Ex-Ante Information and the Design of Keyword Auctions

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Abstract

This paper discusses a class of auctions, weighted unit-price auctions (WUPAs), which capture key features of keyword auctions, a novel mechanism behind the multi-billion-dollar keyword advertising industry. We analyze the equilibrium bidding strategy in the WUPA class and study its two main design parameters – weighting factors and minimum bids – both of which make use of the auctioneer’s ex-ante information on bidders’ ability to generate outcomes. Our results indicate equilibrium bidding functions in WUPAs may have kinks and jumps. WUPAs can be efficient when an auctioneer weights unit-price bids by bidders’ expected yield and imposes the same minimum score (but not the same minimum bid-price) across all bidders. Optimally weighted WUPAs can generate more revenue than generalized first-price auctions, and optimal minimum bids generally differ from those prescribed in the mechanism design literature.

Keywords: keyword advertising, weighted unit-price auctions, ex-ante information, score rule, Google and Yahoo!

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1 Introduction

Keyword advertising is a form of online advertising that allows advertisers to target Internet users by keywords they use in searches or by keywords embedded in the content they view.¹ In recent years, auctions have been adopted by keyword advertising providers such as Yahoo!, Google, and MSN, to sell keyword advertising slots on search-result and regular content pages. These include the familiar “pay-per-click” auctions, in which advertisers bid their willingness-to-pay per click and pay for the clicks they actually receive, and the relatively new “pay-per-call” or “pay-per-action” auctions, in which advertisers pay for each phone call made by viewers of advertisements,² or for per-occurrence of certain consumer actions, such as signing up or making a purchase at the advertised website.³

In these auctions, keyword advertising providers often learn about advertisers’ abilities to generate outcomes. In pay-per-click auctions, for instance, keyword advertising providers can infer advertisers’ ability to generate clicks by observing their past click-through rates. Such information has been gradually integrated into keyword auction designs. In 2002, Google first introduced a design that ranks advertisers by the product of per-click prices they bid and their historical click-through rates. Yahoo!, which has used a price-only ranking mechanism since the beginning of its keyword advertising program, has recently announced plans to switch to a mechanism similar to Google’s. The availability of click-through information also enables alternative minimum-bid policies. For example, Google has recently abandoned its one-size-fits-all minimum-bid policy in favor of a new policy that imposes higher minimum bids for advertisers with low historical click-through rates. In contrast, Yahoo! sets a fixed minimum bid for all keywords and all advertisers. Vise and Malseed, authors of The Google Story, note that Google displays a “socialist tendency” because in Google’s approach,

¹The search-based keyword advertising is also referred to as “search advertising” or “sponsored search.” The content-based keyword advertising is also referred to as “contextual advertising.”
²An alternative pay-per-call format is that advertisers call consumers with their consent.
³For example, SNAP.com has developed a pay-for-performance scheme in which advertisers pay only when a customer converts — in other words, when the customer actually buys a product or performs a specific action deemed valuable by the advertiser, like providing an email address or registering for more information.
advertisements frequently clicked on by Internet users are more likely to show up in top-tier positions [19]. Besides the “socialist tendency” Google’s approach might embody, are there economic justifications for keyword advertising providers to weight advertisers differently and to impose different minimum bids? Do these new auction designs result in higher profits or allocate advertising resources more efficiently? What is the impact of the above weighting schemes and minimum-bid policies on bidders’ equilibrium bidding behavior? The goal of this paper is to address these questions.

The above questions are important, given the rapid growth of the keyword advertising industry. Since 2002, Google’s total revenue has increased almost 15-fold, to $6.1 billion in 2005, most of which comes from keyword advertising. In just a few years, keyword advertising has become a leading form of online advertising. According to the Interactive Advertising Bureau and PricewaterhouseCoopers, keyword advertising revenue reached about $10 billion in 2005 and accounted for 41% of the total online advertising revenue.

In order to address the above questions, we study a class of auction mechanisms, called weighted unit-price auctions (WUPAs), in which bidders bid on unit-prices, and the auctioneer weights their bids differently based on their outcome-generating abilities. Bidders may receive a high or low expected-yield signal, which the auctioneer can observe before the auction. The novelty of the WUPA lies in that it provides an intuitive way for the auctioneer to take advantage of bidders’ yield signals, that is, by assigning different weighting factors and imposing different minimum bids depending on the signal.

Our model applies not only to pay-per-click auctions but also to pay-per-call or pay-per-action auctions, since the “yield” in our model can also be interpreted as the number of calls or other consumer actions. WUPAs may also be applied to other settings, such as selling computer power on a grid-computing facility, auctioning off Internet bandwidth, and procurement auctions with uncertain but verifiable outcomes. In each of these settings, auctioneers are likely to have some information on bidders’ future yield.

Our analysis of WUPAs generates the following novel findings:
• Unlike in standard auction settings, there may be kinks in WUPA bidding functions. The kinks occur when one type of bidder bids so high (low) that the other type of bidder is no longer able to match it (“type” in our study setting refers exclusively to a bidder’s outcome-generating ability).

• Under certain minimum bids, there may be a “jump” in one type of bidder’s bidding function. The reason is that when two types of bidders face different minimum bids, the type facing a more constraining minimum bid will “cluster” near the minimum bids (its bidding function is nearly flat near the minimum bid), which makes it profitable for the other type to either leap-frog all of them or none. The jump will not occur when minimum bids are equally constraining; that is, the implied minimum score (minimum bids times the respective weighting factors) is the same across bidder types.

• Remarkably, although bidders generally do not bid their true valuation (except in the case of single-object WUPAs with the second-score rule), the efficient weighting scheme is simple and the same in both the first-score and the second-score WUPAs: each unit-price bid is weighted by its respective expected yield, as if bidders are bidding their true valuation.

• It is weakly efficient (i.e., the allocation is efficient among participating bidders) to impose the same minimum score across bidder types, provided the auctioneer also uses the efficient weighting factor. As we have suggested, setting different minimum scores for different bidder types causes clustering (among the more-constrained type) and jump-bidding (among the less-constrained), which distorts allocation efficiency locally.

• The revenue-maximizing weighting scheme is generally different from the efficient one. The revenue-maximizing weighting scheme may favor high-yield or low-yield bidders compared with the efficient weighting scheme. When high-yield bidders and low-yield bidders differ only in their expected yield (their unit-valuation distributions are the
same), the optimal weighting scheme always discriminates against high-yield bidders. But as low-yield bidders become stronger — their valuation increases by the hazard-rate order — the revenue-maximizing weighting scheme generally will favor low-yield bidders less and may even favor the high-yield bidders. This finding reflects that it is harder to tell which bidder type is “stronger” overall when they differ both in the expected yield and in the value distribution.

- With an appropriate weighting scheme and minimum-bid policy, WUPAs may generate higher revenue than standard auctions. The “abnormal” revenue arises precisely because the interim information on bidders’ future yield allows auctioneers to further discriminate among bidders, often at the cost of allocation efficiency.

- An optimal minimum-bid policy is generally different from what we see in the mechanism design literature. The reason is that besides excluding bidders of certain valuation, the minimum bids also affect the resource allocation among the participating bidders, particularly through its impact on the size of the “cluster” and the “jump” point. Meanwhile, optimal minimum bids are generally not a uniform minimum score either, implying distortion on allocation efficiency.

The rest of the paper is organized as follows. In section 2 we discuss the related literature. In section 3 we lay out our auction model. We analyze the equilibrium bidding in section 4 and auction efficiency and auctioneers’ revenue in section 5. In section 6, we examine the impact of minimum bids on the bidders’ equilibrium bidding behavior and on efficiency and revenue. We conclude the paper in section 7.

2 Related Literature

Several authors have studied keyword auctions. Varian [20] analyzes the equilibrium of keyword auctions in a complete information setting. Edelman et al. [6] study the equilibrium in
both static complete information and dynamic incomplete information settings. These two papers assume no differences in advertisers’ abilities to generate clicks and characterize bidders’ equilibrium behavior under the “generalized second-price auction” framework, in which slots are allocated by the rank order of unit-price bids, and winners pay their next highest unit-prices. In contrast, we assume advertisers differ in their abilities to generate clicks and focus on the optimal design of auction parameters, namely allocation rules and minimum-bid policies, in an incomplete-information setting. Lahaie [11] compares the ranking mechanisms of Google and Yahoo!. But he focuses on two special cases in which advertisers are ranked solely by prices they bid or by a product of prices and expected click-through rates. Thus, his analysis does not address the issues of optimal resource allocation and revenue maximization. Liu and Chen [12] raise issues similar to those discussed in this paper. However, Liu and Chen [12] consider only the first-score, single-object setting, and do not study minimum-bid policies. In addition, in Liu and Chen [12], bidders’ unit-valuations are independent of their yield signals, whereas in this paper they may be correlated. Feng [8] studies the optimal allocation mechanism in a multiple-slot setting where advertisers’ valuations decrease in the ranks of slots at different speeds. Her work therefore complements ours.

This research contributes to the literature on the role of information in auction designs. Early literature has focused on how auction design can use ex post information, such as bidders’ actual cost (in procurement auctions) ([10], [13], [16], [17]) and bidders’ realized valuation (in drilling rights auctions). In this paper, we study a mechanism that allows the auctioneer to use not only ex post outcome information (the actual number of clicks) but also ex ante information (bidders’ click-generating abilities).

Our paper is related to research on “scoring auctions” in procurement settings([5], [2]). In this literature, suppliers (bidders) submit multi-dimensional bids, such as price $p$ and quality $q$, and are ranked by a score function $s(p, q)$. One of the key questions is whether the auctioneer should announce a score function that corresponds to its true preference. Che [5] shows that the optimal scoring function under-rewards quality (relative to buyer’s prefer-
ence). Asker and Cantillon [2] extend Che [5] to allow bidders to have multi-dimensional cost information. They show that as long as the score function is quasi-linear in price, bidders’ multi-dimensional information can be summarized by single-dimensional sufficient statistics. In both Asker and Cantillon’s [2] settings and ours, the score rule plays a critical role of reducing an otherwise multi-dimensional problem into a single-dimensional one. However, there are also many differences. For example, a similar sufficient statistic does not exist in our model setting; scoring auctions use a quasi-linear function, whereas keyword auctions use a multiplicative one.

