The min mean-weight cycle in a random network

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Abstract

The mean weight of a cycle in an edge-weighted graph is the sum of the cycle’s edge weights divided by the cycle’s length. We study the minimum mean-weight cycle on the complete graph on $n$ vertices, with random i.i.d. edge weights drawn from an exponential distribution with mean 1. We show that the probability of the min mean-weight being at most $c/n$ tends to a limiting function of $c$ which is analytic for $c \leq 1/e$, discontinuous at $c = 1/e$, and equal to 1 for $c > 1/e$. We further show that if the min mean-weight is $\leq 1/(en)$, then the length of the relevant cycle is $\Theta(1)$ (i.e., it has a limiting probability distribution which does not scale with $n$), but that if the min mean-weight is $> 1/(en)$, then the relevant cycle almost always has mean weight $(1 + o(1))/(en)$ and length at least $(2/\pi^2 - o(1)) \log^2 n \log \log n$.

1 Introduction

Many combinatorial optimization problems have been studied when the input is a complete (directed or undirected) graph with independent random weights on the edges. This line of work has been active since the mid-1980s for problems such as the minimum spanning tree [Fri85, FM89], shortest path [EG85, HVM08, Jan99, HHVM06, HHVM07], traveling salesman path [Fri04], minimum weight perfect matching (the assignment problem) [Ald01, LW04, NPS03], spanners [CFMS09], and Steiner tree [BGRS04, AFW08]. In this paper, we study the minimum mean-weight cycle.

Given a directed graph with arc weights, the minimum mean-weight cycle problem is that of finding a cycle with minimum mean weight. The mean weight of a cycle is the ratio between its total weight and its number of arcs. The min mean weight cycle problem, and the closely related minimum ratio cycle problem (where each arc has a cost and a transit time, and the mean ratio of a cycle is the total cost divided by the total transit time), have applications in areas ranging from discrete event systems and computer-aided design to graph theory; see Dasdan [Das04] for a detailed discussion and references. An experimental study of various algorithms for min mean cycle can be found in [GGTW09], including experiments on random graphs. An algorithm by Orlin, Tarjan, and Young [YTO91] emerges as particularly efficient. Their algorithm is based on the parametric shortest path problem, which is the
problem of finding shortest paths in graph where the edge costs are of the form $w_{i,j} + \lambda$, where each $w_{i,j}$ is constant and $\lambda$ is a parameter that varies. This problem is well-defined when $\lambda$ is at least

$$- \min_{\text{cycle } C} \frac{\sum_{ij \in C} w_{i,j}}{|C|},$$

but when $\lambda$ is below this value, there is a negative cycle, so the problem becomes ill-defined. The authors of [YTO91] conjectured that their algorithm is faster on average than in the worst case, by a factor of $n$; analyzing the structure of the min mean cycle is an intermediate step towards that conjecture.

In this paper, we study the min mean-weight cycle in the complete graph on $n$ vertices, with random i.i.d. edge weights drawn from an exponential distribution with mean 1, so that $\Pr[w_e > x] = e^{-x}$. We do this for both the directed complete graph, which is relevant to the experiments of Young, Tarjan, and Orlin [YTO91] and subsequent experiments, and for the undirected complete graph, so that we can more readily compare our results to earlier work on cycles in the random graph $G_{n,p}$ [Jan87, FKP89].

The min max-weight cycle has been studied by Janson [Jan87] and others [FKP89]. One way to instantiate the random graph $G_{n,p}$ is to put each edge in $G_{n,p}$ if its weight is smaller than $\log 1/(1 - p)$ (or if we instead use weights that are uniform in $[0, 1]$, the edge is included if its weight is smaller than $p$). As the parameter $p$ is increased from 0 to 1, the first cycle to appear is the min max-weight cycle. Janson [Jan87] gives formulas for when that cycle occurs (i.e., its max-weight), and for its length distribution: the probability that the min max-weight cycle has max weight less than $c/n$ tends to a continuous function of $c$, which is analytic and increases from 0 to 1 as $c$ increases from 0 to 1, is non-analytic but continuous at $c = 1$, and equals 1 for $c > 1$ (see Figure 1). The limiting length distribution (see Table 1) is completely supported on finite values (i.e., which don’t grow with $n$), but this distribution has a fat tail which gives it an infinite expected value. (For finite $n$, the expected length is order $n^{1/6}$ [FKP89].)

We find that the min mean-weight cycle has a qualitatively different behavior: the probability that the min mean-weight cycle has mean weight at most $c/n$ tends to a function of $c$ which is piece-wise analytic, but which is discontinuous at $c = 1/e$ (see Figure 1). For finite $n$, the mean weight of the min mean-weight cycle is with constant probability within an interval $(1/e, 1/e + o(1))/n$. Furthermore, the limiting length distribution of the min mean-weight cycle is not supported on finite values. To be more precise, the probability that the length of the min mean-weight cycle is length $k$ tends to a positive limiting value $p_k$, but $\sum_k p_k < 1$. (See Table 1) The interpretation of this for finite $n$ is that, with constant probability the cycle has length order 1, and with constant probability the cycle has a length which is a function of $n$ tending to infinity.

It is natural to ask what this function of $n$ is. The behavior of the long cycles is complicated, and we do not conjecture a value for the true answer. The best that we could prove is that the length of the min mean-weight cycle is almost always either $O(1)$ or else at least $(2/\pi^2 - o(1)) \log^2 n \log \log n$. 
Figure 1: The probability that the min max/mean-weight cycle has max/mean-weight less than $c/n$. In the left panels are the graphs for the min max-weight cycle, in the right panels are the graphs for the min mean-weight cycle. In the upper panels are the graphs for the undirected complete graph, in the lower panels are the graphs for the directed complete graph. (The upper left panel was computed by Janson [Jan87].) For the min max-weight cycle, the function is non-analytic at $c = 1$, but is continuous. For the min mean-weight cycle, the function is discontinuous at $c = 1/e$.

