Analyzing the Held-Karp TSP Bound: A Monotonicity Property with Application

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Abstract

In their 1971 paper on the Traveling Salesman Problem and Minimum Spanning Trees, Held and Karp showed that finding an optimally weighted 1-tree is equivalent to solving a linear program for the Traveling Salesman Problem (TSP) with only node-degree constraints and subtour elimination constraints. In this paper we show that the Held-Karp 1-trees have a certain monotonicity property: given a particular instance of the symmetric TSP with triangle inequality, the cost of the minimum weighted 1-tree is monotonic with respect to the set of nodes included. As a consequence, we obtain an alternate proof of a result of Wolsey and show that linear programs with node-degree and subtour elimination constraints must have a cost at least $\frac{2}{3}OPT$, where $OPT$ is the cost of the optimum solution to the TSP instance.

The traveling salesman problem is one of the most notorious in the field of combinatorial optimization, and one of the most well-studied [7]. Currently, the most successful approach to finding optimal solutions to large-scale problems is based on formulating the problem as a linear program and finding explicit partial descriptions of this linear polytope [5], [8]. The most natural constraints are derived from an integer linear programming formulation that uses node-degree constraints and subtour elimination constraints. We focus our attention on symmetric instances of the TSP that obey the triangle inequality. Let $V = \{1, 2, \ldots, n\}$ denote the set of nodes. For any distinct $i$ and $j$, assign a cost $c_{ij}$ such that $c_{ij} = c_{ji}$, and for any $k$ distinct from $i$ and $j$, $c_{ij} \leq c_{ik} + c_{kj}$. Then the Subtour LP on this instance is

$$B = \min \left\{ \sum_{1 \leq i < j \leq n} c_{ij}x_{ij} \right\}$$
subject to: $$\sum_{j > i} x_{ij} + \sum_{j < i} x_{ji} = 2, \quad i = 1, 2, \ldots, n,$$
$$\sum_{i \in S, j \in S, i < j} x_{ij} \leq |S| - 1, \quad \text{for any proper subset } S \subset V,$$
$$0 \leq x_{ij} \leq 1, \quad 1 \leq i < j \leq n. \quad (1)$$

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If $\text{OPT}$ denotes the minimum cost of a traveling salesman tour, i.e., an optimal integral solution to (1), then clearly, $\mathcal{B} \leq \text{OPT}$.

Held and Karp [6] show that $\mathcal{B}$ is related to finding a minimum cost 1-tree. A 1-tree is composed of a tree on nodes $\{2, \ldots, n\}$ and two distinct edges incident to node 1. Thus a 1-tree has exactly one cycle, which contains node 1, and node 1 always has degree two. A minimum-cost 1-tree can be obtained by finding a minimum-cost spanning tree on $\{2, \ldots, n\}$ and adding the two lowest cost edges incident to node 1. If $\pi = (\pi_1, \ldots, \pi_n)$ is a real $n$-vector, a minimum-cost 1-tree with respect to the Lagrangean multipliers $\pi$ is defined to be the minimum-cost 1-tree with respect to the reduced costs $c_{ij} = c_{ij} + \pi_i + \pi_j$. Let $T_1, \ldots, T_t$ be an enumeration of all 1-trees. If $T_k$ is a 1-tree, define the adjusted cost of $T_k$ with respect to $\pi$ to be

$$\sum_{(i,j) \in T_k} \pi_{ij} - 2 \sum_{i=1}^{n} \pi_i$$

and let $w(\pi)$ denote the adjusted cost (with respect to $\pi$) of the minimum-cost 1-tree with respect to $\pi$. It will be convenient to let $c_k$ be the cost of $T_k$, $d_{ik}$ be the degree of node $i$ in $T_k$, and $v_{ik} = d_{ik} - 2$. Then

$$w(\pi) = \min_k [c_k + \sum_{i=1}^{n} \pi_i v_{ik}].$$

Held and Karp proved that if $\mathcal{W} = \max_{\pi} w(\pi)$, then $\mathcal{W} = \mathcal{B}$. If we express $\max_{\pi} w(\pi)$ as the linear program,

$$\mathcal{W} = \max w \text{ subject to: } w \leq c_k + \sum_{i=1}^{n} \pi_i v_{ik}, \quad \forall k = 1, \ldots, t, \quad (2)$$

it is easy to see that there is a polynomial-time algorithm that, given a solution, either finds a violated constraint, or proves that the solution is feasible. Thus, the ellipsoid method can be used to find an optimal set of multipliers and the corresponding tree [4].

