can obtain a higher profit if substitution is taken place. Or alternatively, for any value of \(\lambda > 0\), \(\mathbb{E}(Q_1^*, Q_2^*) \geq \mathbb{E}(\tilde{P}(\lambda > 0))\).

It is observed from (14) and (17) that when \(\lambda = 0\), \(\mathbb{E}(\tilde{P}_2)\) reduces to \(\mathbb{E}(\tilde{P}_2)\). Therefore, the above phenomena gives further support to the idea that by substituting the value of \(Q_2^*\) and \(Q_2^*\) into Eq. (17) or (18) or (19), it can be easily realized that \(\mathbb{E}(\tilde{P}_2)(\lambda > 0)\) will give higher profit than \(\mathbb{E}(\tilde{P}_2)\). Consequently, if we put the value of \(Q_1^*\) and \(Q_2^*\) into Eq. (22) or (23), \(\mathbb{E}(Q_1^*, Q_2^*) (\lambda > 0)\) will also give the higher profit than the solution of the model with substitution by putting \(\lambda = 0\). Thus, we conclude that \(\mathbb{E}(Q_1, Q_2) \geq \mathbb{E}(\tilde{P}(\lambda > 0)) + \mathbb{E}(\tilde{P}_2(\lambda > 0))\), i.e., the decision-maker obtains a higher expected profit when substitution is taken into consideration.

**4. Search procedure**

Our purpose here is to present a numerical search procedure for evaluating the optimal order quantities used to illustrate the benefits of incorporating substitution effects.

From (22) and (23), it is observed that the objective function \(\mathbb{E}(Q_1, Q_2)\) is a complicated function of \(Q_1\) and \(Q_2\). Further using derivatives it is rather difficult to show the concavity of the objective. Again, in Proposition 3, we argue that the solution \((Q_1^*, Q_2^*)\) of the two-item newsvendor problem without substitution is feasible but not optimal to the model with substitution. For this reason, we are going to employ a numerical search procedure to the solution of the problem with substitution.

To obtain the optimal \(Q_1^* \in [D_1, D_1]\) and \(Q_2^* \in [D_2, D_2]\), one needs to consider the following four situations (Fig. 6).

For each situation, a complete search procedure is performed and finally the optimal \((Q_1^*, Q_2^*)\) is found along with maximum \(\mathbb{E}(Q_1^*, Q_2^*)\).

Suppose situation 1 is realized, then the bounds for \(Q_1^*\) reduces to \([D_1, D_1]\) and that of \(Q_2^*\) is \([D_2, D_2]\). For each combination of \(Q_1\) and \(Q_2\) we need to do the following steps:

**Step 1.** Compute \(L_i(Q_1)\) and \(L_j(Q_2)\) using (1).

**Step 2.** Use (6) to compute \(\mathbb{E}(\tilde{P}_2)\). Use (15) to compute \(\mathbb{E}(\tilde{D}_1, \tilde{Q} - Q_1)\).

**Step 3.** Find the value of \(d_{2,1}\) using (14).

**Step 4.** Check the position of \(d_{2,0}\) in \([D_2, D_2]\)

(a)Compute \(\mathbb{E}(\tilde{P}_2)\) using (17).

(b) Using (22) compute the expected resultant profit \(\mathbb{E}(\tilde{Q}_1, Q_2)\).

Otherwise, \(Q_2 < d_{2,0} \leq Q_2\) and then

(c) Compute \(L_i(d_{2,0})\) using (1) and find \(\mathbb{E}(\tilde{P}_2)\) using (18).

(d) Using (22) compute the expected resultant profit \(\mathbb{E}(\tilde{Q}_1, Q_2)\).

**Step 5.** Set the maximum of \(\mathbb{E}(\tilde{Q}_1, Q_2)\).
For a satisfactory performance, one can initialize $Q_1 = D_1$ and $Q_2 = D_2$ and perform an iterative algorithm with increment 1 or $(D_i - D_{i-1})/n$ where $n$ is a user defined real number depends upon the spreads of the fuzzy demands. A large $n$ indicates a higher degree of optimism.