In the related problem of finding the maximum length path whose mean weight is at most $c/n$, Aldous [Ald98] found that there is a transition point at $c = 1/e$, where for fixed $c < 1/e$ the length is $o(n)$, and for fixed $c > 1/e$ the length is order $n$. Recently Ding [Din11] studied the behavior of this path length when $c$ is at or near $1/e$, and proved that the length exhibits a transition at $c = 1/e + \Theta(1/\log^2 n)$, with unspecified constants. By comparison, we prove that with probability $1 - o(1)$ the min mean-weight cycle has mean weight at most $(1/e + (\pi^2 + o(1))/(2e \log^2 n))/n$, but we do not know if the $O(1/\log^2 n)$ correction term is sharp.

Whether the complete graph is directed or undirected will affect the length distribution and the max/mean-weight of the min max/mean-weight cycle, but each of the qualitative
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Table 1: The limiting length distribution of the min max/mean-weight cycle. The left-most column is due to Janson [Jan87]. Here $T(c) = \sum_{k=1}^{\infty} k^{k-1} e^k / k!$ is the “tree function”. For the min max-weight cycle, the length distribution is supported on finite values, while for the min mean-weight cycle, a constant fraction of the probability mass $(1 - \sum k p_k)$ drifts off to infinity. The size of the jumps at the discontinuities in Figure 1 is $1 - \sum k p_k$.

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<td>$p_2$</td>
<td>0.281718</td>
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<td>$p_9$</td>
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behaviors discussed above are unaffected by whether the graph is directed or undirected (as shown in Figure 1 and Table 1). (Though the exponent characterizing the fatness of the tail of the length distribution does change for the min max-weight cycle.)

We call a cycle $c$-light if its mean weight is $< c/n$. We start with an elementary calculation of the expected number of $c$-light cycles of length $k$. Then we show that for $c \leq 1/e$, the set of light cycles is well approximated by a Poisson process with intensity given by the first-moment computation. For $c > 1/e$, the number of $c$-light cycles diverges. Given this Poisson approximation, it is straightforward to do the computations illustrated in Figure 1.
A key difference between the min mean-length cycle and min max-length cycle is that the expected number of $c$-light cycles is finite at the critical value $c = 1/e$, while for the max-length cycles, the expected number of light cycles diverges at the critical value of $c$. As we will explain, the finite expected number of light cycles at the critical value of $c$ is what leads to the discontinuity in the curves in Figure 1 and it is also why $\sum_k p_k < 1$. With probability tending to $1 - \sum_k p_k$ the min mean-weight cycle is long (has length tending to infinity with $n$) and has mean weight $(1/e + o(1))/n$; analyzing its length is difficult because the Poisson approximation breaks down in this regime, but we lower bound it by $(2/\pi^2 - o(1)) \log^2 n \log \log n$.

2 Review of the tree function

Because it plays a key role in the formulas for min mean-weight cycles in the subcritical regime, i.e., for weight $< 1/(en)$, we briefly review the tree function and the closely related Lambert $W$ function. The tree function $T$ is the exponential generating function for rooted spanning trees. Recalling Cayley’s formula that there are $k^{k-1}$ rooted spanning trees on $k$ nodes, we have

$$T(z) = \sum_{k=1}^{\infty} \frac{k^{k-1} z^k}{k!}.$$

From Stirling’s formula, this sum converges when $|z| \leq 1/e$. Using techniques from the theory of generating functions, one can see that

$$T(z) = z e^{T(z)}$$

(see e.g., [Sta99, Proposition 5.3.1]). It is straightforward to check that $T(1/e) = 1$. Near this critical point, using (1), one can deduce

$$T\left(\frac{1-\delta}{e}\right) = 1 - \sqrt{2\delta} + O(\delta).$$

The Lambert $W$ function is defined by the equation

$$z = W(z) e^{W(z)}.$$

This is a multivalued function, but the principal branch is defined so that $W(z) = -T(-z)$ when $|z| \leq 1/e$, and by analytic continuation elsewhere. The tree function figures prominently in the analysis of random graphs near the phase transition (see e.g., [JKLP93]), and the Lambert $W$ function is an important function in applied mathematics; for further background see [CGH+96].
3 The expected number of light cycles

Given $c > 0$, say that a directed or undirected $k$-cycle $C$ or $k$-path $P$ is \textit{c-light} if its mean weight $w(C)/k$ is at most $c/n$.

\textbf{Lemma 3.1.} If $c_1 \leq c_2$ then

$$
\Pr[\text{k-cycle or k-path is c}_2\text{-light but not c}_1\text{-light}] = \begin{cases}
\sim \frac{k^k c_2^k - c_1^k}{k! n^k} & \text{if } k = o(n) \\
\leq \frac{k^k c_2^k - c_1^k}{k! n^k} & \text{for any } k.
\end{cases}
$$

\textbf{Proof.} The weight $w(C)$ of a $k$-cycle or $k$-path $C$ is distributed as the sum of $k$ independent exponential random variables, that is, according to the Gamma distribution with shape parameter $k$, which has density function

$$
\phi : x \mapsto e^{-x} x^{k-1}/\Gamma(k),
$$

(3)

where $\Gamma$ is the gamma function, which is $\Gamma(k) = (k-1)!$ for positive integers $k$. Thus $\Pr[c_1k/n < w(C) \leq c_2k/n] = \int_{c_1k/n}^{c_2k/n} \phi(x) \, dx$. Now $e^{-k/n} \leq 1$ and when $k = o(n)$ we have $e^{-k/n} = 1 - o(1)$. \[\square\]

\textbf{Lemma 3.2.} Let $N_k$ denote the number of directed $k$-cycles. For $k \geq 2$ we have

$$
N_k \begin{cases}
\sim \frac{n^k}{k} & \text{if } k = o(\sqrt{n}) \\
\leq \frac{n^k}{k} & \text{for any } k.
\end{cases}
$$