Our paper is also related to the literature on “tender auctions” [7], in which a buyer (the auctioneer) requests suppliers to bid a unit price for each input factor (e.g., labor and materials) and ranks suppliers by the weighted sum of their unit-price bids. Ewerhart and Eieseler [7] show that optimal tender auctions are generally inefficient but can generate higher revenues than efficient mechanisms such as first-price auctions. However, because tender auctions use an additive score rule and WUPAs use a multiplicative one, the two auction formats apply to different settings and differ significantly in equilibrium bidding strategies. For example, Ewerhart and Eieseler [7] show that bidding in tender auctions typically is non-monotonic, whereas we show that the bidding in WUPAs is monotonic but may have kinks or jumps.

Finally, our paper is also related to the optimal auction and mechanism design literature ([4], [14], [15]). For example, we use the “virtual value” from this literature to explain the intuition of our findings. However, our results concerning how various design features affect the equilibrium bidding behavior, efficiency, and revenue are both novel and specific to WUPAs.
3 Model Setup

We consider a problem of assigning $m$ assets to $n$ ($n \geq m$) risk-neutral bidders. The assets are indexed by $j$, and the bidders are indexed by $i$. Each bidder can exploit only one asset. Upon exploiting asset $j$, bidder $i$ generates a stochastic but verifiable yield $q_i \delta_j$, for which the bidder has a unit-valuation of $v_i$. $\delta_j$ is deterministic, interpreted as the size (or quality) of asset $j$. We assume $\delta_1 \geq \delta_2 \ldots \geq \delta_m$ and normalize $\delta_1 = 1$. $q_i$ is stochastic and exogenous, interpreted as bidder $i$’s yield rate. We assume $q_i$ is affected by bidder $i$’s intrinsic attributes but not by the bidder’s actions (at least not during the auction). In sum, the bidder’s payoff from exploiting asset $j$ is $v_i q_i \delta_j$. In the pay-per-click keyword advertising setting, assets are advertising slots; yields are clicks by the Internet users. The number of clicks advertiser $i$’s advertisement can attract depends both on the relevance of the advertisement $q_i$ and on the prominence of the slot $\delta_j$.\footnote{The prominence of the slot to Web users may be determined by many factors, such as position, size, shape, or media format (text, image, or video) of the slot.} $v_i$ is the advertiser’s valuation per click.

We make the following assumptions about the bidders’ and the auctioneer’s information. Each bidder $i$ learns his/her own unit valuation $v_i$ before the auction, but not others’. Bidder $i$ doesn’t know the future yield $q_i$ but receives a signal $\theta_i \in \{H, L\}$ about $q_i$ before the auction. We call bidders who receive $H$ ($L$) high-yield (low-yield) bidders. We assume $\theta_i$ is also known by the auctioneer,\footnote{This is equivalent to assuming that the auctioneer is able to learn bidders’ abilities to generate outcomes. The assumption of learning is widely used. For example, in Shapiro [18], consumers can learn the product quality gradually. In addition, as we show later, learning is for the auctioneer’s own benefit.} but not by other bidders. The assumption that the auctioneer and the bidder $i$ both observe $\theta_i$ is motivated by the e-commerce technologies that allow the auctioneer to track bidders’ past yields and thus predict the bidders’ future yields.

All bidders and the auctioneer hold a common belief about the distribution of unit-valuations ($v$) and yield signals ($\theta$). $v_i$ and $\theta_i$ may be correlated, but $(v_i, \theta_i)$ is independently distributed from $(v_j, \theta_j)$, $j \neq i$. Let $F_{\theta}(v), \theta \in \{H, L\}$,\footnote{With a slight abuse of the notation, we also use $\theta$ to index the signal types.} denote the cumulative distribution
of $v$ conditional on receiving signal $\theta$. We assume $F_\theta(v)$ has a fixed support $[0, 1]$ and the density function, $f_\theta(v)$, is positive and differentiable everywhere within the support. By the convention of distribution functions, we have $F_\theta(v) = 1$ for $v > 1$.

We assume it is common knowledge that the probabilities of receiving $H$ and $L$ signals are $\alpha$ and $1 - \alpha$, respectively. We assume that a bidder $i$’s private information $v_i$ does not convey further information about $q_i$ so that $E[q|\theta, v] = E[q|\theta]$. Denote $Q_H \equiv E[q|\theta = H]$ and $Q_L \equiv E[q|\theta = L]$ as the expected yields for a bidder who receives signal $H$ and signal $L$, respectively. We assume $Q_H > Q_L$ without loss of generality.

We assume bidders’ payoff functions are additive in their total valuation and the payment. In particular, conditional on winning asset $j$ and making total payment $P$, bidder $i$’s payoff is

$$v_i q_i \delta_j - P$$

The asset is sold through a weighted unit-price auction, which we describe below. Each bidder $i$ is asked to submit his/her willingness-to-pay for per-unit yield (hereafter, unit price), $b_i$, and the auctioneer allocates the asset by a score rule that weights bidders’ unit-price bids based on the yield signals they receive. In particular, let the weighting factor for low-yield bidders be $w$ and normalize the weighting factor for high-yield bidders to 1. The score for bidder $i$ is

$$s_i = \begin{cases} 
  b_i, & \text{if } i \text{ receives yield signal } H \\
  wb_i, & \text{if } i \text{ receives yield signal } L
\end{cases}$$

The bidder with the highest score wins the first asset; the bidder with the second highest score wins the second asset, and so on. Winners pay for their realized yields at unit-prices determined by the auction payment rule (which we will discuss shortly). We call such an auction format weighted unit-price auctions.

By allowing $w$ to take different values, we can accommodate the following stylized auction

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7This assumption can be generalized to $[\bar{v}_L, \bar{v}_L]$ and $[\bar{v}_H, \bar{v}_H]$ for low-yield and high-yield bidders, respectively.
formats. When \( w = 1 \), the winners are determined solely by the prices they bid. One example is Yahoo!’s auction format. When \( w < 1 \), bid prices from low-yield bidders are weighted less than those from high-yield bidders. Google’s auction fits in this category.\(^8\)

A WUPA may be first-score or second-score depending on whether winning bidders are required to “fulfill” their own scores or next highest scores. Formally, let \( b_1w_1, b_2w_2, \ldots, b_mw_m \) be the winning scores, arranged from high to low (where \( b_i \) and \( w_i \) are the bid and the weighting factor of the \( i \)th winner, respectively). In the first-score WUPA, winning bidders fulfill their own scores and thus pay the prices they bid; whereas in the second-score WUPA, the highest bidder pays \( \frac{b_2w_2}{w_1} \) to fulfill the second highest score; the second-highest bidder pays \( \frac{b_3w_3}{w_2} \) to fulfill the third highest score, and so on.

In this paper, we also allow the auctioneer to impose different minimum bids (or reserve prices) based on bidders’ yield signals. In particular, we let \( b_L \) and \( b_H \) be the minimum bids for low-yield and high-yield bidders, respectively.

To summarize, the auction proceeds as follows. First, the auctioneer announces the score rule (the weights) and the payment rule (first-score or second-score). All bidders get a signal about their expected yield and learn their unit-valuation before the auction. Then, each bidder bids a unit-price, and the winners are determined by the announced score rule. The winners pay the auctioneer according to the actual yields at unit-prices determined by the auction payment rule.

4 Equilibrium Bidding Strategy

We consider a symmetric, pure-strategy Bayesian-Nash equilibrium of WUPAs. By “symmetric,” we mean that bidders with the same unit-valuation and yield signal will bid the same.

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\(^8\)To the best of our knowledge, Google weights advertisers’ bid prices on a particular keyword by their click-through rates (CTRs) over their entire history on this keyword, which, if we believe high past CTRs lead to high future CTRs, translates to the case in which bid prices from advertisers with high expected CTRs are weighted more.
4.1 First-Score WUPAs

Let $b_H(v)$ denote the equilibrium bidding function (i.e., a mapping from unit-valuation to bid price) for bidders who receive signal $H$, and $b_L(v)$ for bidders who receive signal $L$. We conjecture that both $b_L(v)$ and $b_H(v)$ are strictly increasing (we verify this in the Appendix) so that inverse bidding functions exist and strictly increase. We denote the inverse bidding functions as $\phi_L(b)$ and $\phi_H(b)$, respectively. The following relationship exists between high-yield’s and low-yield’s bidding functions.

**Lemma 1** In first-score WUPAs, for any pair $(v_L, v_H) \in [0, 1] \times [0, 1]$, if $v_H = wv_L$, then

$$b_H(v_H) = wb_L(v_L) \quad (3)$$

**Proof.** Consider a high-yield bidder with unit-valuation $wv$ who bids $wb$ and a lower-yield bidder with $v$ who bids $b$. Both bidders get a score $wb$, and their payoff functions are

$$U_L(v, b) = Q_L(v - b) \sum_{j=1}^{m} \delta_j \Pr\{wb \text{ ranks } j\th \} \quad (4)$$

$$U_H(wv, wb) = Q_H(wv - wb) \sum_{j=1}^{m} \delta_j \Pr\{wb \text{ ranks } j\th \} \quad (5)$$

It is easy to establish that

$$U_H(wv, wb) = \frac{wQ_H}{Q_L} U_L(v, b). \quad (6)$$

For $b_L(v)$ and $b_H(v)$ to be equilibrium bidding functions, at any $v$, $b = b_L(v)$ must maximize $U_L(v, b)$ and $b = b_H(v)$ must maximize $U_H(v, b)$. So, (6) suggests that if bidding $b$ is the best choice for a low-yield bidder with unit-valuation $v$, bidding $wb$ must be the best choice for a high-yield bidder with unit-valuation $wv$, which implies $b_H(wv) = wb_L(v)$. $\blacksquare$

Lemma 1 says in any equilibrium a high-yield bidder with unit-valuation $wv$ and a low-yield bidder with unit-valuation $v$ will score the same. The intuition for Lemma 1 is as follows. Consider a high-yield bidder with unit-valuation $wv$ who bids $wb$ and a low-yield bidder with unit-valuation $v$ who bids $b$. By the score rule, the former has the same winning probability as the latter. Meanwhile, the payoff of the former (conditional on winning) differs.
from that of the latter only by a scalar. By (4) and (5), their expected payoffs differ only by a scalar too. The fact that multiplying a payoff function by a scalar does not alter the solution to an optimization problem implies that $b$ is the solution to the low-yield bidder’s optimization problem if and only if $wb$ is the solution to the high-yield bidder’s. Therefore, Lemma 1 holds.