The main result of this note is that the Held and Karp bound obeys a nice monotonicity property. Let $O \subseteq V$ and let $\mathcal{W}_O$ be the cost of the optimal solution to the LP (2) on the node set $O$.

**Theorem 1** $\mathcal{W}_O \leq \mathcal{W}$.

**Proof.** If $n \leq 5$, then it is well known that the extreme points of the polytope defined by (1) are integral [5]; i.e., they correspond to tours. Thus, in this case, the triangle inequality implies that an optimal tour on $[n]$ can be shortcut to

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1In fact, Held and Karp consider a linear program with the subtour elimination constraints only for those subsets not containing node 1. However, it is easy to see that the additional constraints are implied by the corresponding constraint on the complementary set of nodes.
yield a tour on \( O \) that is no longer. This yields the desired result. Consider next \( n > 5 \). By observing that the formulation (1) is independent of the choice of the special node 1, we can assume that, without loss of generality, \( O = \{1, \ldots, n - 1\} = [n - 1] \). We shall assume that \( W_{[n-1]} > W \) and show that this leads to a contradiction.

Let \( \overline{T} = T_k \) and \( \pi = (\pi_1, \ldots, \pi_{n-1}) \) be the optimal 1-tree and the optimal Lagrangean multipliers for \([n-1]\), respectively, so that

\[
W_{[n-1]} = c_k + \sum_{i=1}^{n-1} \pi_i v_{ik}. \tag{3}
\]

We first show that \( \overline{T} \) and \( \pi \) can be picked such that the two edges adjacent to node 1 have the same reduced cost.

**Lemma 1** There exist Lagrangean multipliers \( \pi \) for \([n-1]\) for an optimal weighted 1-tree \( \overline{T} \) such that if node 1 is adjacent to nodes \( x \) and \( z \),

\[
c_1 x = c_1 z.
\]

**Proof.** Suppose that in \( \overline{T} \), node 1 is adjacent to nodes \( x \) and \( z \), and \( \tau_{1x} < \tau_{1z} \). This implies that \((1, x)\) must be the single cheapest edge adjacent to 1, so all optimal 1-trees with respect to the Lagrangean multipliers \( \pi \) must include \((1, x)\). Consider the dual to the LP (2), as follows:

\[
\mathcal{Y} = \min \sum_k c_k y_k, \quad \text{subject to: \quad} \sum_k y_k = 1, \\
\sum_i v_{ik} y_k = 0 \quad i = 1, \ldots, n-1, \\
y_k \geq 0. \tag{4}
\]

As Held and Karp observed, this dual LP finds the minimum cost convex combination of 1-trees such that each node has average degree two. By complementary slackness, each tree \( T_k \) for which \( y_k \neq 0 \) in the optimal dual solution is an optimal 1-tree with respect to \( \pi \). As noted above, \((1, x)\) must be in each of these trees. Since \( x \) will have at least degree two for each tree, it must have exactly degree two for each tree in the dual solution.

Pick one such \( T_k \). Since \( x \) has degree two, increasing \( \pi_x \) will not change the adjusted cost of \( T_k \) from the optimum value, \( W_{[n-1]} \). We show that this does not affect the optimality of the spanning tree of \( T_k \) on \( 2, \ldots, n-1 \). Increasing \( \pi_x \) does not affect the relative order of the reduced cost of edges incident to \( x \), and does not affect the reduced cost of any other edge. Since \( x \) is a leaf in this spanning tree, the edge incident to \( x \) is the cheapest such edge, and if \( \pi_x \) is increased a minimum spanning tree will contain this edge. Clearly, all other edges will remain in the spanning tree as well.