\text{(For } k \geq 3, \text{ the number of undirected } k\text{-cycles is of course } \frac{1}{2}N_k.\text{)}

\textbf{Proof.}

$$
\frac{kN_k}{n^k} = \frac{n(n-1) \cdots (n-k+1)}{n^k} = \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right) = \exp\left[-\frac{k^2}{2n} + O\left(\frac{k}{n}\right) + O\left(\frac{k^3}{n^2}\right) + \cdots\right]
$$

which is $1 - o(1)$ when $k = o(\sqrt{n})$, and at most $1$ in all cases. \[\square\]

\textbf{Theorem 3.3.} Let $Z_c^{(k)}$ denote the number of directed $k$-cycles with mean weight less than $c/n$, and $Z = \sum_k Z_c^{(k)}$ denote the total number of $c$-light directed cycles. If $c_1 \leq c_2$ then

$$
\mathbb{E}[Z_{c_2}^{(k)} - Z_{c_1}^{(k)}] = \begin{cases}
\sim \frac{(c_2^k - c_1^k)k^k}{k \times k!} & \text{if } k = o(\sqrt{n}) \\
\leq \frac{(c_2^k - c_1^k)k^k}{k \times k!} & \text{for any } k,
\end{cases}
$$

(4)
and

\[
\lim_{n \to \infty} \mathbb{E}[Z_c] = \begin{cases} 
\sum_{2 \leq k < \infty} \frac{(ck)^k}{k!} = T(c) - c & \text{for } c \leq 1/e, \\
\infty & \text{for fixed } c > 1/e.
\end{cases} \tag{5}
\]

Proof. Equation (4) is immediate from Lemmas 3.1 and 3.2. For large \(k\), by Stirling’s formula the expression in (4) is asymptotic to \(\frac{(ce)^k}{\sqrt{2\pi k^{3/2}}}\).

Thus for fixed \(c > 1/e\), the number \(Z_c\) of \(c\)-light cycles tends to \(\infty\). For \(c \leq 1/e\), note that \(\mathbb{E}[Z_c] \leq T(c) - c\), and that \(\mathbb{E}[Z_c] \geq \sum_{k \leq k_0} \mathbb{E}[Z_c^{(k)}]\), which converges to the \(k \leq k_0\) terms for the series for \(T(c) - c\). Taking the \(k_0 \to \infty\) limit then yields (5).

For the undirected complete graph, the expected number of undirected \(c\)-light \(k\)-cycles is then of course \(\frac{1}{2} \mathbb{E}[Z_c^{(k)}]\) for \(k \geq 3\), and the total expected number of \(c\)-light cycles is

\[
\lim_{n \to \infty} \mathbb{E}[\# \text{ undirected } c\text{-light cycles}] = \begin{cases} 
\frac{T(c) - c - c^2}{2} & \text{for } c \leq 1/e, \\
\infty & \text{for fixed } c > 1/e.
\end{cases}
\]

4 Poisson approximation for short light cycles

Next we show that the short \(c\)-light cycles are well approximated by a Poisson process. Here “short” means length at most \(L_0 = \log n/(2 \log \log n)\), though for our main results it would suffice to prove this Poisson approximation for any \(L_0\) which tends to infinity as \(n \to \infty\). For \(c \leq 1/e\), we know from the first-moment bounds in Theorem 3.3 that with high probability there are no cycles of length \(\omega(1)\). It will then follow that the set of all \(c\)-light cycles is well approximated by a Poisson process when \(c \leq 1/e\).

For our purposes, the most convenient method to show Poisson approximation is the Chen-Stein method, as formulated by Arratia, Goldstein, and Gordon [AGG89, Theorem 2]:

**Theorem 4.1** ([AGG89]). Let \(\{X_\alpha : \alpha \in I\}\) be a finite set of indicator random variables of dependent events, and let \(\{Y_\alpha : \alpha \in I\}\) be a set of mutually independent Poisson random variables such that \(\mathbb{E}[Y_\alpha] = \mathbb{E}[X_\alpha]\) for each \(\alpha\). For each \(\alpha\) let \(B_\alpha\) be a subset of \(I\), which is interpreted as the “neighborhood of \(\alpha\)”. Then the total variation distance between the dependent Bernoulli process \((X_\alpha)_{\alpha \in I}\) and the independent Poisson process \((Y_\alpha)_{\alpha \in I}\) is at most

\[
2 \sum_{\alpha \in I} \sum_{\beta \in B_\alpha} \mathbb{E}[X_\alpha | X_\beta] + 2 \sum_{\alpha \in I} \sum_{\beta \in B_\alpha \setminus \beta \neq \alpha} \mathbb{E}[X_\alpha X_\beta] + \sum_{\alpha \in I} \mathbb{E}\left[|\mathbb{E}[X_\alpha | \{X_\beta : \beta \notin B_\alpha\}] - \mathbb{E}[X_\alpha]|\right].
\]

With a suitable choice of the neighborhood sets \(B_\alpha\), the third term above can easily be made zero, and analyzing the first two terms above is manageable.
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Theorem 4.2. Suppose \( L_0 = \log n/(2 \log \log n) \). For any \( \varepsilon \), for sufficiently large \( n \), the collection of \( 1 \)-light cycles with length at most \( L_0 \) is within total variation distance \( \varepsilon \) of a Poisson process whose intensity is the expected number of such cycles. (Except with probability \( \varepsilon \), for all \( k \leq L_0 \) and all \( c \leq 1 \), the number of \( c \)-light cycles of length \( k \) equals the number of points in the corresponding region of the Poisson process.)

Proof. We divide the interval \((0, 1]\) into subintervals of length \( \Delta \). To apply Theorem 4.1, let \( I \) denote the set of pairs \((C, c)\), where \( C \) is a directed \( k \)-cycles with \( k \leq L_0 \), and \((c - \Delta, c]\) is one of the subintervals. Let \( X_{C,c} \) be the indicator random variable for cycle \( C \) being \( c \)-light (i.e., a cycle with mean weight at most \( c/n \)), but not \((c - \Delta)\)-light.