By the monotonicity of bidding functions, a low-yield bidder $i$ with unit-valuation $v$ can beat a bidder $j$ if and only if $j$ receives a low-yield signal and has unit valuations less than $v$ or $j$ receives a high-yield signal and has unit-valuation less than $wv$ (by Lemma 1). So the probability for a low-yield bidder with unit-valuation $v$ to beat another bidder can be represented by

$$G_L(v) \equiv \alpha F_H(wv) + (1 - \alpha)F_L(v)$$

Similarly, $G_H(v) \equiv \alpha F_H(v) + (1 - \alpha)F_L(v/w)$ is the probability for a high-yield bidder with unit-valuation $v$ to beat another bidder. We define

$$P_j^\theta(v) \equiv \binom{n-1}{n-j} [G_\theta(v)]^{n-j} [1 - G_\theta(v)]^{j-1}, \theta \in \{L, H\}$$

$$\rho_\theta(v) \equiv \sum_{j=1}^{m} P_j^\theta(v) \delta_j, \theta \in \{L, H\}$$

$P_j^\theta(v)$ is the equilibrium probability of winning the $j$th asset for a bidder who receives signal $\theta$ and has a unit-valuation of $v$. $\rho_\theta(v)$ is the bidder’s expected size of the asset awarded, or expected winning, in equilibrium.

**Proposition 1** Given $w$ ($w > 0$), equilibrium bidding functions in the first score WUPA are given by

$$b_\theta(v) = v - \int_0^v \frac{\rho_\theta(t) dt}{\rho_\theta(v)}, \text{ for } v \in [0, 1], \text{ and for } \theta \in \{L, H\}$$

where $\rho_\theta$ is defined in (9).

**Proof.** See appendix. ■

When $w < 1$, $G_H(v)$ changes from $[\alpha F_H(v) + (1 - \alpha)F_L(v/w)]$ to $[\alpha F_H(v) + (1 - \alpha)]$ as $v$ goes beyond $w$. As a result, the expected winning for high-yield bidders $\rho_H(v)$ has a
kink at \( v = w \). Consequently, the bidding function for high-yield bidders also has a kink at \( v = w \). Figure 1 illustrates a pair of equilibrium bidding functions with a kink in high-yield bidders’ bidding function. Intuitively, the kink occurs because the high-yield bidders with unit-valuation less than \( w \) face competition both from low-yield bidders and from high-yield bidders, whereas high-yield bidders with unit-valuation more than \( w \) will face competition only from high-yield bidders. By the same argument, when \( w > 1 \), the bidding function for low-yield bidders has a kink at \( v = 1/w \).

With Proposition 1, we can explicitly evaluate the expected revenue of the auctioneer.

\[
\pi = n(1 - \alpha)Q_L \int_0^1 \rho_L(v) \left( v - \frac{1 - F_L(v)}{f_L(v)} \right) f_L(v) dv + n\alpha Q_H \int_0^1 \rho_H(v) \left( v - \frac{1 - F_H(v)}{f_H(v)} \right) f_H(v) dv
\]

The first term on the right side of the equation is the expected revenue from the low-yield bidders, and the second term is the expected revenue from high-yield bidders. We interpret \( Q_L \left[ v - \frac{1-F_L(v)}{f_L(v)} \right] \) and \( Q_H \left[ v - \frac{1-F_H(v)}{f_H(v)} \right] \) as the “virtual value” [4] of low-yield and high-yield

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9Under general supports \([\bar{v}_L, \tilde{v}_L]\) and \([\bar{v}_H, \tilde{v}_H]\), there may be as many as two kinks in the bidding functions. For example, with general supports \([1, 3]\) for low-yield and \([1, 2]\) for high-yield and \( w = 0.5 \), there will be a kink at low-yield bidders’ bidding function at \( v = 2 \) and a kink at high-yield bidders’ bidding functions at \( v = 1.5 \).

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bidders, respectively. Thus, the total expected revenue is the expected virtual value of the winning bidder (which can be either low-yield or high-yield).

4.2 Second-Score WUPAs

**Lemma 2** In second-score WUPAs, for any pair \((v_L, v_H) \in [0, 1] \times [0, 1]\), if \(v_H = wvL\), then

\[
b_H(v_H) = wb_L(v_L) \tag{12}
\]

**Proof.** Denote \(s_{j:n-1}\) as the random variable for \(j\)th highest score among \(n-1\) bidders in equilibrium. Consider a high-yield bidder with unit-valuation \(wv\) bidding \(wb\) and a lower-yield bidder with unit-valuation \(v\) bidding \(b\). So both bidders get a score \(s = wb\).

\[
U_H(wv, wb) = Q_H \left[ \sum_{j=1}^{m} \delta_j \Pr\{wb \text{ ranks } j\text{th}\} \left( wv - \frac{1}{w} \mathbb{E}[s_{j:n-1}|s_{j-1:n-1} \geq s, s_{j:n-1} \leq s] \right) \right]
\]

\[
U_L(v, b) = Q_L \left[ \sum_{j=1}^{m} \delta_j \Pr\{wb \text{ ranks } j\text{th}\} \left( v - \frac{1}{w} \mathbb{E}[s_{j:n-1}|s_{j-1:n-1} \geq s, s_{j:n-1} \leq s] \right) \right]
\]

Similar to the proof of Lemma 1, we have \(U_H(wv, wb) = \frac{wQ_H}{Q_L} U_L(v, b)\) and \(b_H(wv) = wb_L(v)\).  

If only one asset is to be allocated \((m = 1)\), we can show, by an argument similar to those made in standard second-price auctions, bidding one’s true unit-valuation is a (weakly) dominant strategy. In multi-asset settings, however, bidding one’s true unit-valuation is not an equilibrium strategy. In fact, bidders have an incentive to bid below their true unit-valuation. This is because in the event of losing the first asset, the bidder may still win other assets.

Unfortunately, there is generally no close-form bidding function in second-score WUPAs. Nevertheless, the above lemma allows us to establish the revenue equivalence between first-score WUPAs and second-score WUPAs and thus to study second-score WUPAs without obtaining their bidding functions.

**Proposition 2** The first-score WUPA and the second-score WUPA are revenue-equivalent.
Proof. $b_H(wv) = wb_L(v)$ implies lower-yield bidder with unit-valuation $v$ and high-yield bidder with $wv$ will tie in both first-score and second-score WUPAs. Therefore, first-score WUPAs and second-score WUPAs allocate the assets in the same way. By revenue equivalence theorem (e.g., Proposition 14.1 of Krishna’s book [9]), they must also generate the same amount of revenue. ■

Because the first-score WUAP and the second-score WUAP allocate the assets in the same way and generate the same expected revenue, we do not distinguish between them in the following sections.

5 Efficiency and Revenue

5.1 Allocation Efficiency

One important aspect of a mechanism is whether it can allocate the resource to those who value it the most. Here we consider \textit{ex ante} efficiency because a bidder’s valuation of an asset is random at the time of auction. When a bidder with unit-valuation $v$ and yield signal $\theta$ wins the asset $j$, it creates an expected social surplus of $v\delta_j Q_\theta$. Given that the expected winnings for high-yield and low-yield bidders are $\rho_H(v)$ and $\rho_L(v)$, respectively, the total social surplus generated by a WUAP is calculated as

$$n(1-\alpha)Q_L \int_0^1 v\rho_L(v) f_L(v) \, dv + n\alpha Q_H \int_0^1 v\rho_H(v) f_H(v) \, dv$$

We define the efficient weighting factor, $w_{eff}$, as one that maximizes (13) and call the WUAP with an efficient weighting factor \textit{an efficient WUAP}.

\textbf{Proposition 3} \ $w_{eff} = \frac{Q_L}{Q_H}$.

\textbf{Proof.} See appendix. ■

Proposition 3 says even though bidders may not bid their true unit-valuation, it is still efficient to weight their bids by their expected yield \textit{as if bidders are bidding their true unit-valuations}.Remarkably, the efficient weighting factor is independent of the distribution of
both unit-valuation and the yield-signals. This feature makes it straightforward to implement an efficient weighting scheme.

Actually, Proposition 3 has an intuitive explanation. According to Lemma 1, when \( w = \frac{Q_L}{Q_H} \), a high-yield bidder with unit-valuation \( \frac{Q_L}{Q_H} v \) ties with a low-yield bidder with unit-valuation \( v \) in equilibrium. Since they also have the same total expected valuation for the asset, the ranking of bidders is in fact consistent with their total valuation, thereby assuring allocation efficiency.

In order to compare WUPAs and auctions where bidders bid on their total payment, we introduce the generalized first-price auction, an extension of the standard first-price auction into a multiple-asset setting. In a generalized first-price auction, bidders bid on their willingness-to-pay for the first asset, and assets are assigned based on the ranking of bids. The highest bidder wins the first asset and pays its bid; the second highest bidder wins the second asset and pays its bid times asset size \( \delta_2 \), and so on. Denote \( x \) as a bidder’s expected valuation for the first asset (unit-valuation times his/her expected yield). By our model setting, a bidder’s belief about the distribution of another bidder’s expected valuation for the first asset (without knowing that bidder’s yield-signal) is

\[
H(x) = \alpha F_H \left( \frac{x}{Q_H} \right) + (1 - \alpha) F_L \left( \frac{x}{Q_L} \right)
\]  

(14)

Obviously, when there is only one asset \( (m = 1) \), the generalized first-price auction reduces to a standard first-price auction. It is easy to see that the generalized first-price auction inherits its efficiency property; that is, assets are allocated in a way that achieves the maximum total valuation.