If node \( z \) is also adjacent to node 1, and we increase \( \pi_x \) by \( \tau_{1z} - \tau_{1x} \), then \((1, x)\) and \((1, z)\) are still the two cheapest edges adjacent to 1, but \( \tau_{1x} = \tau_{1z} \). By the arguments above, \( T_k \) is an optimal 1-tree with respect to the modified
multipliers $\pi$ such that $T_k$ has adjusted cost $W_{[n-1]}$. Thus $T_k$ and the new $\pi$ are optimal solutions to the LP (2). $\square$

Let $T(\pi_n)$ be the minimum-cost 1-tree on $V$ with respect to $\pi$ for nodes in $[n-1]$ and $\pi_n$ for node $n$. If the adjusted cost of $T(\pi_n)$ is greater than or equal to $W_{[n-1]}$ for any $\pi_n$, then by supposition it is greater than $W$. Thus we have found a vector $\pi$ for which the minimum-cost 1-tree on $V$ has adjusted cost greater than $W$, which contradicts the maximality of $W$.

Thus, $T(\pi_n)$ must have adjusted cost less than $W_{[n-1]}$. We will show that we can delete node $n$ from some $T(\pi_n)$ such that the adjusted cost of the resulting 1-tree is no greater, which contradicts the minimality of $T$ with respect to $\pi$. Thus the supposition $W < W_{[n-1]}$ must be false.

We now show that there exists $\pi_n$ such that $n$ has degree two in $T(\pi_n)$.

**Lemma 2** If node $n$ in $T(\pi_n)$ has degree $k < n - 1$, then there exists $\delta \geq 0$ such that $n$ has degree $k + 1$ in $T(\pi_n - \delta)$.

**Proof.** By the definition of an optimal 1-tree, $T(\pi_n)$ is a minimum spanning tree on $V - \{1\}$ plus the two cheapest edges adjacent to node 1, all with respect to the reduced costs $\pi_{ij}$. We can assume that the minimum spanning tree is constructed as follows: sort the elements in non-decreasing order; include in the tree those edges that connect two connected components in the graph induced by the edges that come earlier in the ordering. Note that by changing $\delta$, only the costs of edges incident to $n$ are altered, and these changes can only move those edges earlier in the order. Furthermore, if an edge is included, it will still be included after moving it earlier in the order.

For a particular value of $\delta$, there may be many orderings of the edges consistent with the reduced costs (due to ties in the values). Perform a series of interchanges, bringing the edges incident to $n$ earlier in the order, one step at a time. If the degree of $n$ increases as a result of one of these interchanges, we have proved the lemma. Next consider the edges incident to node 1, and check if the edge $(1, n)$ is of the same cost as one of the edges in the current solution. Again, if the degree of node $n$ increases, we are done.

Apply the above argument with $\delta = 0$. If this fails to produce the desired tree, increase $\delta$ until the reduced cost of one of the edges incident to $n$ equals the reduced cost of one of the other edges in the graph, and then repeat the procedure given above for the new value of $\delta$. Note that if $\delta$ is sufficiently large (greater than $\max_j\{\pi_{jn}\} - \min_{i,j}\{\pi_{ij}\}$) then the degree of node $n$ must become $n - 1$. Therefore, the procedure given above must terminate and give a 1-tree in which node $n$ has degree $k + 1$. $\square$

**Corollary 1** There exists a value $\pi_n$ such that node $n$ has degree two in $T(\pi_n)$.

**Proof.** This follows from Lemma 2 and the observation that if $\pi_n$ is sufficiently large, then it must have degree 1 in any minimum-cost 1-tree. $\square$
Let $\pi_n$ be such that node $n$ has degree two in $T(\pi_n) = T_a$, and let $w$ and $x$ be the two nodes adjacent to $n$. If $(w, x)$ is not in $T_a$, then form the 1-tree $T_b$ by removing edges $(w, n)$ and $(x, n)$, and adding $(w, x)$. Since $v_{na} = 0$, $v_{ia} = v_{ib}$, and $c_{wx} \leq c_{nx} + c_{nw}$, 

$$c_b + \sum_{i=1}^{n-1} \pi_i v_{ib} \leq c_a + \pi_n v_{na} + \sum_{i=1}^{n-1} \pi_i v_{ia}$$

$$< c_b + \sum_{i=1}^{n-1} \pi_i v_{ib},$$

which contradicts the minimality of $T(= T_k)$ with respect to the multipliers $\pi$.