For instance, if $D_1$ for $P-I$ is $(400, 500, 600)$ and $D_2$ for $P-II$ is $(600, 700, 800)$, then for situation 1, the search space for $Q_1$ and $Q_2$ may be considered as $(400, 401, 402 \ldots 500)$ and $(600, 601, 602 \ldots 700)$, respectively.

On the other hand, if situation 2 is realized, then the bounds for $Q_1$ reduces to $[D_1, D_1]$ and $Q_2$ is in $[D_2, D_2]$. In this case, for each combination of $Q_1$ and $Q_2$ we need to do the following steps:

Step 1. Compute $L_1(Q_1)$ and $R_2(Q_2)$ using (1).
Step 2. Use (7) to compute $E(PS_1)$. Use (15) to compute $E(\tilde{D}_1, PS_1)$.
Step 3. Find the value of $d_{1,0}$ using (14).
Step 4. Check the position of $d_{1,0}$ in $[D_2, Q_2]$. If $d_{2,0} < D_2$, then
  (a) Compute $E(\tilde{P}_1)$ using (17).
  (b) Using (22) compute the expected resultant profit $E(Q_1, Q_2)$.
  else if $D_2 < d_{2,0} < D_2$ then
  (c) Compute $L_2(d_{2,0})$ using (1) and find $E(\tilde{P}_2)$ using (18).
  (d) Using (22) compute the expected resultant profit $E(Q_1, Q_2)$.
  else, $D_2 < d_{2,0} < Q_2$ and then
  (e) Compute $R_2(d_{2,0})$ using (1) and find $E(\tilde{P}_2)$ using (19).
  (f) Using (22) compute the expected resultant profit $E(Q_1, Q_2)$.
Step 5. Set the maximum of $E(Q_1, Q_2)$.

A similar approach may be taken for situation 3 and situation 4. In these cases, $E(Q_1, Q_2)$ is to be considered from (23). Based on this numerical optimization scheme among all of these four situations, one would achieve the optimal expected profit $E(Q_1, Q_2)$.

To look into the advantage of product substitution, we compute the percentage increase in expected profit (PIEP) as

$$\text{PIEP} = 100 \times \frac{(E_{\text{sub}} - E_{\text{non-sub}})}{E_{\text{non-sub}}}$$

where, $E_{\text{sub}}$ and $E_{\text{non-sub}}$ represent the expected resultant profit for substitutable and non-substitutable case, respectively. The results of the computational study are discussed in the following section.

5. Numerical illustration

In this section, we perform an extensive computational study using MATLAB to demonstrate the above fuzzy model in which the use of one-way substitution policy increases the average expected profit. Assuming that the expert’s opinion about the customer demand is linguistic, we consider a two-item newsboy-type problem with triangular fuzzy demands where the vendor sells two grades of products with different brand, one with high brand ($P-I$) and the other with low brand ($P-II$). We only illustrate the model with upwards substitution (i.e., the unmet demand of $P-I$ can be supplied by the excess item of $P-II$, if any, but not vice-versa).

Suppose the demand for brand name item is about 500 and for the low graded item is around 700 (as given in Table 1). In the computational study, we (i) first explore the conditions of the parameters, which initially determine the order quantities for each product when the products are not substitutable, (ii) restrict the state space for $Q_1$ and $Q_2$ for each situation so that a complete search procedure can be applied to determine the optimal solution when substitution is taken into account and (iii) finally compare the gains in profits between the

