Uncertainty analysis of the phase and RMS value by non-coherent sampling in the frequency domain

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Abstract – The determination of phase and RMS value belongs to the most important tasks of signal analysis in the frequency domain. Evaluating these parameters is frequently performed by the DFT (Discrete Fourier Transform) algorithm. In practice, signals are mostly sampled non-coherently. This leads to the well known effect called leakage, which results in spreading frequency spectral component energy to neighboring frequency lines. Therefore, time windows are used for leakage suppression. This paper introduces the problematic of uncertainty analysis of phase and RMS value computed from the DFT spectrum by non-coherent sampling using cosine windows. The analysis is aimed to investigate the influence of quantization noise.

Keywords – frequency spectrum, uncertainty analysis, cosine window, RMS value, phase, DFT.

I. INTRODUCTION

More and more analog signals are converted to the digital form because of many advantages of digital processing and storage. Evaluation of the signal characteristics is mostly performed as the analysis of phases and RMS values in the frequency domain. By coherent sampling, phases as well as RMS values (which correspond to modules in this case) can be directly read from the frequency spectrum. Uncertainty analysis of phase and RMS value belongs to the most important tasks of signal analysis in the frequency domain.

II. DFT MODULE AND PHASE UNCERTAINTY ANALYSIS

Phases as well as modules are determined by frequency spectrum \( X(k) \) which can be computed from a sequence of samples \( x(n) \) by the DFT algorithm. As non-coherent sampling is assumed, widely used cosine windows are considered to be applied to the sampled signal. Cosine time window \( w_{\cos}(n) \) is defined as

\[
w_{\cos}(n) = \sum_{k=0}^{L} a_k \cos \left( \frac{2\pi}{N} nk \right)
\]

where \( L \) is the window order, \( N \) its length and \( a_k \) are the coefficients specific for each window.

Frequency spectrum can be computed using the DFT algorithm given by

\[
X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) w_{\cos}(n) e^{-j \frac{2\pi}{N} nk}
\]

\( X(k) \) can be gained as the composition of real \( R(k) \) and imaginary \( I(k) \) part [1]

\[
X(k) = R(k) + j I(k)
\]

where

\[
R(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) w_{\cos}(n) \cos \left( \frac{2\pi}{N} nk \right)
\]

\[
I(k) = -\frac{1}{N} \sum_{n=0}^{N-1} x(n) w_{\cos}(n) \sin \left( \frac{2\pi}{N} nk \right)
\]

Summarizing all uncertainties of the acquisition process to quantization uncertainty, uncertainties of the real and imaginary parts are given by

\[
u_{\text{E}(k)}^2 = \left[ \frac{\partial R(k)}{\partial x(n)} \right]^2 u_{\text{E}(n)}^2 + \left[ \frac{\partial I(k)}{\partial x(n)} \right]^2 u_{\text{E}(n)}^2 + \sum_{n=0}^{N-1} \cos \left( \frac{4\pi}{N} nk \right) u_{\text{E}(n)}^2
\]

\[
u_{\text{Q}(k)}^2 = \left[ \frac{\partial R(k)}{\partial x(n)} \right]^2 u_{\text{Q}(n)}^2 + \left[ \frac{\partial I(k)}{\partial x(n)} \right]^2 u_{\text{Q}(n)}^2 + \sum_{n=0}^{N-1} \cos \left( \frac{4\pi}{N} nk \right) u_{\text{Q}(n)}^2
\]

where \( u_{\text{E}(n)} \) is the quantization uncertainty and \( u_{\text{Q}(n)} \) the quantization uncertainty.