In efficient WUPAs, a bidder is assigned an asset if and only if it has the highest total valuation for the asset among those who have not been assigned an asset. Thus Proposition 3 also implies that efficient WUPAs allocate the same way as generalized first-price auctions and generate the same expected revenue to auctioneers. Thus:

**Proposition 4** The efficient WUPA is revenue-equivalent to a generalized first-price auction where bidders bid their total payment.
Proof. See appendix. ■

Note that in generalized first-price auctions, winners pay upfront, whereas in WUPAs, winners pay ex post based on realized outcomes. In this sense, bidders bear less risk in WUPAs than in generalized first-price auctions (or other bid-on-total-payment auctions). The risk-sharing feature of unit-price auctions is considered as advantageous, for example, by McAfee and McMillan [13] in the study of procurement auctions. The fact that WUPAs impose little risk on advertisers (“you’re only charged if people click your ads”) is often used as a tag line to attract participation in keyword advertising.

5.2 Revenue

The analysis on revenue serves two purposes. First, we examine whether the auctioneer has incentives to deviate from the efficient weighting scheme for the sake of profitability. Second, because the efficient WUPA generates the same expected revenue as bid-on-total-payment auctions (Proposition 4), we can easily compare revenues of WUPAs and bid-on-total-payment auctions by conducting a comparative static analysis on the weighting factor.

We define the optimal weighting factor, $w_{opt}$, as one that maximizes the total expected revenue of the auctioneer (11). We call the WUPA with an optimal weighting factor an optimal WUPA. We first consider a setting in which the unit-valuation is independent of the yield signal, so that $F_L(v) = F_H(v) = F(v)$. We say a cumulative distribution function $F(v)$ is increasing-hazard-rate (IHR) if its hazard rate $\frac{f(v)}{1-F(v)}$ increases in $v$ within the support. Many distributions, including uniform, normal, and exponential, are IHR.

Proposition 5 If the unit-valuation and yield signal are independent and $F(v)$ is IHR, then $w_{opt} > \frac{Q_L}{Q_H}$.

Proof. See appendix. ■

Proposition 5 implies that the optimal WUPA is generally inefficient: it discriminates against high-yield bidders. The reason is that for any $w \leq w_{eff}$, $Q_L \left(v - \frac{1-F(v)}{f(v)}\right)$, the virtual value of a low-yield bidder with unit-valuation $v$, is always higher than $Q_H \left(wv - \frac{1-F(w)}{f(wv)}\right)$,
the virtual value of a corresponding high-yield bidder with unit-valuation $wv$. Thus, a profit-
maximizing auctioneer is better off by raising $w$ to allocate the asset more often to low-yield
bidders. Thus, we conclude $w_{opt}$ will be higher than $w_{eff}$. This result is consistent with
findings in [7]: revenue-maximizing tender auctions subsidize low types in order to reduce
winning bidders’ informational rents.

When the unit-valuation is correlated with the yield signal, the optimal weighting factor
may be higher or lower than the efficient weighting factor, as illustrated in the following
example.

**Example 1** Assume $m = 1$ and the unit-valuations of low-yield and high-yield bidders are
uniformly distributed on $[0, z]$ and $[0, 1]$, respectively. We can derive the explicit formula for
$w_{opt}$:

$$w_{opt} = \frac{2 Q_L}{Q_H} \left( \frac{Q_L z - 1}{n+1} \right) + 2$$

Let $\alpha = 0.5$, $Q_L = 0.5$, $Q_H = 1$, $z = 4$, and $n = 5$, we have $w_{opt} = \frac{5}{16} < w_{eff} = 0.5$.

When bidders’ unit-valuations are correlated with their yield signals, a bidder’s virtual
valuation, $Q_\theta \left[ v - \frac{1 - F_\theta(v)}{f_\theta(v)} \right]$, is affected both by the bidder’s expected yield and the distri-
bution of unit-valuation. Thus, it is difficult to characterize when the optimal weighting
factor should be higher (or lower) than the efficient weighting factor. To circumvent this
difficulty and to gain some understanding about the relationship between the distribution
of unit-valuation and the optimal weighting factor, we study the sufficient condition for the
optimal weighting factor to decrease below.

**Proposition 6** Let $w_{opt}$ be the optimal weighting factors under unit-valuation distribution
pair $\{F_L, F_H\}$. Assume $F_H(v)$ is IHR. If $\tilde{F}_L$ satisfies

$$Q_L \left[ v - \frac{1 - \tilde{F}_L(v)}{\tilde{f}_L(v)} \right] > Q_H \left[ w_{opt} v - \frac{1 - F_H(w_{opt}v)}{f_H(w_{opt}v)} \right], \forall v \in [0, 1]$$

then under $\{\tilde{F}_L, F_H\}$, then $\tilde{w}_{opt} > w_{opt}$.

**Proof.** See appendix. □
The above proposition implies that when there is a substantial increase in low-yield bidder’s virtual value (as given by condition (16)), the optimal weighting factor should increase. An increase in virtual value corresponds to “weaker” low-yield bidders – in the sense that they can claim less rent for themselves. Thus, this proposition shows that, in general, the auctioneer should discriminate more against high-yield bidders as low-yield bidders become weaker.

Because the efficient WUPA generates the same expected revenue as bid-on-total-payment auctions (Proposition 4) and the optimal WUPAs can generate more revenue than the efficient WUPA (Proposition 5 and discussion on the case of correlated case), we immediately have:

**Corollary 1** Optimal WUPAs generate more revenue than generalized first-price auctions.

Consider the case with $m = 1$, in which the generalized first-price auctions reduce to standard first-price auctions. According to the optimal mechanism design literature, the standard auctions (with appropriately set reserve price) can achieve the highest revenue among all mechanisms. The above corollary indicates, however, that WUPAs can achieve even higher revenue. This of course is not a violation of the result in the optimal mechanism design. The key is that WUPAs allow the auctioneer to discriminate bidders based on information on their future yield signals, which is not available in the standard mechanism design setting.

This corollary illustrates that information on bidders’ future yield can be exploited to enhance the auctioneer’s revenue. The key to this “abnormal” revenue lies in the availability of interim information (yield signal) that enables further discrimination among bidders.

### 6 Minimum Bids

The optimal mechanism design literature suggests that a revenue-maximizing mechanism often involves imposing minimum bids (or entry fees) to exclude bidders whose participation
reduces the auctioneer’s revenue. In the following, we first examine the equilibrium bidding with minimum bids, and then the impact of minimum bids on efficiency and revenue.

6.1 Equilibrium Bidding under Minimum Bids

We consider a first-score WUPA with minimum bids $b_L$ and $b_H$ for low-yield and high-yield bidders, respectively. We will focus on the case with $w \leq 1$ and $b_L \leq b_H/w$. The cases with $w > 1$ and/or $b_L > b_H/w$ can be analyzed in a similar way. In order to avoid the trivial case in which no low-yield bidders can win over a participating high-yield bidder, we assume $b_H/w < 1$.

We conjecture that a pure-strategy equilibrium exists. The equilibrium bidding functions $b_L(v)$ and $b_H(v)$ must satisfy the following criteria: (a) the lowest participating bidder of yield-signal $\theta$ must have a unit-valuation of $b_\theta$ and bid his/her true unit-valuation; that is, $b_\theta(b_\theta) = b_\theta$. (b) $b_L(v)$ and $b_H(v)$ must be strictly increasing. The criterion (a) is simply the consequence of minimum bids, and the criterion (b) is required by the incentive compatibility condition.\footnote{See Appendix A.7 for a proof.}

By the monotonicity of the bidding functions, there must exist $v_0$ ($v_0 \in [b_L, 1]$) such that all participating low-yield bidders with $v \geq v_0$ score at least as high as the lowest participating high-yield bidder, i.e., $wb_L(v) \geq b_H$, and all participating low-yield bidders with $v < v_0$ score lower than that. In other words, $v_0$ is the point where low-yield bidders start competing with high-yield bidders.

If $v_0 = 1$, no low-yield bidders can win over a participating high-yield bidder, and the problem degenerates. We will focus on a more interesting case where $v_0 < 1$. In the following, we assume the condition for $v_0 < 1$ is satisfied.\footnote{See footnote 12 for such a condition.}

As in the no-minimum-bid case, a mapping similar to Lemma 1 holds between a low-yield bidder with unit-valuation above $v_0$ and the corresponding high-yield bidder.
**Lemma 3** Given minimum bids $b_L$ and $b_H$ ($b_L \leq b_H/w$) and $w \leq 1$, for any pair $(v_L, v_H) \in [0, 1] \times [0, 1]$, if $v_H = wv_L$ and $v_L > v_0$, then

$$b_H(v_H) = wb_L(v_L)$$ (17)

**Proof.** See appendix. ■

Lemma 3 is an extension of Lemma 1 into the minimum-bid setting. As before, Lemma 3 implies a low-yield bidder with unit-valuation $v \in (v_0, 1]$ ties with a high-yield bidder with unit-valuation $wv$ in equilibrium.

If $b_L = b_H/w$, the lowest participating low-yield bidder ties with the lowest participating high-yield bidder, and thus $v_0 = b_L$. In such a case, Lemma 3 holds for any participating pair $(v_L, v_H)$ that satisfies $v_H = wv_L$. Thus, our previous analysis for the no-minimum-bid case shall apply here too.

If $b_L < b_H/w$, however, the equilibrium mapping (17) no longer holds for all participating pairs that satisfy $v_H = wv_L$. Suppose the otherwise. We have $b_L(b_H/w) = b_H(b_H)/w$. By the definition of $v_0$, we have $v_0 = b_H/w$. Note that by criterion a, $b_H(b_H)/w = b_H/w$. Thus, $b_L(b_H/w) = b_H/w$ and the low-yield bidder with unit-valuation $b_H/w$ earns zero payoff. However, this cannot be an equilibrium because the low-yield bidder can be better off by bidding $b_L < b_H/w$ to earn positive payoff. Therefore, we conclude $v_0 > b_H/w$.

The fact that $v_0 > b_H/w$ also implies that the two bidding functions cannot both be continuous. If both bidding functions were continuous, by the definition of $v_0$, we have $wb_L(v_0) = b_H$, which, together with Lemma 3, implies $v_0 = b_H/w$. The following proposition establishes that the low-yield bidders’ bidding function jumps at $v_0$.