Let \( B_{C,c} \) denote the subset of pairs \((C', c') \in I \) for which cycles \( C \) and \( C' \) have at least one edge in common. We make this choice of the \( B_{C,c} \)'s to simplify the evaluation of the third term from Theorem 4.1. The variables \( \{X_{C',c'} : (C', c') \notin B_C\} \) only depend on edges that are disjoint from cycle \( C \), so conditioning on them has no effect on the weight of cycle \( C \).

Thus

\[
E[X_{C,c}|\{X_{C',c'} : (C', c') \notin B_C\}] - E[X_{C,c}] = 0,
\]

and so the third term from Theorem 4.1 is zero.

Next we observe that the first two terms of Theorem 4.1 are unaffected by the subdivision of the interval \((0, 1]\): We can define \( X_C = \sum_c X_{C,c} \), where the sum is over the right endpoints of the intervals in the subdivision of \((0, 1]\), which are still Bernoulli random variables, and define \( B_C \) to be the set of cycles that have at least one edge in common with \( C \). Then

\[
\sum_C \sum_{C' \in B_C} E[X_C]E[X_{C'}] = \sum_{C,c} \sum_{(C', c') \in B_C} E[X_{C,c}]E[X_{C',c'}],
\]

and similarly for the second term. Therefore we work with the \( X_C \)'s and \( B_C \)'s.

For the first term of Theorem 4.1, we write:

\[
\sum_C \sum_{C' \in B_C} E[X_C]E[X_{C'}] = \sum_{k \leq L_0} \sum_{\ell \leq L_0} q_k q_{\ell} N_k N_{\ell} \Pr[k\text{-cycle and } \ell\text{-cycle overlap}],
\]

where \( q_k \) is the probability that a \( k \)-cycle is \( 1 \)-light. The expected number of intersections between a \( k \)-cycle and an \( \ell \)-cycle is \( k\ell/n \), so they intersect with probability at most \( k\ell/n \). Thus the first term is at most

\[
\frac{2}{n} \left[ \sum_{k \leq L_0} q_k N_k k \right]^2.
\]

But from Theorem 3.3, \( q_k N_k \leq e^k/\sqrt{2\pi k^3} \). So the first term of Theorem 4.1 is bounded by

\[
O\left(\frac{e^{2L_0}}{n}\right),
\]

which tends to 0 as \( n \to \infty \).

For the second term of Theorem 4.1 we consider all possible pairs of distinct non-edge-disjoint cycles \( C, C' \) of \( I \). Let \( k \) be the number of edges common to \( C \) and to \( C' \), \( k + \ell \) be
the length of $C$ and $k + m$ be the length of $C'$. We let $w/n$ denote the total weight of the edges $C$ and $C'$ share, $v/n$ denote the total weight of edges in $C$ but not $C'$, and $x/n$ denote the total weight of edges in $C'$ but not $C$. The probability that both cycles are $c$-light is:

$$
\mathbb{E}[X_C X_{C'}] = \int_{w+v<(k+\ell)c} \int_{w+x<(k+m)c} \frac{e^{-v/n}(w/n)^{k-1}}{\Gamma(k)} \frac{e^{-v/n}(v/n)^{\ell-1}}{\Gamma(\ell)} \frac{e^{-x/n}(x/n)^{m-1}}{\Gamma(m)} \frac{1}{n^3} \, dw \, dv \, dx.
$$

We can bound the $e^{-v/n}$, $e^{-v/n}$, and $e^{-x/n}$ terms by 1:

$$
\mathbb{E}[X_C X_{C'}] \leq \frac{1}{n^{k+\ell+m}} \int_{w+v<(k+\ell)c} \int_{w+x<(k+m)c} \frac{w^{k-1} v^{\ell-1} x^{m-1}}{\Gamma(k) \Gamma(\ell) \Gamma(m)} \, dw \, dv \, dx.
$$

We enlarge the domain of integration to the $(w, v, x)$ such that $w < (k + \ell)c$, $v < (k + \ell)c$, and $x < (k + m)c$, so that the triple integral has a product form that can be evaluated explicitly, and then we take $c = 1$:

$$
\mathbb{E}[X_C X_{C'}] \leq \frac{1}{n^{k+\ell+m}} \frac{((k+\ell)c)^k ((k+\ell)c)^\ell ((k+m)c)^m}{k! \ell! m!} \leq \frac{L_0^{2L_0}}{n^{k+\ell+m}}.
$$

We now count the number of cycle pairs $(C, C')$ that have at least one edge in common given $k, \ell, m$.

Suppose $C' \setminus C$ consists of $i \geq 1$ paths. There are at most $L_0$ possibilities for the lengths $m_1, m_2, \ldots, m_i$ of the paths of $C' \setminus C$. With those lengths specified, we can list the $k + \ell$ vertices of $C$ in order from some arbitrary starting point, specify where along each path of $C' \setminus C$ starts and ends, and specify the $m_j - 1$ vertices of each path. Thus the number of such configurations is at most $n^{k+\ell} L_0^i L_0^{2i} n^{m-i}$.

Altogether the number of overlapping cycles $(C, C')$ is bounded by

$$
\sum_{i=1}^{L_0} \frac{L_0^i}{n^i} n^{k+\ell+m} \leq 2 \frac{L_0^3}{n} n^{k+\ell+m},
$$

provided $L_0 \leq \sqrt[3]{n/2}$. There are at most $L_0$ choices for each of $k$, $\ell$, and $m$. The second term of Theorem 4.1 is then bounded by $4L_0^{2L_0+6}/n$ (provided $L_0 \leq \sqrt[3]{n/2}$). When $L_0 = \frac{1}{2} \log n / \log \log n$, we have

$$
\frac{4}{n} L_0^{2L_0+6} = \frac{4}{n} \exp \left[ \left( \log \log n - \log \log \log n \right) \left( \log n / \log \log n + 6 \right) \right]
= 4 \exp \left[ 6 \left( \log \log n - \log \log \log n \right) \right],
$$

which tends to 0 as $n \to \infty$. \qed
5 Below the critical point: short light cycles

Given the Poisson approximation result in Theorem 4.2 and the first-moment estimate in Theorem 3.3, it is straightforward to derive the formulas for the mean-weight of the min mean-weight cycle (shown in Figure 1), and the probability that the length of the cycle is \(k\) for any fixed \(k\) (in Table 1). Similar computations were done by Janson [Jan87] for the min max-weight cycle.