Module \( M(k) \) and phase \( \phi(k) \) are

\[
M(k) = \sqrt{R^2(k) + I^2(k)} , \quad \phi(k) = \arctan \frac{I(k)}{R(k)}
\]

Module uncertainty is given by

\[
u_{\text{M}(k)}^2 = \left[ \frac{\partial M(k)}{\partial R(k)} \right]^2 u_{\text{M}(k)}^2 + \left[ \frac{\partial M(k)}{\partial I(k)} \right]^2 u_{\text{M}(k)}^2 + 2 \left[ \frac{\partial M(k)}{\partial R(k)} \frac{\partial M(k)}{\partial I(k)} \right] u_{\text{M}(k)} u_{\text{E}(k)}
\]

where

\[
u_{\text{M}(k)}^2 = \frac{1}{2} \left[ \frac{\partial R(k)}{\partial x(n)} \right]^2 u_{\text{M}(k)}^2 + \frac{1}{2} \left[ \frac{\partial I(k)}{\partial x(n)} \right]^2 u_{\text{M}(k)}^2 + \sum_{n=0}^{N-1} \sin \left( \frac{4\pi}{N} nk \right) u_{\text{M}(k)}^2
\]

\[
u_{\text{M}(k)}^2 = \frac{1}{2} \left[ \frac{\partial R(k)}{\partial x(n)} \right]^2 u_{\text{M}(k)}^2 + \frac{1}{2} \left[ \frac{\partial I(k)}{\partial x(n)} \right]^2 u_{\text{M}(k)}^2 + \sum_{n=0}^{N-1} \sin \left( \frac{4\pi}{N} nk \right) u_{\text{M}(k)}^2
\]
because \( u^2_{\text{rms}}(k) \) is an odd function, sinus is an even function and the sum of the sinus function in (10) is zero. Then

\[
u^2_{\text{rms}} = \frac{1}{M(k)^2} \left[ R^2(k) u^2_{\text{rms}} + \phi^2(k) u^2_{\text{rms}} \right] = \frac{u^2_{\text{rms}}}{2M^2(k)N} \left[ M^2(k)NNPG + \frac{1}{N} \sum_{(\pi}^{N} \left( \frac{\pi}{N} \right) \right] w_{\text{rms}}(R^2(k) - \phi^2(k)) \]

(11)

In further analysis, only the interval \((0; N/4)\) is considered as well as even \(N\) and \(N > 2L + 1\). It can be derived [5] that the second term (summation) is zero for common case when values of \(k\) are within the interval \((L; N/2 - L)\). Assuming this criterion, the module uncertainty can be simplified as

\[
u^2_{\text{rms}} u^2_{\text{rms}} = \frac{u^2_{\text{rms}}}{2N} \]

(12)

The phase uncertainty is similarly

\[
u^2_{\text{rms}} = \frac{1}{M(k)^2} \left[ R^2(k) u^2_{\text{rms}} + \phi^2(k) u^2_{\text{rms}} \right] = \frac{u^2_{\text{rms}}}{2M^2(k)N} \left[ M^2(k)NNPG + \frac{1}{N} \sum_{(\pi}^{N} \left( \frac{\pi}{N} \right) \right] w_{\text{rms}}(R^2(k) - \phi^2(k)) \]

(13)

With the same conditions like for the module uncertainty, the phase uncertainty is given by

\[
u^2_{\text{rms}} = \frac{u^2_{\text{rms}}}{2M^2(k)N} \]

(14)

III. RMS VALUE UNCERTAINTY

The RMS value of a sine wave, \( S_{\text{rms}} \), can be computed from the positive and negative frequency bands of \(2L + 1\) components centered at the frequency of interest \(k\) [2] which can be found e.g. as the local maximum in the amplitude frequency spectrum

\[
S_{\text{rms}} = \frac{1}{NNPG} \left( \sum_{i=-L}^{L} M^2(i) + \sum_{j=-L}^{L} M^2(j) \right) \]

(15)

where \(M(i), M(j)\) are modules according to (8) computed from (2). For real signals, the frequency spectrum is an even function. Thus, formula (15) can be simplified as

\[
S_{\text{rms}} = \sqrt{\frac{2}{NNPG} \sum_{i=-L}^{L} M^2(i)} \]

(16)

To compute the uncertainties simpler, equation (16) is decomposed as

\[
S_{\text{rms}} = \sqrt{y}, \quad y = \frac{2}{NNPG} \sum_{i=-L}^{L} M^2(i) \]

(17)