**Proposition 7** Given minimum bids $b_L$ and $b_H$ ($b_L \leq b_H/w \leq 1$) and $w \leq 1$, bidding functions are given by

$$b_\theta(v) = v - \frac{\int_{b_\theta}^v \rho_\theta(t) \, dt}{\rho_\theta(v)}, \text{ for } v \in [b_\theta, 1], \text{ and for } \theta \in \{L, H\}$$ (18)
where \( \rho_\theta(v) = \sum_{j=1}^{m} P^j_\theta(v) \delta_j \), \( P^j_\theta(v) = \binom{n-1}{n-j} [G_\theta(v)]^{n-j} [1 - G_\theta(v)]^{j-1} \),

\[
G_L(v) = \begin{cases} 
\alpha F_H(b_H) + (1 - \alpha) F_L(v) & \text{for } v \in [b_L, v_0) \\
\alpha F_H(wv) + (1 - \alpha) F_L(v) & \text{for } v \in [v_0, 1] 
\end{cases}
\]

\[
G_H(v) = \begin{cases} 
\alpha F_H(v) + (1 - \alpha) F_L(v_0) & \text{for } v \in [b_H, wv_0] \\
\alpha F_H(v) + (1 - \alpha) F_L(w) & \text{for } v \in [wv_0, w] \\
\alpha F_H(v) + (1 - \alpha) & \text{for } v \in [w, 1]
\end{cases}
\]

and \( v_0 \) is determined by\(^{12}\)

\[
w \int_{b_L}^{v_0} \rho_L(t) dt = \int_{b_H}^{wv_0} \rho_H(t) dt
\]

**Proof.** See appendix. 

The above equilibrium features a jump in the bidding function of the bidders with a lower minimum score (in the stated case, the low-yield bidders).\(^{13}\) For example, let \( m = 1 \), \( n = 5 \), \( \alpha = 0.5 \), \( F_L(v) = F_H(v) = v \), \( w = 0.5 \), \( b_L = 0 \), and \( b_H = 0.1 \). The jump in the low-yield bidders’ bidding function occurs at \( v_0 = 0.33 \) as shown in Figure 2(a). In this figure, low-yield bidders in region I compete among themselves; low-yield bidders in region II compete with both low- and high-yield bidders. High-yield bidders in region I beat any low-yield bidders in region I, yet cannot compete with low-yield bidders in region II due to the jump at low-yield bidders’ bidding function at \( v_0 \). High-yield bidders in region II compete with low-yield bidders in region II, while high-yield bidders in region III compete only with high-yield bidders, since low-yield bidders can no longer match their bids. Thus, there exist two kinks in high-yield bidders’ bidding function.

Intuitively, high-yield bidders with valuations near minimum bids \( b_H \) are unable to bid as low as they want due to the minimum-bid constraint. Therefore, their bids tend to cluster near the minimum bid. Consequently, it becomes profitable for some low-yield bidders to boost their bids to leap-frog all the clustered bids from high-yield bidders with whom they

\(^{12}\)As implied by (21), the condition for \( v_0 < 1 \) is \( w \int_{b_L}^{1} \rho_L(t) dt < \int_{b_H}^{w} \rho_H(t) dt \).

\(^{13}\)Strictly speaking, the low-yield bidder with \( v = v_0 \) is indifferent between bidding low and bidding high, and hence could use a mixed strategy. We assume in such a case the bidder always bids high. This assumption does not change the equilibrium outcome, since the probability measure is zero for the event a bidder has a unit-valuation \( v_0 \).
otherwise cannot compete. The benefit from leap-frogging decreases for low-yield bidders with lower unit-valuation. The last low-yield bidder to leap-frog is indifferent about bidding higher than all the clustered bids from high-yield bidders or bidding lower than them.

The case $b_L > b_H/w$ and/or $w > 1$ can be similarly analyzed except that when $b_L > b_H/w$, high-yield bidders’ bidding function (instead of low-yield bidders’) has a jump, and when $w > 1$, the low-yield bidders with unit-valuation $v \in [1/w, 1]$ beat any high-yield bidders.

### 6.2 Efficiency under Minimum Bids

We now consider the impact of minimum bids on allocation efficiency. The notion of efficiency we use is similar to Mark Armstrong’s [1]:

**Definition 1 (Weak Efficiency)** A mechanism is weakly efficient if it allocates assets in a way that maximizes the total expected valuation of all participating bidders.

Note that weak efficiency is slightly different from the typical notion of efficiency in that weak efficiency concerns the total valuation of participating bidders while the typical efficiency concerns the total valuation of both bidders and the auctioneer. An efficient mechanism must also be weakly efficient.

We say the auctioneer imposes a uniform minimum score if a minimum bid pair $(b_L, b_H)$ satisfies $b_L = b_H/w$ (i.e., the minimum score is the same for both low-yield and high-yield
Proposition 8  (a) A WUPA is weakly efficient if the auctioneer sets \( w = Q_L/Q_H \) and imposes a uniform minimum score.  (b) Given \( b_L < 1 \) and \( b_H < Q_L/Q_H \),\(^{14} \) a WUPA is weakly efficient only if \( w = Q_L/Q_H \) and \( b_L = b_H/w \).

Proof.  (a) If the auctioneer imposes a uniform minimum score, \( b_L = v_0 = b_H/w \), and Lemma 3 holds for all participating low-yield bidders \( v \in [b_L, 1] \). By the same argument in Proposition 3, \( w = Q_L/Q_H \) leads to an efficient allocation. Thus, a WUPA with a uniform minimum score and \( w = Q_L/Q_H \) is weakly efficient.

(b) We first show that a WUPA with \( b_L \neq b_H/w \) is inefficient. We first consider the case of \( b_L < b_H/w \) (the minimum bid for high-yield bidders is more constraining). In such a case, any \( w \) resulting in \( v_0 = 1 \) is not efficient, since a low-yield bidder with unit-valuation 1 have higher expected valuation but lower winning probability than a high-yield bidder with unit-valuation \( b_H \). For \( v_0 < 1 \), by Lemma 3, a low-yield bidder with unit-valuation \( v > v_0 \) ties with a high-yield bidder with unit-valuation \( wv \). By the same argument in Proposition 3, the WUPA is efficient at the high unit-valuation only if \( w = Q_L/Q_H \). However, if \( w = Q_L/Q_H \), by Proposition 7, high-yield bidders with unit-valuation \( v \in (b_H, wv_0) \) are unrivaled by low-yield bidders with unit-valuation \( v/w \), implying that high-yield bidders are inefficiently favored in the resource allocation by the current minimum bids. So, it is not possible for an auctioneer to achieve allocation efficiency with \( b_L < b_H/w \). By a similar argument, we can show that it is not possible for an auctioneer to achieve allocation efficiency with \( b_L > b_H/w \).

If \( b_L = b_H/w, v_0 = b_L = b_H/w \). By the same argument in Proposition 3, any \( w \neq Q_L/Q_H \) is not efficient. So, a WUPA is weakly efficient only if \( w = Q_L/Q_H \) and \( b_L = b_H/w \).

Proposition 8 shows that a simple and intuitive uniform minimum-score rule, together with an expected-yield-based weighting scheme, allows the auctioneer to achieve (weak) allocation efficiency despite bidders acting strategically. Because multiple minimum scores

\(^{14}b_L < 1 \) rules out a trivial case where only high-yield bidders participate. \( b_H < Q_L/Q_H \) assures that at least some low-yield bidders can match high-yield bidders in terms of the expected valuation.
can lead to weak efficiency, the auctioneer may compare the aggregate social surplus under different minimum scores to select the most efficient one (i.e., the one that generates the highest social surplus).

### 6.3 Minimum Bids and Revenue

Similar to the derivation of (11), we can explicitly evaluate the expected revenue of the auctioneer with minimum bids

\[
\pi = n(1 - \alpha)Q_L \int_{b_L}^{1} \rho_L(v) \left( v - \frac{1 - F_L(v)}{f_L(v)} \right) f_L(v)dv + n\alpha Q_H \int_{b_H}^{1} \rho_H(v) \left( v - \frac{1 - F_H(v)}{f_H(v)} \right) f_H(v)dv
\]

(22)

One may think the optimal minimum bids, as in standard auctions, should be chosen to admit only the bidders with positive virtual values, which requires the optimal minimum bids to satisfy:

\[
b_L - \frac{1 - F_L(b_L)}{f_L(b_L)} = 0 \quad \text{and} \quad b_H - \frac{1 - F_H(b_H)}{f_H(b_H)} = 0
\]

(23)

This, however, ignores the fact that in our setting, minimum bids also affect \(v_0\) and the expected winnings of participating bidders (refer to Proposition 7). In general, the traditional optimal minimum-bid policy, which solely aims to exclude bidders with negative virtual values, is not optimal in our model setting.

We derive the optimal minimum bids in the following proposition. For simplicity, we define \(J_\theta(v) \equiv v - \frac{1 - F_\theta(v)}{f_\theta(v)}, \theta \in \{H, L\}\).

**Proposition 9** Assume \(b_L \leq b_H/w \leq 1\) and \(w \leq 1\). The optimal minimum bids \((b_L^*, b_H^*)\) are the solution to the following system of equations

\[
(1 - \alpha)Q_L \int_{b_L^*}^{v_0} \frac{\partial \rho_L(v)}{\partial b_L} J_L(v) f_L(v) dv - \alpha Q_H \rho_H(b_H^*) J_H(b_H^*) f_H(b_H^*)
\]

\[+
\partial v_0 \left[ \alpha Q_H \int_{b_H^*}^{v_0} J_H(v) f_H(v) \frac{\partial \rho_H(v)}{\partial v_0} dv - (1 - \alpha) Q_L J_L(v_0) f_L(v_0) \rho_H(v) \left|_{v_0}^{v_0} \right| \right] = 0 \]  

(24)

\[
(1 - \alpha)Q_L J_L(b_L^*) f_L(b_L^*) \rho_L(b_L^*)
\]

\[+
\partial v_0 \left[ \alpha Q_H \int_{b_H^*}^{v_0} J_H(v) f_H(v) \frac{\partial \rho_H(v)}{\partial v_0} dv - (1 - \alpha) Q_L J_L(v_0) f_L(v_0) \rho_H(v) \left|_{v_0}^{v_0} \right| \right] = 0 \]  

(25)
where
\[
\frac{\partial v_0}{\partial b_H} = \frac{w \int_{b_L}^{v_0} \frac{d\rho_H(t)}{dt} dt - \rho_H(b_H)}{w \rho_H(wv_0) - w \rho_H(b_H) + \int_{b_L}^{v_0} \frac{d\rho_H(t)}{dt} dt}
\] (26)
\[
\frac{\partial v_0}{\partial b_L} = \frac{-w \rho_L(b_L)}{w \rho_H(wv_0) - w \rho_H(b_H) + \int_{b_L}^{v_0} \frac{d\rho_H(t)}{dt} dt}
\] (27)

Furthermore, the optimal minimum bids are generally not the solution to (23) or a uniform minimum score policy.