<table>
<thead>
<tr>
<th>Table 1</th>
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<tbody>
<tr>
<td><strong>Fuzzy information for demands.</strong></td>
</tr>
<tr>
<td><strong>Product</strong></td>
</tr>
<tr>
<td><strong>Demand</strong></td>
</tr>
</tbody>
</table>
optimal solutions with the profits obtained when order points are determined from non-substitutable case. Table 2 provides the results when products are not substitutable, whereas Table 3 shows the results with substitution. In Table 3, the optimal values are given for different combination of cost parameters, which covers all the four possible situations. More precisely, we cover the cases where (1) the unit cost and the price of both the products satisfying the conditions $2c_1 - p_1 > 0$ and $2c_2 - p_2 > 0$, (2) the unit cost and the price of P-I satisfying the condition $2c_1 - p_1 > 0$, but that of P-II satisfying $2c_2 - p_2 < 0$, (3) $2c_1 - p_1 < 0$ for P-I, but $2c_2 - p_2 > 0$ for P-II and (4) $2c_1 - p_1 < 0$ for P-I, and $2c_2 - p_2 < 0$ for P-II.

For instance, consider a case where $c_1 = 10$, $p_1 = 15$, $c_2 = 6$ and $p_2 = 10$. We first compute the optimal values for $Q_{PI}^1$ and $Q_{PI}^2$, without considering substitution, by using (Proposition 1). In this case, $2c_1 - p_1 = 5 > 0$ and $2c_2 - p_2 = 2 > 0$, consequently, the optimal $Q_{PI}^1$ lies between 400 and 500 and is given by $L_1(Q_{PI}^1) = 0.66$, whereas the optimal $Q_{PI}^2$ lies between 600 and 700 and is determined by $L_2(Q_{PI}^2) = 0.8$. First row of Table 2 shows the corresponding results without substitutability. Notice that in this case $Q_{PI}^1 < [400, 500]$ and $Q_{PI}^2 < [600, 700]$.

Now let us assume that the upward substitution is taken place with a specified degree of substitutability. To perform the computation four possible situations are identified, namely, (a) $(400 \leq Q_1 < 500, 600 < Q_2 \leq 700)$, (b) $(400 < Q_1 < 500, 700 < Q_2 < 800)$, (c) $(500 < Q_1 < 600, 600 < Q_2 < 700)$ and (d) $(500 < Q_1 < 600, 700 < Q_2 < 800)$. In each situation, we run the associated algorithm to compute the resulting profit.

For each combination of $Q_1$ and $Q_2$, it first checks the position of $d_{P2}$ in $[D_2,Q_2]$, and then by computing $E(P_{P2})$ it generates the resulting expected profit. By comparing the four possible situations we record $E(Q_1, Q_2)$ for an optimal $(Q_1, Q_2)$ pair.

Since results of judgment in real problems vary from manager to manager, their subjectivity should be reflected in computing the expected resultant profit. In each setting, we vary $\tau$ from 0.0 to 1.0. The percentage of profit gain due to product substitutability increases as $\tau$ increases.

Now, let us consider the case when ordering according to the independent newsvendor problem is taken into account. In Table 4, we compute the values of expected profit $E(Q_1, Q_2)$ for various sets of $(c_1,p_1,c_2,p_2)$. The percent performance in expected profit from non-substitutable case to substitution case with ordering point $(Q_{PI}^1, Q_{PI}^2)$ and from $E(Q_1, Q_2)$ to $E(Q_1, Q_2)$ can be measured by: PIEP1 = $100 \times \frac{(E(Q_1, Q_2) - E(Q_{PI}^1, Q_{PI}^2))/E(Q_{PI}^1, Q_{PI}^2)}$, and PIEP2 = $100 \times \frac{(E(Q_1, Q_2) - E(Q_1, Q_2))/E(Q_1, Q_2)}$, respectively.