The uncertainty of the RMS value is

\[
u^2_{\text{rms}} u^2_{\text{rms}} = \left[ \frac{\partial S_{\text{rms}}(k)}{\partial y(k)} \right]^2 u^2_{\text{rms}}(k) = \frac{u^2_{\text{rms}}(k)}{4S_{\text{rms}}(k)} \]

(18)

where

\[
\begin{array}{l}
u^2_{\text{rms}}(k) = \sum_{i=2}^{\infty} \left[ \frac{\partial^2 S_{\text{rms}}}{\partial y^2}(k) \right] u^2_{\text{rms}}(k) + 2 \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \frac{\partial S_{\text{rms}}}{\partial y}(k) \frac{\partial S_{\text{rms}}}{\partial y}(k) \end{array} \]

\[
- \frac{4}{NNPG} \sum_{i=2}^{\infty} \sum_{j=2}^{\infty} \left[ \frac{\partial S_{\text{rms}}}{\partial y}(k) \right] u^2_{\text{rms}}(k) \]

\[
= \frac{u^2_{\text{rms}}(k)}{4S_{\text{rms}}(k)} \]

(19)

The uncertainty \(u_{\text{rms}}(i, M(j))\) is the estimated covariance associated with modules \(M(i)\) and \(M(j)\). These lines are dependent on each other because of the influence of applied time window (frequency spectrum of windowed signal is given by the convolution of signal frequency spectrum and window frequency spectrum). Direct computation of the module covariance would be difficult since the output module does not correspond to the sum of neighboring frequency lines. As the convolution is a linear operation, the output real and imaginary parts of frequency spectrum are given as the summation of real (imaginary) parts of frequency spectra—input signal frequency spectrum and time window frequency spectrum. Consequently, it is more advantageous to express the uncertainty \(u_{\text{rms}}(i, M(j))\) by means of the covariance between real (imaginary) parts of the output frequency spectrum

\[
u_{\text{rms}}(i, M(j)) = \frac{\partial M(i)}{\partial y(i)} \frac{\partial M(j)}{\partial y(j)} u_{\text{rms}}(i, M(j)) + \frac{\partial M(i)}{\partial y(i)} \frac{\partial M(j)}{\partial y(j)} u_{\text{rms}}(i, M(j)) + \frac{\partial M(i)}{\partial y(i)} \frac{\partial M(j)}{\partial y(j)} u_{\text{rms}}(i, M(j))

(20)

and and \(r(R(i), R(j)), r(i, j))\) are correlation coefficients defined as

\[
\begin{array}{l}
r(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{cov}(X, X) \text{cov}(Y, Y)}} \end{array} \]

(22)

\[
\text{cov}(X, Y) = \text{E}[X - \text{E}(X)][Y - \text{E}(Y)] \]

(23)

Uncertainty of real and imaginary parts (5), (6) can be simplified similarly as in the case of module uncertainty (12). Accordingly, in the interval \((L; N/2 - L)\), it is valid

\[
\begin{array}{l}
u_{\text{rms}}(i) = \frac{\partial y}{\partial y(i)} u_{\text{rms}}(i) \end{array} \]

(24)

As a result of the convolution linearity, correlation coefficients of real part and imaginary part are the same

\[
r(R(i), R(j)) = r(i, j) \]

(25)

Another effect of the convolution is that each frequency line of the output frequency spectrum consists of contributions of neighboring frequency lines modified by window coefficients

\[
X_{\text{out}}(i) = a_i X_{\text{in}}(i) + \sum_{l=1}^{L} a_i \left( X_{\text{in}}(i+l) + X_{\text{in}}(i-l) \right) \]

(26)

where \(X_{\text{in}}\) is the frequency spectrum of the input signal (before windowing), \(X_{\text{out}}\) is the output frequency spectrum (after windowing) and \(a_i\) are window coefficients. Equation (26) shows the dependence of neighboring spectral lines. Substituting (26) into (22), it is possible to obtain the correlation coefficient between frequency spectral lines.
For example, the determination of correlation coefficient between two adjacent frequency lines for first order cosine windows is shown. Equation (26) becomes for first order windows as follows

\[
X_{in}(i) = a_0X_{in}(i) + \frac{a_1}{2}(X_{in}(i+1) + X_{in}(i-1))
\]  