5.1 Weight of the cycle

**Theorem 5.1.** For the directed complete graph, for fixed \(c\), there is a cycle with mean weight \(\leq c/n\) with probability

\[
\lim_{n \to \infty} \Pr[\exists \text{ cycle with mean-weight } \leq c/n] = \begin{cases} 
1 - \exp[-T(c) + c] & c \leq 1/e \\
1 & c > 1/e,
\end{cases}
\]

while for the undirected complete graph the probability is

\[
\lim_{n \to \infty} \Pr[\exists \text{ cycle with mean-weight } \leq c/n] = \begin{cases} 
1 - \exp[-T(c) + c + c^2]/2] & c \leq 1/e \\
1 & c > 1/e.
\end{cases}
\]

**Proof.** For \(c \leq 1/e\), by the first moment estimate, with probability \(1 - o(1)\) there are no \(c\)-light cycles with length \(> L_0 = \log n / (2 \log \log n)\). By the Poisson approximation, there is a \(c\)-light cycle of length \(\leq L_0\) with probability \(\exp[-\mu + o(1)]\), where \(\mu = (1 + o(1)) \sum_{k=2}^{L_0} k^{k-1} e^k / k! = T(c) - c + o(1)\) for the directed complete graph, and \(\mu = (T(c) - c - c^2)/2 + o(1)\) for the undirected complete graph. For fixed \(c > 1/e\), the sum \((1 + o(1)) \sum_{k=2}^{L_0} k^{k-1} e^k / k!\) tends to infinity with \(n\), and the Poisson approximation still holds, so with probability \(1 - o(1)\) there is a \(c\)-light cycle.

So the finiteness of \(T(c) - c\) and \((T(c) - c - c^2)/2\) at \(c = 1/e\) accounts for the discontinuities in the curves in Figure 1. Recalling the behavior of the tree function near \(c = 1/e\), we see that these curves for the min mean-weight cycle have a square-root plus constant behavior to the left of the critical point.

5.2 Length of the cycle

**Theorem 5.2.** Suppose \(k\) is fixed as \(n \to \infty\). For the directed complete graph, for \(k \geq 2\)

\[
\lim_{n \to \infty} \Pr[\min \text{ mean-weight cycle has length } k] = \lim_{n \to \infty} \Pr\left[ \min \text{ mean-weight cycle has length } k \text{ and weight } \leq \frac{k^k}{e} \right] = \int_0^{1/e} \frac{e^{k-1} k^k}{k!} e^{-T(c) + c} dc.
\]
For the undirected complete graph, for $k \geq 3$

$$\lim_{n \to \infty} \Pr[\text{min mean-weight cycle has length } k] =$$

$$\lim_{n \to \infty} \Pr \left[ \text{min mean-weight cycle has length } k \text{ and weight } \leq \frac{k}{e} \right] = \int_{0}^{1/e} \frac{c^{k-1}k^k}{2k!} e^{-(T(c)+c+c^2)/2} dc.$$

Proof. Let us consider the Poisson process which approximates the light cycles. Suppose that $k$ is fixed and $c \leq 1/e$. By the first-moment estimate, the probability that there is cycle of length $k$ that is $c + dc$-light but no cycle of any size that is $c$-light is

$$\frac{c^{k-1}k^k}{k!} \times e^{-(T(c)+c+c^2)/2} dc.$$

for the directed complete graph, and

$$\frac{1}{2} \frac{c^{k-1}k^k}{k!} \times e^{-(T(c)+c+c^2)/2} dc.$$

for the undirected complete graph. By integrating, we obtain the probability that, in the Poisson approximation, the min mean-weight cycle has length $k$ and mean weight at most $1/e$. But as $n \to \infty$, with probability $1 - o(1)$ the Poisson approximation exactly equals the true process of cycles with mean weight at most $1/e$.

Next we consider the possibility that the min mean-weight cycle has length $k$ and mean weight $> 1/e$. Suppose $0 < \delta < 1$. With probability tending to 1 as $n \to \infty$, there is a $(1 + \delta/k)/e$-light cycle. But the expected number of $k$-cycles that $(1 + \delta/k)/e$-light but not $1/e$-light tends to 0 as $\delta \to 0$. So the probability that the min mean-weight cycle has length $k$ and mean weight $> 1/e$ tends to 0 as $n \to \infty$.

The formulas in Table 1 are rewritten slightly using Equation 1 to write $e^{-T(c)} = c/T(c)$. Theorem 5.1 and Theorem 5.2 imply

$$\lim_{n \to \infty} \Pr[\text{min mean-weight cycle has mean weight } > 1/e] > 0$$

but

$$\sum_{k} \lim_{n \to \infty} \Pr[\text{min mean-weight cycle has length } k \text{ and mean weight } > 1/e] = 0.$$ 

There is no contradiction of course. In Section 6 we further investigate the length of the min mean-weight cycle when its mean weight is $> 1/e$. 