The observations made in this study result in the following managerial insights:

- As the degree of product substitutability $\tau$ increases from 0.0 to 1.0, the utility of product substitution becomes more valuable. When $\tau = 0$, it is shown that the above fuzzy model with substitution reduces to the model without substitution and the results coincides with each other (see Tables 2 and 3). The highest proportion gains in expected profit from the substitutable case are accrued when $\tau = 1$. Each experiment set, we observe that when $\tau$ increases, the optimal order quantity $Q_1$ for P-I decreases but this trend changes in the opposite direction for $Q_2$. A strong reason for such opposite trend of changes is due to the upward substitution in which the unit cost of purchasing P-II is cheaper as compared to P-I ($c_1 > c_2$). The practical implication is that the shop-keeper can expect more business from low graded product if upward substitution is taken place.
Table 4
Percentage performance in expected profit with \( \tau = 0.0, 0.2, 0.5, 0.8 \) and 1.0.

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>( p_1 )</th>
<th>( c_2 )</th>
<th>( p_2 )</th>
<th>( \tau )</th>
<th>( \mathbb{E}[Q_1^2, Q_2^2])</th>
<th>PEP1</th>
<th>PEP2</th>
</tr>
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<tbody>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
<td>10</td>
<td>0.0</td>
<td>4726.67</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
<td>10</td>
<td>0.2</td>
<td>4763.50</td>
<td>0.70%</td>
<td>0.08%</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
<td>10</td>
<td>0.5</td>
<td>4799.10</td>
<td>1.53%</td>
<td>0.52%</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
<td>10</td>
<td>0.8</td>
<td>4826.76</td>
<td>2.18%</td>
<td>1.41%</td>
</tr>
<tr>
<td>10</td>
<td>15</td>
<td>6</td>
<td>10</td>
<td>1.0</td>
<td>4838.61</td>
<td>2.37%</td>
<td>2.50%</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>3</td>
<td>11</td>
<td>0.0</td>
<td>6093.94</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>3</td>
<td>11</td>
<td>0.2</td>
<td>6119.88</td>
<td>0.42%</td>
<td>0.49%</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>3</td>
<td>11</td>
<td>0.5</td>
<td>6209.17</td>
<td>1.89%</td>
<td>0.72%</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>3</td>
<td>11</td>
<td>0.8</td>
<td>6260.93</td>
<td>2.74%</td>
<td>2.31%</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>3</td>
<td>11</td>
<td>1.0</td>
<td>6288.19</td>
<td>3.19%</td>
<td>3.77%</td>
</tr>
</tbody>
</table>

- Table 3 shows that the percentage performance achieved for the second experiment set averaged around 3.14%, ranging from 0.0% to 7.07%. Similarly, the average percentage performance for the first, third and fourth sets are 2.27%, 0.59% and 0.82%, respectively. These results imply that the upward substitution is more beneficial when the profit margin per unit of \( P-II \) is high and that of \( P-I \) is low.
- From Table 4 we observe that in a real business environment when imprecise demand is taken into account, the ordering point \( \{Q_1^2, Q_2^2\} \) gives the higher profit to the model with substitution, but it is not optimal to the model. These results prove Proposition 3.

6. Conclusion

In this study, we analyze a fuzzy inventory model based on single-period products in which the opportunity for product substitution is taken into consideration. It is supposed that the single-period problem operates under uncertainty in customer demand, which is described by imprecise terms and modelled by fuzzy sets. We demonstrate a situation – how the consideration of one-way substitution affects the inventory system. Optimization of complex resultant profit using derivatives gives high degree of difficulty and hence we propose an efficient numerical search technique which provides a useful tool for finding the optimum using fuzzy logic.

The work may be applicable in modeling of unidirectional lateral transshipments in warehouses or products remanufacturing policies in fuzzy environment. Though the specific fuzzy model developed here considers a newsboy problem without considering holding cost, salvage value, etc., however, our analysis can be extended including such complicating factors. The future research may also be extended for both-way substitution policy.

Acknowledgement

We are gratefully acknowledging the anonymous reviewers and the editor for their valuable suggestions.