(27)

The covariance between two adjacent frequency lines is

\[
\text{cov}(X_{in}(i), X_{in}(i+1)) = E[(X_{in}(i) - E(X_{in}(i)))(X_{in}(i+1) - E(X_{in}(i+1)))]
\]  

(28)

By the subtraction of mean values, only the quantization noise remains. Therefore in further analysis, only the quantization noise is considered and frequency spectral lines \(X_{in}\) and \(X_{out}\) represent only the contribution of quantization noise. As a result of zero mean value of quantization noise. As a result of zero mean value of quantization noise is frequency independent, only the product of contributions corresponding to the same frequency lines \((X_{in}(i)X_{in}(i)\) and \(X_{in}(i+1)X_{in}(i+1))\) is nonzero. In addition, the influence of the quantization noise is the same in all frequency lines; 

\[
E(X_{in}^2(i)) = E(X_{in}^2)
\]  

(29)

\[
\text{cov}(X_{in}(i), X_{in}(i+1)) = a_0a_1E(X_{in}^2)
\]  

(30)

Similarly, it can be get

\[
\text{cov}(X_{in}(i), X_{in}(i)) = \left[ a_0^2 + \frac{a_1^2}{2} \right]E(X_{in}^2) = NNPG \times E(X_{in}^2)
\]  

(31)

and final correlation coefficient is (22)

\[
r(i, j + 1) = \frac{a_0a_1}{NNPG}
\]  

(32)

Correlation coefficient \(r(i, j)\) for cosine window of any order can be derived as

\[
r(i, j) = \frac{C(d)}{Q(L)}
\]  

(33)

where \(d = |i - j|\),

\[
Q(L) = a_0^2 + \frac{1}{2} \sum_{n=1}^{L} a_n^2 = NNPG
\]  

(34)

\[
C(d) = \left\{ \begin{array}{ll}
\sum_{n=0}^{d} b_n b_{d-n} & d \in \{1, L\} \\
\sum_{n=d+1}^{L} b_n b_{d-n} & d \in (L, 2L) \\
0 & d > 2L
\end{array} \right.
\]  

(35)

and coefficients \(b_i\) are

\[
b_i = \begin{cases} 2^{-i} & i = 0 \\ \frac{1}{2} & i \in \{1, L\} \end{cases}
\]  

(36)

For further simplification let’s express real and imaginary parts of (20) in polar coordinates

\[
u_{iM(j)} = M(i) \cos(\phi(i)) M(j) \cos(\phi(j)) + \frac{M(i) \sin(\phi(i)) M(j) \sin(\phi(j))}{M(i) M(j)} R_{iM(j)}
\]  

(37)

Phase characteristic of cosine windows is changing in steps of \(\pi\) in the location of the main lobe [4]. This results in

\[
u_{iM(j)} = u_{iM(i)} u_{M(j)} (-1)^{i-j} r(i, j)
\]  

(38)

Since the odd coefficients of cosine windows have the negative sign (and correlation coefficients for odd \(d\) are negative), equation (40) can be simplified

\[
u_{iM(j)} = M(i) \cos(\phi(j)) M(j) \cos(\phi(j)) + \sin(\phi(i)) \sin(\phi(j))
\]  

(39)

In the interval \(\{L, N/2 - L\}\) equation (41) becomes

\[
u_{iM(j)} = u_{iM(i)} r(i, j)
\]  

(40)

The last step necessary for the expression of the RMS value uncertainty is to determine the last term of equation (19). In the case of coherent sampling, the ratio of two neighboring amplitude frequency lines of time window main lobe is given by the ratio of the window coefficients. For the module \(M(i)\) it is valid

\[
2M(i) = |k_{L-L}| \sqrt{2} S_{RMS}
\]  

(41)