The equation (24) shows that the optimal minimum-bid for low-yield bidders is a tradeoff among three different effects: the effect on low-yield bidders’ participation \(b_L\), the effect on the jump point \(v_0\), and the indirect effect on high-yield bidders’ optimal participation \(b_H\).

To see that a uniform minimum score is not optimal, we note that under a uniform minimum score policy, \(v_0 = b_L\) and \(wv_0 = b_H\), and the conditions (24) and (25) reduce to
\[
J_L(b_L) = 0 \text{ and } J_H(wb_L) = 0
\] (28)
which generally do not hold at the same time. So the uniform minimum score is not optimal in general.

The following example illustrates how the optimal minimum bids in our setting differ from the uniform minimum score and the traditional ones, and the improvement of revenue from imposing the optimal minimum bids.

**Example 2** Let \(m = 1\), \(n = 5\), \(\alpha = 0.5\), \(Q_L = 0.8\), \(Q_H = 1\), and \(w = 0.8\). Assume that \(F_H(v) = v\) and \(F_L(v) = 2v - v^2\). (1) Under the traditionally “optimal” minimum bids \((b_L = \frac{1}{3}, b_H = \frac{1}{2})\), the expected revenue is \(\pi = 0.5213\). (2) Under the optimal uniform minimum score \((b_L = 0.4812, b_H = wb_L)\), the expected revenue is \(\pi = 0.5211\). (3) Under the optimal minimum bids \((b_L = 0.334, b_H = 0.6152)\), the expected revenue is \(\pi = 0.5309\).

7 Concluding Remarks

Motivated by the practices of selling keyword advertising slots, we investigate an auction setting in which the auctioneer learns not only bidders’ willingness to pay for per-unit yield
but also the differences in their abilities to generate yields. We focus on a unique class of auction mechanisms, weighted unit-price auctions, which allows the auctioneer to act on signals of bidders’ future yield by setting different weighting factors for bids and imposing different minimum bids. An important feature of WUPAs is that they can use the ex-ante non-price information on bidders, for example, the accumulated knowledge on advertisers’ performance (in generating clicks, calls, or purchases) in the keyword advertising context. The application of WUPAs in keyword advertising has lead to Google’s dramatic success. We believe as the state-of-the-art on measuring advertisement performances advances, WUPAs will continue to grow in importance.

We investigate how WUPA design dimensions, namely, the weighting scheme and minimum-bid policy, affect equilibrium bidding strategies, resource allocation efficiency, and the auctioneer’s revenue. One of our main findings is that auctioneers can achieve resource allocation efficiency with WUPAs by using estimates of future yields as weighting factors and setting a uniform minimum score for bidders with different yield signals. Such a weighting scheme and a minimum-bid policy are both intuitive and easy to implement as they require no knowledge on how bidders’ unit-valuations are distributed. Our findings also put (efficient) WUPAs on a par with standard auctions in terms of efficiency.

Auctioneers can achieve superior revenue by strategically setting the weighting scheme. The task of choosing a profit-maximizing weighting scheme generally requires knowledge both on estimates of bidders’ future yields and their unit-valuation distribution. If bidders differ only in expected yield (i.e., yields are independent of unit-valuations), the optimal weighting scheme involves subsidizing the disadvantaged, that is, the low-yield bidders. If bidders differ in both unit-valuation distributions and expected yields, the wisdom of “subsidizing the disadvantaged” still works, but in a twisted way: when bidders differ in two dimensions (unit-valuations and expected yields), it may not be easy to tell who is the disadvantaged. As the unit-valuation of the low-yield bidders increases (stochastically), they gain in competitive advantage, and it may even be optimal for the auctioneer to subsidize the high-yield bidders.
The latter case is possible when, for instance, advertisements with higher click-through rates (higher yield) have lower conversion rates (thus lower unit-valuation).

Setting minimum bids for bidders with different yield signals may help the auctioneer to obtain extra revenue. These minimum bids not only set the bars to entry; they can also cause additional distortion among participating bidders—a phenomena not observed in standard auction settings. The optimal minimum-bid policy in WUPAs is generally different from those suggested in the optimal mechanism design literature. As with weighting schemes, implementing minimum bids also requires knowledge about estimates of bidders’ future yields and distribution of their unit-valuations. We note that no matter what the auctioneer’s objective is (maximizing efficiency or maximizing revenue), the quality of information on bidders’ future yields is important. It pays off to invest in technologies and procedures to improve the estimation of bidders’ future yields.

WUPAs are generally not optimal mechanisms. Nevertheless, WUPAs may be preferred for several reasons. Besides the revenue and efficiency features we discuss above, WUPAs also impose fewer risks on bidders than bid-on-total-payment auctions. For example, in keyword auctions, advertisers pay only when advertisements lead Internet users to click or to purchase. Such an arrangement hedges some of the advertisers’ risks and is more sensible and acceptable to advertisers. WUPAs are also more intuitive and easier to work with than more elaborate (non-linear) bidding and payment schedules.

Finally, our analysis of WUPAs may have implications for designing online procurement auctions. The use of online procurement auctions has grown dramatically in recent years. However, current online auctions are controversial in that they tend to select winners based on low prices and pay inadequate attention to other dimensions such as quality, delivery, and services (Beall et al. [3]). We contend that as buyers accumulate knowledge on bidders’ non-price attributes over the years, they may base weighting factors on such knowledge in determining winners and payments. Of course, procurement settings have special needs that may require adaptation of WUPAs. For instance, buyers may have to consider the cost of
switching suppliers, and they may be reluctant to reveal their true preferences for business secrecy reasons.

References


A Appendix

A.1 Proof of Proposition 1

Lemma 1 implies that $\phi_H(wb) = w\phi_L(b)$ for $b \in [0, b_L(1)]$. Substituting this into (4) and (5), we can uniformly write the payoff functions as

$$U_\theta(v, b) = Q_\theta(v - b)\rho_\theta(\phi_\theta(b))$$  \hspace{1cm} (29)

We denote

$$V_\theta(v) \equiv U_\theta(v, b_\theta(v)) = Q_\theta(v - b_\theta(v))\rho_\theta(\phi_\theta(b_\theta(v)))$$  \hspace{1cm} (30)

as the equilibrium payoff of a bidder with unit-valuation $v$.

$$\frac{dV_\theta(v)}{dv} = \frac{\partial U_\theta(v, b_\theta(v))}{\partial v} + \frac{\partial U_\theta(v, b_\theta(v))}{\partial b} \frac{db_\theta(v)}{dv}$$  
$$= \frac{\partial U_\theta(v, b_\theta(v))}{\partial v}$$  
$$= Q_\theta\rho_\theta(v)$$  \hspace{1cm} (31)

where the second equality is due to $\frac{\partial U_\theta(v, b_\theta(v))}{\partial b} = 0$ (the first-order condition). Applying the boundary condition $V_\theta(0) = 0$, we get

$$V_\theta(v) = Q_\theta \int_0^v \rho_\theta(t)dt$$  \hspace{1cm} (32)

Combining (30) (note $\phi_\theta(b_\theta(v)) = v$) and (32), we can solve the bidding function as

$$b_\theta(v) = v - \int_0^v \frac{\rho_\theta(t)dt}{\rho_\theta(v)}$$  \hspace{1cm} (33)

Now we show that $\frac{db_\theta(v)}{dv} > 0$.

$$\frac{db_\theta(v)}{dv} = \frac{\rho_\theta'(v) \int_0^v \rho_\theta(t)dt}{\rho_\theta^2(v)}$$  \hspace{1cm} (34)

The sign of the above first-order derivative is solely determined by that of $\rho_\theta'(v)$. It is sufficient to show $\rho_\theta'(v) > 0$, or $\sum_{j=1}^m \delta_j P_\theta^{ij}(v) > 0$.

$$P_\theta^{ij}(v) = (\frac{n-1}{n-j})G_\theta(v)^{n-j-1} (1 - G_\theta(v))^{j-2} [(n - j) - (n - 1)G_\theta(v)] G_\theta'(v)$$  \hspace{1cm} (35)
Notice that $P_{\theta}^{1}(v) \geq 0$ and $P_{\theta}^{j}(v) \leq 0$ for all $v$; $P_{\theta}^{j}(v) (1 < j < n)$ crosses zero only once from positive to negative on $(0, 1)$. The crossing point, $v_{c}^{j}$, is the solution to $G_{\theta}(v_{c}^{j}) = \frac{n-j}{n-1}$. It is clear that $0 < v_{c,n-1}^{c} < \ldots < v_{c}^{3} < v_{c}^{2} < 1$. Thus, for a given $v \in (0, 1)$, there exists $j_{v} \in \{1, 2, \ldots, n-1\}$ such that

$$P_{\theta}^{j_{v}}(v) > 0, \text{ for } j = 1, \ldots, j_{v}, \text{ and } P_{\theta}^{j_{v}}(v) \leq 0, \text{ for } j = j_{v} + 1, \ldots, n. \quad (36)$$

Let $\delta_{m+1} = \delta_{m+2} = \ldots = \delta_{n} = 0$. We have

$$\sum_{j=1}^{m} \delta_{j}P_{\theta}^{j}(v) = \sum_{j=1}^{n} \delta_{j}P_{\theta}^{j}(v) > \delta_{j_{v}} \sum_{j=1}^{n} P_{\theta}^{j}(v) = 0 \quad (37)$$

where the inequality is due to $\delta_{1} \geq \delta_{2} \geq \ldots \geq \delta_{n}$ and (36), and the last equality is due to the fact that $\sum_{j=1}^{n} P_{\theta}^{j}(v) = (G_{\theta}(v) + 1 - G_{\theta}(v))^{n-1} = 1$.