Appendix A. Analysis of \( \mathbb{E}[\tilde{P}_2^2] \)

We recall the profit function earned from \( P-II \) when it is being used for substitution as

\[
P_2^2 = [(p_2 D_{2,0.5} - c_2 Q_2) + p_2 \min\{\tau (D_{1,0.5} - Q_1), (Q_2 - D_{2,0.5})\}] \cup [p_2 - c_2]Q_2
\]

To predict the expected value of fuzzy profit function \( \tilde{P}_2^2 \) there are two cases to be considered according to the position of \( Q_2 \) in \( [D_2, D_2] \).

Case 1. When \( Q_2 \) lies between \( D_2 \) and \( D_2 \) (Fig. 4), using Eq. (14) and (Proposition 2), we have

(i) For \( d_{2,0.5} < D_2 \), the \( \beta \)-level set of \( \tilde{P}_2^2 \) is derived as

\[
P_{2,0.5}^2(Q_2, \beta) = \{p_2 - c_2\}Q_2 \cup \{p_2 - c_2\}Q_2 \quad \forall \beta \in [0, 1]
\]

and

\[
P_{2,0.5}^2(Q_2, \beta) = \{p_2 - c_2\}Q_2 \cup \{p_2 - c_2\}Q_2 \quad \forall \beta \in [0, 1]
\]

Therefore, the expected resultant profit from \( P-II \) after substitution is given by

\[
\mathbb{E}[\tilde{P}_2^2] = \int_0^1 0.5((p_2 - c_2)Q_2 + (p_2 - c_2)Q_2) d\beta = (p_2 - c_2)Q_2.
\]

(ii) For \( D_2 < d_{2,0.5} < Q_2 \), the \( \beta \)-level set of \( \tilde{P}_2^2 \) is derived as
Hence, the expected value of $P_2^s$, for $0 \leq \beta \leq L_2(d_{2,0})$,

一步步推导如下：

\[
\begin{align*}
P_{2,0}^s(Q_2, \beta) &= \left\{ \begin{array}{ll}
(p_2 - c_2)Q_2 + p_2\tau(d_1 - Q_1), & \text{if } L_2(d_{2,0}) \leq \beta \leq L_2(Q_2), \\
(p_2 - c_2)Q_2, & \text{if } L_2(Q_2) < \beta \leq 1.
\end{array} \right.
\end{align*}
\]

Therefore, $P_{2,0}^s(Q_2, \beta)$ for $0 \leq \beta \leq 1$.

Hence, the expected resultant profit $P_2^s$ from $P$-II after substitution is computed by

\[
E(P_2^s) = \int_{L_2(d_{2,0})}^{L_2(Q_2)} \left( 0.5(p_2 - c_2)Q_2 + p_2\tau(d_1 - Q_1) + (p_2 - c_2)Q_2 \right) d\beta.
\]

Case 2. When $Q_2$ lies between $D_2$ and $D_2$ (Fig. 5), using Eq. (14) and (Proposition 2), we have

(i) For $d_{2,0} < D_2$, the $\beta$-level set of $P_2^s$ is derived as

\[
P_{2,0}^s(Q_2, \beta) = \left\{ \begin{array}{ll}
(p_2 - c_2)Q_2, & \text{for } L_2(d_{2,0}) \leq \beta \leq L_2(Q_2), \\
(p_2 - c_2)Q_2, & \text{for } L_2(Q_2) < \beta \leq 1.
\end{array} \right.
\]

and $P_{2,0}^s(Q_2, \beta)$ for $0 \leq \beta \leq 1$.