Due to the convolution, window frequency spectrum is shifted to the frequency of all significant sine waves contained in the signal. Substituting (42), (43) into (19), following formula can be get

\[
u_{iM(j)} = \frac{8}{NNPG} u_{iM(i)}^2 S_{RMS} + \frac{4}{NNPG} u_{jM(j)}^2 S_{RMS} \sum_{k=-L}^{L} |k_i|^2 |r(i, j)|
\]  

(42)

By the substitution (12) and (44) into (18), the final formula of RMS value uncertainty follows
\[
\begin{align*}
    u_{x_{\text{rms}}}^2 &= \frac{2}{N} \sum_{i=1}^{N-1} \sum_{j=1}^{i} |r_{(i,j)}|^2 \\
    &= \frac{1}{N \times NPG} u_s^2
\end{align*}
\]

Equation (45) was obtained with the assumption of coherent sampling. In case of non-coherency, the output modules are not given directly by window coefficients. Nevertheless, experimental computations showed that the differences of results by coherent and non-coherent sampling are negligible; thus, equation (45) is applicable also for the case of non-coherent sampling.

Equation (45) is relatively complicated for the common usage. A rough estimate of the RMS value uncertainty can be get using a simplified formula [5]

\[
    u_{x_{\text{rms}}}^2 = \frac{1}{N \times NPG} u_s^2
\]

This formula was derived with the assumption that correlation coefficients (34) equal one; therefore, the simplified estimate of RMS uncertainty is greater than actual one (experiments showed an increase of uncertainty estimation of about 20%).

IV. NUMERICAL VERIFICATION

Validity of the formula derived above was verified by simulation. Whereas the results of theoretical analysis correspond to the uncertainty type “B”, simulation results were obtained by statistical analysis, so they correspond to the uncertainty type “A”. The ratio of both uncertainties for five cosine windows is computed in Tables 1 and 2.

Table 1. Phase uncertainty – ratio of theoretical uncertainty \(u_B\) to the uncertainty gained by simulation \(u_A\)

<table>
<thead>
<tr>
<th>Window</th>
<th>Window order (L)</th>
<th>(u_B / u_A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hann</td>
<td>1</td>
<td>0.996</td>
</tr>
<tr>
<td>Hamming</td>
<td>1</td>
<td>1.015</td>
</tr>
<tr>
<td>Blackman</td>
<td>2</td>
<td>0.999</td>
</tr>
<tr>
<td>4 Term Blackman-Harris</td>
<td>3</td>
<td>0.996</td>
</tr>
<tr>
<td>7 Term Blackman-Harris</td>
<td>6</td>
<td>1.007</td>
</tr>
</tbody>
</table>

Table 2. RMS value uncertainty – ratio of theoretical uncertainty \(u_B\) to the uncertainty gained by simulation \(u_A\)

<table>
<thead>
<tr>
<th>Window</th>
<th>Window order (L)</th>
<th>(u_B / u_A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hann</td>
<td>1</td>
<td>0.991</td>
</tr>
<tr>
<td>Hamming</td>
<td>1</td>
<td>1.006</td>
</tr>
<tr>
<td>Blackman</td>
<td>2</td>
<td>1.013</td>
</tr>
<tr>
<td>4 Term Blackman-Harris</td>
<td>3</td>
<td>0.997</td>
</tr>
<tr>
<td>7 Term Blackman-Harris</td>
<td>6</td>
<td>1.010</td>
</tr>
</tbody>
</table>

Simulations were processed for 10,000 signals with uniformly distributed quantization noise. The simulation uncertainty was computed as a standard deviation of particular results.

V. CONCLUSION

Uncertainty analysis introduced in this paper enables the uncertainty determination of phases of separate frequency spectral lines as well as RMS values in the frequency spectrum. As the majority of data records are sampled by non-coherent sampling, this analysis was performed with regard to signal windowing in the time domain. Cosine windows, which are widely applied for the signal processing, are assumed. The analysis was performed with regard to quantization uncertainty as the only source of uncertainties. Other sources are considered to be negligible or a part of quantization uncertainty. The approach presented in GUM [6] was used for uncertainty evaluation. Experimental simulations showed to be in a very good agreement with the theoretical results.

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