5.3 Tail behavior of the length distribution

We can approximate the large-$k$ behavior of the probability $p_k$ that the min mean-weight cycle has length $k$ (sending $n$ to infinity first and then $k$). We make the substitution $c = (1 - \delta)/e$ to obtain, for the directed complete graph,

$$p_k = \frac{e^{-k}k^k}{k!}e^{1/e} \int_0^1 (1 - \delta)^k \frac{e^{-\delta/e}}{T((1 - \delta)/e)} \frac{d\delta}{e}.$$  

The integrand is approximately $e^{-k\delta}$ for small $\delta$, and large $\delta$ contribute negligibly, so the integral is approximately $1/(ke)$, and so for large $k$

$$p_k = (1 + o(1))e^{-1+1/e}\sqrt{2\pi}k^{-3/2}.$$  

For the undirected complete graph, a similar computation yields

$$p_k = (1 + o(1))e^{-1/2+1/(2e)+1/(2e^2)}2\sqrt{2\pi}k^{-3/2}.$$  

By comparison, Janson [Jan87] shows that for the min max-weight cycle on the undirected complete graph, the expected number of cycles with max weight at most $c/n$ is $\frac{1}{2}(\log \frac{1}{1-c} - c - c^2/2)$, so the probability that such a cycle exists is $1 - (1 - c)^{1/2}e^{c/2+c^2/4}$ which has its threshold at 1, and so

$$p_k = \frac{1}{2} \int_0^1 c^{k-1}(1 - c)^{1/2}e^{c/2+c^2/4} dc,$$

which for large $k$ is

$$p_k = (1 + o(1))\frac{\sqrt{\pi}}{4}e^{3/4}k^{-3/2}.$$  

The computations for the min max-weight cycle on the directed complete graph are similar, though it is perhaps surprising that unlike the previous three cases, the asymptotics of $p_k$ in this case are $p_k = \Theta(k^{-2})$. More specifically, we have

$$p_k = \int_0^1 c^{k-1}(1 - c)e^c dc,$$

Letting $c = 1 - \delta/k$, the integrand is for small $\delta$ asymptotically

$$e^{-\delta}\frac{\delta}{k}e\frac{d\delta}{k},$$

so we see that the integral is asymptotically $e/k^2$.  

6 Above the critical point: long light cycles

Recall that a cycle $C$ is $c$-light if $\text{weight}(C) \leq \text{length}(C)c/n$. We say that a cycle $C$ is $A$-uniformly $c$-light if $C$ is $c$-light, and in addition, for every subpath $P$ of $C$,

$$\text{weight}(P) \leq \left\lfloor \text{length}(P) + A \right\rfloor \frac{c}{n}.$$ 

In the directed complete graph, let $Z_{L,\delta}$ denote the number of $(1 + \delta)/e$-light cycles of length between $L - 1/\delta$ and $L$, and let $Y_{L,\delta,A}$ denote the number of $A$-uniformly $(1 + \delta)/e$-light cycles of length between $L - 1/\delta$ and $L$. We will eventually choose the parameters $\delta$, $L$, and $A$ so that

$$\delta = \Theta(1/\log^2 n)$$
$$L = \Theta(\log^2 n \log \log n)$$
$$A \approx \log n.$$ 

We aim to show that with high probability such cycles exist.

To prove the next lemma, we use one of the Komlós-Major-Tusnády theorems [KMT76, Theorem 1] which relate random walk to Brownian motion:

**Theorem 6.1** ([KMT76]). Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with expected value 0, variance 1, and finite exponential moments, i.e., their distribution function $f(x)$ satisfies $\int e^{tx} f(x) \, dx < \infty$ for $|t| \leq t_0 > 0$. Then these random variables can be coupled to i.i.d. standard normal random variables $Y_1, Y_2, \ldots$ such that for any $\lambda$ there are constants $K_1$ and $K_2$ for which

$$\Pr\left[ \max_{k \leq n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Y_i \right| > K_1 \log n + x \right] < K_2 e^{-\lambda x}.$$ 

**Lemma 6.2.** If a cycle of length $L$ has mean weight $c/n$, then it is $A$-uniformly $c$-light with probability at least

$$\exp[-(\pi^2/2 + o(1))L/A^2].$$

Here the $o(1)$ term tends to 0 as $L/A^2 \to \infty$ and $L/A^3 \to 0$.

**Proof.** The edge weights $W_1, \ldots, W_L$ are distributed according $L$ independent exponential random variables with mean 1. The mean weight of the cycle is $c/n = \sum_{i=1}^{L} W_i / L$. The cycle is $A$-uniformly $c$-light when

$$\sum_{i=1}^{L} W_{a+i \mod L} \leq (\ell + A) \frac{\sum_{i=1}^{L} W_i}{L}$$

(6)
for each \( a, \ell \in \{1, \ldots, L\} \). (The parameters \( c \) and \( n \) play no role here. Because the edge weights are distributed according to exponential random variables, regardless of the cycle’s mean weight \( c/n \), the probability that it is \( A \)-uniformly \( c \)-light will be the same.)

Let \( X_i = W_i - 1 \). Suppose that \( 0 < \varepsilon \leq 1/4 \) and

\[
0 \leq \sum_{i=1}^{L} X_i \leq \varepsilon A, \quad (7)
\]

and that for each \( k \in \{1, 2, \ldots, L\} \),

\[
-\frac{1 - \varepsilon}{2} A \leq \sum_{i=1}^{k} X_i \leq \frac{1 - \varepsilon}{2} A. \quad (8)
\]

Equation (7) implies \( \sum_{i=1}^{L} W_i \geq L \). When \( a + \ell \leq L \), (8) implies \( \sum_{i=1}^{\ell} W_{a+i} \leq \ell + (1 - \varepsilon)A \), which then implies (6) when \( a + \ell \leq L \). If \( a + \ell > L \), then \( \sum_{i=1}^{\ell} W_{a+i \mod L} = \sum_{i=1}^{L} W_i - \sum_{i=1+a+\ell \mod L}^{a} W_i \leq \varepsilon A + (1 - \varepsilon)A + \ell \) by (7) and (8), so again we have (6).