In this case, the expected value of $P_2^s$ is given by

\[
E(P_2^s) = (p_2 - c_2)Q_2.
\]

(ii) For $D_2 < d_{2,0} \leq D_2$, the $\beta$-level set of $P_2^s$ is derived as

a) If $L_2(d_{2,0}) \leq R_2(Q_2)$ then

\[
\begin{align*}
P_{2,0}^s(Q_2, \beta) &= \left\{ \begin{array}{ll}
(p_2 - c_2)Q_2, & \text{for } L_2(d_{2,0}) \leq \beta \leq L_2(Q_2), \\
(p_2 - c_2)Q_2, & \text{for } L_2(Q_2) < \beta \leq 1.
\end{array} \right.
\end{align*}
\]

Therefore, $P_{2,0}^s(Q_2, \beta)$ for $0 \leq \beta \leq 1$.

and $P_{2,0}^s(Q_2, \beta)$ for $0 \leq \beta \leq 1$.

Hence, the expected value of $P_2^s$ is computed by

\[
E(P_2^s) = \int_{L_2(d_{2,0})}^{L_2(Q_2)} \left( 0.5(p_2 - c_2)Q_2 + p_2\tau(d_1 - Q_1) + (p_2 - c_2)Q_2 \right) d\beta.
\]

b) If $L_2(d_{2,0}) > R_2(Q_2)$ then, in a similar way, we derive the expected value of $P_2^s$ as
\[ \mathbb{E}(\hat{P}_2) = p_2 \int_{R_{2}(d_{2,0})}^{R_{1}(d_{2,0})} 0.5D_{2,0}(\beta) d\beta + 0.5p_2 \tau L_2(d_{2,0})(d_1 - Q_1) + (1 - 0.5L_2(d_{2,0}))p_2 Q_2 - c_2 Q_2. \]

(iii) For \( D_2 < d_{2,0} \leq Q_2 \), the \( \beta \)-level set of \( \hat{P}_2 \) is derived as

\[
\{ \begin{aligned}
\hat{P}_2^0(Q_2, \beta) &= \left[ p_2 D_{2,0}(\beta) - c_2 Q_2 + p_2 \tau(d_1 - Q_1) \right] \text{ for } 0 \leq \beta \leq R_2(Q_2), \\
\hat{P}_2^0(Q_2, \beta) &= \left[ (p_2 - c_2) Q_2 \right] \text{ for } R_2(Q_2) \leq \beta \leq R_2(d_{2,0}) \\
\hat{P}_2^0(Q_2, \beta) &= \phi \text{ for } R_2(d_{2,0}) \leq \beta \leq 1.
\end{aligned} \]

Therefore,

\[ \hat{P}_2^0(Q_2, \beta) = p_2 D_{2,0}(\beta) - c_2 Q_2 + p_2 \tau(d_1 - Q_1) \text{ for } 0 \leq \beta \leq 1. \]

and

\[ \hat{P}_2^0(Q_2, \beta) = \left\{ \begin{array}{ll}
(p_2 - c_2) Q_2 & \text{ for } 0 \leq \beta \leq R_2(Q_2), \\
(p_2 - c_2) Q_2 & \text{ for } R_2(Q_2) \leq \beta \leq R_2(d_{2,0}) \\
p_2 D_{2,0}(\beta) - c_2 Q_2 + p_2 \tau(d_1 - Q_1) & \text{ for } R_2(d_{2,0}) \leq \beta \leq 1.
\end{array} \right. \]

Finally, the expected value of \( \hat{P}_2 \) is computed by

\[ \mathbb{E}(\hat{P}_2) = \int_{R_2(d_{2,0})}^{R_1(d_{2,0})} 0.5(p_2 D_{2,0}(\beta) - c_2 Q_2 + p_2 \tau(d_1 - Q_1)) d\beta + \int_{R_2(d_{2,0})}^{1} 0.5(p_2 D_{2,0}(\beta) - c_2 Q_2 + p_2 \tau(d_1 - Q_1)) d\beta + \int_{R_2(d_{2,0})}^{1} 0.5p_2 \tau L_2(d_{2,0})(d_1 - Q_1) + 0.5p_2 Q_2 - c_2 Q_2. \]

References