### A.2 The Derivation of Expected Revenue

Let $M_{\theta}(v)$ denote the expected payment from a bidder with unit-valuation $v$ and signal $\theta$ in the first-score WUPA. The expected payment from a bidder is equal to the bidder’s total expected valuation upon winning minus the bidder’s expected payoff.

$$M_{\theta}(v) = Q_{\theta}v\rho_{\theta}(v) - V_{\theta}(v) = Q_{\theta} \left[ v\rho_{\theta}(v) - \int_{0}^{v} \rho_{\theta}(t)dt \right] \quad (38)$$

The expected payment from one bidder (with probability $\alpha$ being high-yield and with probability $(1 - \alpha)$ being low-yield) is

$$\alpha E[M_{H}(v)] + (1 - \alpha) E[M_{L}(v)] = \alpha Q_{H} \int_{0}^{1} \left[ v\rho_{H}(v) - \int_{0}^{v} \rho_{H}(t)dt \right] f_{H}(v)dv + (1 - \alpha) Q_{L} \int_{0}^{1} \left[ v\rho_{L}(v) - \int_{0}^{v} \rho_{L}(t)dt \right] f_{L}(v)dv = \alpha Q_{H} \int_{0}^{1} \rho_{H}(v) \left[ v - \frac{1 - F_{H}(v)}{f_{H}(v)} \right] f_{H}(v)dv + (1 - \alpha) Q_{L} \int_{0}^{1} \rho_{L}(v) \left[ v - \frac{1 - F_{L}(v)}{f_{L}(v)} \right] f_{L}(v)dv \quad (39)$$

The total expected revenue from all bidders is $n$ times the above.
A.3 Proof of Proposition 3

First note that $G_H(wv) = G_L(v)$ and $\frac{dG_H(wv)}{dw} \big|_{wv} = -\frac{1-\alpha}{\alpha w} f_L(v) \frac{dG_L(v)}{dw}$. We can establish

$$\frac{d\rho_H(v)}{dw} \big|_{wv} = -\frac{1-\alpha}{\alpha w} f_L(v) \frac{d\rho_L(v)}{dw}$$ (40)

In addition, using the same technique in Proof of Proposition 1, we can show

$$\frac{d\rho_L(v)}{dw} > 0$$ (41)

Taking the first-order derivative of (13) with respect to $w$ yields

$$(1-\alpha) Q_L \int_0^1 v \frac{d\rho_L(v)}{dw} f_L(v) dv + \alpha Q_H \int_0^1 v \frac{d\rho_H(v)}{dw} f_H(v) dv$$ (42)

If $w \leq 1$, noting $\frac{d\rho_H(v)}{dw} = 0$ for $v > w$, we can re-organize (42) as

$$(1-\alpha) Q_L \int_0^1 v \frac{d\rho_L(v)}{dw} f_L(v) dv + \alpha Q_H \int_0^w v \frac{d\rho_H(v)}{dw} f_H(v) dv$$

$$(1-\alpha) \int_0^1 v \frac{d\rho_L(v)}{dw} f_L(v) dv = (1-\alpha) (Q_L - wQ_H) \int_0^1 v \frac{d\rho_L(v)}{dw} f_L(v) dv$$ (43)

where the second equality is due to integration by substitution and (40). Because $\frac{d\rho_L(v)}{dw} > 0$ (41), the above first order derivative is positive if $w < \frac{Q_L}{Q_H}$ and negative if $w > \frac{Q_L}{Q_H}$. So $w = \frac{Q_L}{Q_H}$ maximizes the social welfare among all $w \in [0, 1]$.

Using a similar logic, we can verify $w > 1$ cannot maximize the social welfare. So, $w_{eff} = \frac{Q_L}{Q_H}$.

A.4 Proof of Proposition 4

Using the similar approach as in Proof for Proposition 1, we can derive the bidding function for the generalized first-price auction as

$$b(x) = x - \frac{\sum_{j=1}^m \delta_j \int_0^x \frac{(n-j)}{(n-j)^2-1} H(t)^{n-j} [1 - H(t)]^{j-1} dt}{\sum_{j=1}^m \delta_j \frac{(n-j)}{(n-j)^2-1} H(x)^{n-j} [1 - H(x)]^{j-1}}$$
If this bid is from a low-yield bidder, let $v = x/Q_L$. Noting that $H(Q_Lv) = \alpha F_H\left(\frac{Q_L}{Q_H} v\right) + (1 - \alpha) F_L(v) = \alpha F_H(w_{eff}v) + (1 - \alpha) F_L(v) = G_L(v)$, we have

$$b(x) = Q_Lv - \sum_{j=1}^{m} \delta_j \int_{0}^{Q_Lv} \left(\frac{1 - Q_L}{v - \frac{1 - F_L(v)}{f_L(v)}}\right) f_L(v)dv$$

$$= Q_Lv - \int_{0}^{v} \frac{\rho_L(t)dt}{\rho_L(v)}$$

$$= Q_L b_L(v)$$

which means the total payment the bidder bids is exactly the unit-price he/she would bid under efficient WUPAs times his/her expected yield. Similar argument holds if the bid is from a high-yield bidder. Therefore, efficient WUPAs are revenue-equivalent to generalized first-price auctions.

### A.5 Proof of Proposition 5

Taking the first order derivative of the expected revenue (11) with respect to $w$ yields

$$\frac{d\pi}{dw} = (1 - \alpha) Q_L \int_{0}^{1} \frac{d\rho_L(v)}{dw} \left(v - \frac{1 - F_L(v)}{f_L(v)}\right) f_L(v)dv$$

$$+ \alpha Q_H \int_{0}^{w} \frac{d\rho_H(v)}{dw} \left(v - \frac{1 - F_H(v)}{f_H(v)}\right) f_H(v)dv$$

We only need to check the sign of $\frac{d\pi}{dw}$ for $0 < w \leq \frac{Q_L}{Q_H}$. For $0 < w \leq \frac{Q_L}{Q_H}$,

$$\frac{d\pi}{dw} = (1 - \alpha) Q_L \int_{0}^{1} \frac{d\rho_L(v)}{dw} \left(v - \frac{1 - F_L(v)}{f_L(v)}\right) f_L(v)dv$$

$$+ \alpha Q_H \int_{0}^{w} \frac{d\rho_H(v)}{dw} \left(v - \frac{1 - F_H(v)}{f_H(v)}\right) f_H(v)dv$$

$$= (1 - \alpha) Q_L \int_{0}^{1} \frac{d\rho_L(v)}{dw} \left(v - \frac{1 - F_L(v)}{f_L(v)}\right) f_L(v)dv$$

$$- (1 - \alpha) Q_H \int_{0}^{1} \frac{d\rho_L(v)}{dw} \left(wv - \frac{1 - F_H(wv)}{f_H(wv)}\right) f_L(v)dv$$

$$= (1 - \alpha) \int_{0}^{1} \left(\frac{d\rho_L(v)}{dw} f_L(v)\right) Q_L \left(v - \frac{1 - F_L(v)}{f_L(v)}\right) - Q_H \left(wv - \frac{1 - F_H(wv)}{f_H(wv)}\right) dv$$

$$= (1 - \alpha) \int_{0}^{1} \left(\frac{d\rho_L(v)}{dw} f_L(v)\right) \left[v (Q_L - Q_Hw) + Q_H \frac{1 - F_H(wv)}{f_H(wv)} - Q_L \frac{1 - F_L(v)}{f_L(v)}\right] dv$$

[34]
where the first equality is because for $v > w$, $\frac{d\rho_H(v)}{dw} = 0$ and the second equality is due to (40).

Note that $\frac{d\rho_L(v)}{dw} > 0$ by (41). Clearly, for $0 < w \leq \frac{QL}{QH}$, $v(Q_L - Q_Hw) \geq 0$. By the IHR property (note that $F_L(\cdot) = F_H(\cdot) = F(\cdot)$),

$$Q_H \frac{1 - F_H(wv)}{f_H(wv)} - Q_L \frac{1 - F_L(v)}{f_L(v)} > 0,$$

(47)

So, (46) is greater than 0, which implies $w_{opt} > \frac{QL}{QH}$.

A.6 Proof of Proposition 6

Similar to (45), under $(\tilde{F}_L, F_H)$, the first-order derivative of the expected revenue with respect to $w$

$$\frac{d\tilde{\pi}}{dw} = (1 - \alpha) \int_0^1 \frac{d\tilde{\rho}_L(v)}{dw} f_L(v) \left[ Q_L \left( v - \frac{1 - \tilde{F}_L(v)}{f_L(v)} \right) - Q_H \left( wv - \frac{1 - F_H(wv)}{f_H(wv)} \right) \right] dv$$

(48)

Notice that because $F_H(v)$ is IHR, $v - \frac{1 - F_H(v)}{f_H(v)}$ is strictly increasing. Furthermore, if (16) holds, $Q_L \left( v - \frac{1 - \tilde{F}_L(v)}{f_L(v)} \right) - Q_H \left( wv - \frac{1 - F_H(wv)}{f_H(wv)} \right) > 0$ for all $w \in (0, w_{opt}]$. Also, similar to (41), $\frac{d\tilde{\rho}_L(v)}{dw} > 0$. So $\frac{d\tilde{\pi}}{dw} > 0$ for all $w \in (0, w_{opt}]$, and therefore $\tilde{w}_{opt} > w_{opt}$.

A.7 Proof of the strict monotonicity of bidding functions

Take low-yield bidders as an example. The incentive compatibility conditions requires that for any $v'' > v'$,

$$[v' - b_L(v')] \rho_L(v') \geq [v' - b_L(v'')] \rho_L(v'')$$

(49)

$$[v'' - b_L(v'')] \rho_L(v'') \geq [v'' - \rho_L(v')] \rho_L(v')$$

(50)

Combining (49) and (50), we get $\rho_L(v'') \geq \rho_L(v')$, which implies $b_L(v'') \geq b_L(v')$. Next we show $b_L(v') \neq b_L(v'')$. Actually, if $b_L(v') = b_L(v'')$, then $b_L(v) = b$ for all $v \in [v', v'']$. A low-yield bidder with unit-valuation $v''$ is better off by bidding $b + \epsilon$ ($\epsilon$ is an infinitesimal positive number) since the bidder loses only $\epsilon$ for per-unit resource awarded, yet improves
its expected winning by a significant amount. This contradicts the equilibrium condition. Therefore \( b_L(v) \) must be strictly increasing.