Now \( X_i \) has zero mean, unit variance, and finite exponential moments, so Theorem 6.1 implies that the partial sums \( \sum_{i=1}^{k} X_i \) are well approximated by a standard Brownian motion. It is known how to compute the probability that a standard Brownian motion \( B_t \) stays within an interval up through time \( T \). If the interval is \([-a, a]\), then this probability is

\[
\Theta \left( \exp \left[ -\frac{\pi^2 T}{8 a^2} \right] \right).
\]

Conditional upon the Brownian motion remaining within this interval, its final position within the interval at time \( T \) has a well-behaved distribution which for large \( T/a^2 \) converges to a sine function.

We set \( T = L \) and \( a = A(1 - 2\varepsilon)/2 \), we see that the \( B_t \) stays within the interval \( \pm A(1 - 2\varepsilon)/2 \) and ends within the interval \( (\varepsilon A/3, 2\varepsilon A/3) \) with probability at least

\[
\Theta \left( \varepsilon \times \exp \left[ -\frac{1}{(1 - 2\varepsilon)^2} \frac{\pi^2 L}{2 A^2} \right] \right). \quad (9)
\]

If this event occurs, and the partial sums \( \sum_{i=1}^{k} X_i \) are within \( \varepsilon A/3 \) of the Brownian motion, then equations (8) and (7) hold.

By assumption \( L/A^3 \to 0 \), so let us suppose \( L/A^3 \leq 1 \). Then \( A \geq L^{1/3} \). By assumption \( L/A^2 \to \infty \), so \( L \to \infty \). Let us take

\[
\varepsilon = 6 \max \left( L/A^3, K_1 \log L/A, e^{-L^{1/2}/A} \right),
\]

which by our assumptions tends to 0, and we can suppose that it is at most 1/4 as assumed above.
In the KMT theorem, we choose \( \lambda = 20 \) and \( x = L/A^2 \). By our choice of \( \varepsilon \), the deviation \( K_1 \log L + x \) is smaller than \( \varepsilon A/3 \). The probability that the Brownian motion and the random walk are not within \( \varepsilon A/3 \) of one another is at most \( K_2 e^{-20L/A^2} \). Now \( 20 > \pi^2/(2(1-2\varepsilon)^2) \), so provided \( L/A^2 \) is sufficiently large, even if we condition on the unlikely event that the Brownian motion stays within the interval and ends within an even smaller interval, it is still extremely likely that the random walk does not deviate more than \( \varepsilon A/3 \) from the Brownian motion. Thus, the probability that the cycle is \( A \)-uniformly \( c \)-light is at least as large as the expression in equation (9). Since \( \varepsilon \geq e^{-L_1/2} \) and \( L/A^2 \to \infty \), the factor of \( \varepsilon \) in (9) can be further rewritten as \( \exp[-(\pi^2/2 + o(1))L/A^2] \).

**Lemma 6.3.** For the directed complete graph, the expected number of \( A \)-uniformly \((1+\delta)/e\)-light directed cycles of size between \( L - 1/\delta \) and \( L \) is

\[
\mathbb{E}[Y_{L,\delta,A}] = \frac{e^{L\delta}}{\sqrt{2\pi\delta L^{3/2}}} \exp[-(\pi^2/2 + o(1))L/A^2].
\]

Here the \( o(1) \) term tends to 0 as \( L/A^2 \to \infty \) and \( L/A^3 \to 0 \). (For the undirected graph, the expected value is half as large.)

**Proof.** Immediate from the first moment estimates for \( c \)-light cycles and Lemma 6.2.

**Lemma 6.4.** If \( L^3 e^A/n \leq 1/2 \), then

\[
\text{Var}[Y_{L,\delta,A}] \leq \frac{2L^3 e^A \delta L}{n} \mathbb{E}[Y_{L,\delta,A}].
\]

(This same bound holds for both the directed and undirected complete graph.)

**Proof.** For any cycle \( C \) of length between \( L - 1/\delta \) and \( L \), let \( U_C \) denote the indicator of the event that \( C \) is uniformly light (within this proof, “uniformly light” means \( A \)-uniformly \((1+\delta)/e\)-light). Let \( C_1, C_2 \) be two cycles of lengths \( L_1 \) and \( L_2 \), which are both in the range \( L - 1/\delta \) to \( L \). If \( C_1 \) and \( C_2 \) have no edges in common then \( U_{C_1} \) and \( U_{C_2} \) are independent. So we have

\[
\text{Var}[Y_{L,\delta,A}] = \sum_{C_1} \sum_{C_2} (\mathbb{E}[U_{C_1} U_{C_2}] - \mathbb{E}[U_{C_1}] \mathbb{E}[U_{C_2}])
\]

\[
= \sum_{C_1} \sum_{C_2:C_2 \cap C_1 \geq 1} (\mathbb{E}[U_{C_1} U_{C_2}] - \mathbb{E}[U_{C_1}] \mathbb{E}[U_{C_2}])
\]

\[
\leq \sum_{C_1} \sum_{C_2:C_2 \cap C_1 \geq 1} \mathbb{E}[U_{C_1} U_{C_2}]
\]

\[
= \sum_{C_1} \mathbb{E}[U_{C_1}] \sum_{C_2:C_2 \cap C_1 \geq 1} \mathbb{E}[U_{C_2} | U_{C_1} = 1]
\]
We partition the inner sum into sub-sums depending on the overlaps between $C_1$ and $C_2$.

Suppose $C_2 \setminus C_1$ consists of $i \geq 1$ paths, of lengths $m_1, m_2, \ldots, m_i$. Let $m = \sum_j m_j < L$. To specify the $j$th path, we can specify its start and end points on $C_1$, as well as the internal vertices, so there are $\leq L^2 n^{m_j - 1}$ possible such subpaths. Hence the number of such $C_2$’s is at most $L^2 n^{m-1}$. Conditional on the weights in $C_1$, the probability that $C_2$ is uniformly light is at most the probability that for each $j$, the $j$th subpath $C_2 \setminus C_1$ has weight at most $w_j := (m_j + A)(1 + \delta)/(en)$. The probability that the $j$th subpath is light enough is at most $w_j m_j! \leq \left(\frac{(1 + \delta)(1 + A/m_j)}{n^{m_j}}\right)^{m_j} \leq e^{\delta m_j + A}/n^{m_j}$, and so the conditional probability that $C_2$ is uniformly light is at most $e^{\delta m + A}/n$. We obtain

$$
\sum_{C_2 : C_2 \setminus C_1 = i \text{ paths of lengths } m_1, \ldots, m_i} \mathbb{E}[U_{C_2}|U_{C_1}] \leq \left[\frac{L^2 e^A}{n}\right]^i e^{\delta m}
$$

and

$$
\sum_{m_1, \ldots, m_i} \sum_{C_2 : C_2 \setminus C_1 = i \text{ paths of lengths } m_1, \ldots, m_i} \mathbb{E}[U_{C_2}|U_{C_1}] \leq \left[\frac{L^3 e^A}{n}\right]^i e^{\delta m}
$$

Since by assumption $L^3 e^A \leq n/2$, altogether we have,

$$
\sum_{i \geq 1} \sum_{C_2 : C_2 \setminus C_1 = i \text{ paths}} \mathbb{E}[U_{C_2}|U_{C_1}] \leq 2 \frac{L^3 e^A}{n} e^{\delta L}.
$$

**Lemma 6.5.** If $\delta \ll 1$ and $1 \ll \delta L_0 \leq \frac{1}{2} \log L_0$, then with high probability there are no cycles of length at most $L_0$ that are $(1 + \delta)/e$-light but not $1/e$-light. (This holds for both the directed and undirected complete graph.)

**Proof.** Using the inequality version of the first-moment estimate [4], the expected number of cycles of length $\leq L_0$ that are $(1 + \delta)/e$-light but not $1/e$-light is at most

$$
\sum_{k=1}^{L_0} \frac{1}{\sqrt{2\pi k^{3/2}}} [(1 + \delta)^k - 1].
$$

We break the sum apart into blocks of size $1/\delta$. For the first block, $k < 1/\delta$, we can use the bound

$$(1 + \delta)^k - 1 = \Theta(\delta k),$$

and find that the sum of the first block is to within constants

$$
\int_0^{1/\delta} k^{-1/2} \delta dk = 2\delta^{1/2}.
$$
For each remaining block of size $1/\delta$, we can use the bound
\[(1 + \delta)^k - 1 \leq e^{\delta k}\]
so that the block sum is $O(e^{\delta k}/(\delta k^{3/2}))$. These block-sum bounds increase geometrically, so the total is
\[O \left( \frac{e^{\delta L_0}}{\delta L_0^{3/2}} \right).\]
But $\delta L_0 \leq \frac{1}{2} \log L_0$, so this is $O(1/(\delta L_0))$, and $\delta L_0$ tends to infinity. So the expected number of such cycles is $o(1)$.

We now have the ingredients to prove our results above the critical point:

**Theorem 6.6.** For both the directed and undirected complete graph, conditional upon the min mean-weight cycle having mean weight $> 1/(en)$, with probability $1 - o(1)$ the mean weight of the cycle is at most
\[1 + \frac{\pi^2/2 + o(1)}{\log^2 n},\]
and its length is at least
\[(2/\pi^2 - o(1)) \log^2 n \log \log n.\]

**Proof.** We aim to show that with high probability there are $(1 + \delta)/e$-light cycles, but not of size $\leq L_0$, unless the cycles are also $1/e$-light. We choose the parameters $A$ and $L$ so that $\mathbb{E}[Y_{L,\delta,A}] \gg 1$ and $\text{Var}[Y_{L,\delta,A}] \ll \mathbb{E}[Y_{L,\delta,A}]^2$, which will imply that in fact with high probability there is an $A$-uniformly $(1 + \delta)/e$-light cycle with size between $L - 1/\delta$ and $L$. We now gather our constraints (which are the same for both the directed and undirected complete graph):
\[
\frac{e^{\delta L}}{\sqrt{2\pi \delta L^{3/2}}} \exp[-(\pi^2/2 + o(1)L/A^2)] \gg 1
\]
\[
\frac{e^{\delta L}}{\sqrt{2\pi \delta L^{3/2}}} \exp[-(\pi^2/2 + o(1)L/A^2)] \gg \frac{2L_0^3 e^{A} e^{\delta L}}{n}
\]
\[L \geq L_0\]
\[\delta \ll 1\]
\[1 \ll \delta L_0\]
\[\delta L_0 \leq \frac{1}{2} \log L_0.\]

We aim to maximize $L_0$ and minimize $\delta$. From the above constraints we see $\delta L \geq 1$. Then from the first constraint above we need $\delta > \pi^2/(2A^2)$. Upon cancelling the $e^{\delta L}$ terms in the second constraint, the left-hand-side becomes at most 1, so see that we need $A \leq \log n$, so we need
\[\delta \geq \pi^2/(2 \log^2 n).\]
Starting with $L_0 \leq n$, the last constraint then implies $L_0 \leq O(\log^2 n)$ and then $L_0 \leq O(\log^2 n \log \log n)$ and then that we need

$$L_0 \leq \frac{2}{\pi^2} \log^2 n \log \log n.$$ 

We can make $L_0$ nearly as large and $\delta$ nearly as small as the above bounds, by picking the following parameter values. Let $\varepsilon > 0$ be an arbitrarily small positive constant.

$$A = (1 - \varepsilon) \log n$$

$$\delta = \frac{\pi^2/2 + 11\varepsilon}{\log^2 n}$$

$$L_0 = \frac{2}{\pi^2 - \varepsilon} \log^2 n \log \log n$$

$$L = \frac{1}{\varepsilon} \log^2 n \log \log n.$$ 

It is straightforward to verify that for sufficiently small fixed $\varepsilon$, the above values satisfy the preceding constraints for all sufficiently large $n$. \hfill $\Box$

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## References


The min mean-weight cycle in a random network


The min mean-weight cycle in a random network

Mathieu & Wilson