### A.8 Proof for Lemma 3

We suppose there exists a mapping \( \Lambda : (v_0, 1] \rightarrow [h_H, 1] \) such that \( wb_L(v) = b_H(\Lambda(v)) \). That is, a low-yield bidder with \( v \) will tie with a high-yield bidder \( \Lambda(v) \) in equilibrium. Similarly, we define \( P^j_\theta(v) \equiv \binom{n-1}{n-j} [G_\theta(v)]^{n-j} [1 - G_\theta(v)]^{j-1} \) and \( \rho_\theta(v) \equiv \sum_{j=1}^m P^j_\theta(v) \delta_j, \theta \in \{L, H\} \), where

\[
G_L(v) = \left[ (1 - \alpha) F_L(v) + \alpha F_H(\Lambda(v)) \right], \text{ for all } 1 \geq v > v_0 \\
G_H(v) = \left[ (1 - \alpha) F_L(\Lambda^{-1}(v)) + \alpha F_H(v) \right], \text{ for all } \Lambda(1) \geq v > \Lambda(v_0)
\]

We can then solve the equilibrium bidding for each bidder type as

\[
b_L(v) = v - \frac{U^0_L/Q_L + \int_{v_0}^v \rho_L(t) \, dt}{\rho_L(v)} \\
b_H(v) = v - \frac{U^0_H/Q_H + \int_{\Lambda(v_0)}^{v_\Lambda} \rho_H(t) \, dt}{\rho_H(v)}
\]  

where \( U^0_L \) and \( U^0_H \) are equilibrium payoff of a low-yield bidder with unit-valuation \( v_0 \) and equilibrium payoff of a high-yield bidder with unit-valuation \( \Lambda(v_0) \), respectively. By \( wb_L(v) = b_H(\Lambda(v)) \),

\[
w \left[ v - \frac{U^0_L/Q_L + \int_{v_0}^v \rho_L(t) \, dt}{\rho_L(v)} \right] = \Lambda(v) - \frac{U^0_H/Q_H + \int_{\Lambda(v_0)}^{v_\Lambda} \rho_H(t) \, dt}{\rho_H(\Lambda(v))} \\
= \Lambda(v) - \frac{U^0_H/Q_H + \int_{v_0}^v \rho_L(t) \Lambda'(t) \, dt}{\rho_L(v)}
\]

where the second step is due to \( \rho_L(v) = \rho_H(\Lambda(v)) \). We multiply both sides of (53) by \( \rho_L(v) \) and take the first-order derivative with respect to \( v \),

\[
w [v \rho'_L + \rho_L - \rho_L] = \Lambda' \rho_L + \rho'_L \Lambda - \rho_L \Lambda'
\]

so we get \( \Lambda(v) = rv \).
A.9 Proof of Proposition 7

Our analysis in section 6.1 implies that a low-yield bidder with \( v \in [b_H, v_0) \) participates but cannot compete with any participating high-yield bidders. The probability for such a bidder to beat any other bidder is \( G_L(v) = \alpha F_H(b_H) + (1 - \alpha) F_L(v) \). For a low-yield bidder with \( v \in [v_0, 1] \) (who competes with both low-yield bidders and high-yield bidders), \( G_L(v) = \alpha F_H(wv) + (1 - \alpha) F_L(v) \). Similarly, we can obtain the probability of beating any other bidder for high-yield bidders with unit-valuations in \([b_H, wv_0]\) (who beat any low-yield bidders in \([b_H, v_0]\) but none of the low-yield bidders in \([v_0, 1]\)), in \([wv_0, w]\) (who compete both with high-yield bidders and low-yield bidders), and in \((w, 1]\) (who beat any low-yield bidders). The equilibrium winning and the equilibrium bidding functions follow naturally. The only undetermined variable is \( v_0 \). Notice that Lemma 3 implies for any \( v \in [v_0, 1] \),

\[
V_H(wv) = Q_H(wv - b_H(wv)) \rho_H(wv) = Q_H w(v - b_L(v)) \rho_L(v) = \frac{w Q_H}{Q_L} V_L(v) \quad (55)
\]

Meanwhile, we have (by a similar process in the proof of Proposition 1)

\[
V_L(v_0) = Q_L \int_{b_L}^{v_0} \rho_L(t) dt \quad \text{and} \quad V_H(wv_0) = Q_H \int_{b_H}^{wv_0} \rho_H(t) dt. \quad (56)
\]

Evaluating (55) at \( v = v_0 \) and substituting the above, we immediately have \( w \int_{b_L}^{v_0} \rho_L(t) dt = \int_{b_H}^{wv_0} \rho_H(t) dt \), which determines \( v_0 \). We can verify that the bidding strategies obtained in the above process constitute an equilibrium.

A.10 Proof of Proposition 9

Define \( \rho_L(v_0^-) \equiv \lim_{v \to v_0^-} \rho_L(v) \). Taking the partial derivative of of (22) with respect to \( b_H \) and \( b_L \), respectively, we obtain the first-order conditions (note that \( v_0 \) is a function of \( b_H \) and \( b_L \)).
and \( b_L \)
\[
(1 - \alpha) Q_L \rho_L \left(v_0^-\right) J_L \left(v_0\right) f_L \left(v_0\right) \frac{\partial v_0}{\partial b_H} + (1 - \alpha) Q_L \int_{b_L}^{v_0} \frac{d\rho_L(v)}{dv} J_L \left(v\right) f_L \left(v\right) dv \\
- (1 - \alpha) Q_L \rho_L \left(v_0\right) J_L \left(v_0\right) f_L \left(v_0\right) \frac{\partial v_0}{\partial b_H} - \alpha Q_H \rho_H \left(b_H^-\right) J_H \left(b_H^-\right) f_H \left(b_H^-\right) \\
+ \alpha Q_H \frac{\partial v_0}{\partial b_H} \int_{b_H}^{v_0} \frac{d\rho_H(v)}{dv} J_H \left(v\right) f_H \left(v\right) dv = 0 \\
- (1 - \alpha) Q_L \rho_L \left(b_L^-\right) J_L \left(b_L^-\right) f_L \left(b_L^-\right) + (1 - \alpha) Q_L \rho_L \left(v_0^-\right) J_L \left(v_0\right) f_L \left(v_0\right) \frac{\partial v_0}{\partial b_H} \\
- (1 - \alpha) Q_L \rho_L \left(v_0\right) J_L \left(v_0\right) f_L \left(v_0\right) \frac{\partial v_0}{\partial b_H} + \alpha Q_H \frac{\partial v_0}{\partial b_H} \int_{b_H}^{v_0} \frac{d\rho_H(v)}{dv} J_H \left(v\right) f_H \left(v\right) dv = 0 
\]  
\( (57) \)

Substituting \( \rho_L \left(v_0^-\right) = \rho_H \left(b_H^-\right) \) and \( \rho_L \left(v_0\right) = \rho_H \left(wv_0\right) \) and reorganizing the above two equations
\[
(1 - \alpha) Q_L \int_{b_L}^{v_0} \frac{d\rho_L(v)}{dv} J_L \left(v\right) f_L \left(v\right) dv - \alpha Q_H \rho_H \left(b_H^-\right) J_H \left(b_H^-\right) f_H \left(b_H^-\right) \\
+ \frac{\partial v_0}{\partial b_H} \left[ \alpha Q_H \int_{b_H}^{v_0} J_H \left(v\right) f_H \left(v\right) \frac{d\rho_H(v)}{dv} dv - \left(1 - \alpha\right) Q_L J_L \left(v_0\right) f_L \left(v_0\right) \rho_H \left(v\right) \right]_{b_H^2}^{v_0} = 0 
\]  
\( (59) \)

\[
- (1 - \alpha) Q_L J_L \left(b_L^-\right) f_L \left(b_L^-\right) \rho_L \left(b_L^-\right) + \frac{\partial v_0}{\partial b_H} \left[ \alpha Q_H \int_{b_H}^{v_0} J_H \left(v\right) f_H \left(v\right) \frac{d\rho_H(v)}{dv} dv - \left(1 - \alpha\right) Q_L J_L \left(v_0\right) f_L \left(v_0\right) \rho_H \left(v\right) \right]_{b_H^2}^{v_0} = 0 
\]  
\( (60) \)

where \( \frac{\partial v_0}{\partial b_H} \) and \( \frac{\partial v_0}{\partial b_L} \) can be derived from the partial derivatives of both sides of equation (21) with respect to \( b_H \) and \( b_L \), respectively
\[
w \rho_L \left(v_0^-\right) \frac{\partial v_0}{\partial b_H} + w \int_{b_L}^{v_0} \frac{d\rho_L(t)}{dt} dt = w \rho_H \left(wv_0\right) \frac{\partial v_0}{\partial b_H} - \rho_H \left(b_H^-\right) + \frac{\partial v_0}{\partial b_H} \int_{b_H}^{v_0} \frac{d\rho_H(t)}{dv_0} dt \\
-w \rho_L \left(b_L^-\right) + w \rho_L \left(v_0^-\right) \frac{\partial v_0}{\partial b_L} = w \rho_H \left(wv_0\right) \frac{\partial v_0}{\partial b_L} + \frac{\partial v_0}{\partial b_L} \int_{b_H}^{v_0} \frac{d\rho_H(v)}{dv_0} dv
\]

Substituting \( \rho_L \left(v_0^-\right) = \rho_H \left(b_H^-\right) \) and re-organizing the above, we can get (26) and (27).

The system of equations (59) and (60) allows us to solve the optimal minimum bids for low-yield bidders \( (b_L^-) \) and high-yield bidders \( (b_H^-) \). For example, solving (59) we can get \( b_H^- = b_H^* \left(b_L^-\right) \). Substituting \( b_H^* \left(b_L^-\right) \) into (60), we can derive \( b_L^* \).