Suppose that the edge $(w, x)$ is already in $T(\pi_n)$. This means that there is a cycle $(n, w, x)$, and since node 1 is in the unique cycle in a 1-tree, either $w$ or $x$ must be node 1. Say that $w \equiv 1$. By the optimality of $T(\pi_n)$, $(1, x)$ must be one of the edges adjacent to node 1 in $T$. By Lemma 1, there exists another edge $(1, z)$ with $\pi_{1z} = \pi_{1x}$. So we can remove edge $(1, x)$ and add $(1, z)$ without affecting the optimality of $T(\pi_n)$. $(1, x) \equiv (w, x)$ is no longer in the tree, so we can shortcut node $n$ as above.

This completes the proof. Note that since $B = W$, the theorem also implies that $B_0 \leq B$. $\square$

As a part of their analysis of the probabilistic behavior of the Euclidean traveling salesman problem, Goemans and Bertsimas [3] have independently obtained a similar result.

With just a little more work, we can obtain Wolsey’s lower bound on $B$ (see [10]) by using a result of Christofides. Christofides [1] observed that if $T$ is the cost of a spanning tree, and $M$ is the cost of a matching on the odd nodes of the tree, then $M + T \geq \text{OPT}$. This comes from the fact that a tree plus a matching on the odd nodes yields an Eulerian graph. By starting with an Eulerian circuit of the graph and “shortcutting” any multiply visited nodes, we can obtain a tour no longer than the total length of edges in the Eulerian subgraph. The same holds true if a 1-tree is used instead of a spanning tree.

If we assume that there is an even number of nodes, the cost of a matching can be bounded in terms of $B$.

**Lemma 3** Let $M$ be the cost of the minimum cost matching, assuming that $n = |V|$ is even. Then $M \leq \frac{1}{2}B$.

**Proof.** Let $\pi$ be an optimal solution to (1). Then $\frac{1}{2}\pi$ satisfies the following constraints:

$$\sum_{j<i} x_{ij} + \sum_{j<i} x_{ji} = 1, \quad i = 1, 2, \ldots, n,$$

$$\sum_{i \in S, j \in S, i < j} x_{ij} \leq \frac{1}{2}(|S| - 1), \quad S \subset V, |S| \geq 3, |S| \text{ odd},$$

$$0 \leq x_{ij} \leq 1, \quad 1 \leq i < j \leq n.$$

By a classic result of Edmonds [2], these are exactly the constraints for the linear programming formulation of the matching problem. Since the objective
function for the two LPs is exactly the same (min $\sum_{1 \leq i < j \leq n} c_{ij}x_{ij}$) and $\frac{1}{2} \pi$ is a feasible solution to (6), the cost of the matching is no greater than half the cost of the Subtour LP. Thus $M \leq \frac{1}{2} B$. □

Pick a optimal 1-tree $T_s$ with $\pi_i = 0$, for all $i$. This implies $c_s \leq B$. Let $O \subseteq V$ be the odd degree nodes of $T$. Then

$$OPT \leq c_s + MO$$

$$OPT \leq B + \frac{1}{2} BO$$

$$OPT \leq B + \frac{1}{2} B$$

$$OPT \leq \frac{3}{2} B$$

$$\frac{2}{3} OPT \leq B$$

Equation (7) follows from Christofides’ technique, (8) follows from Lemma 3, and (9) follows from the monotonicity theorem. Therefore, $B$, the value of the Subtour LP, is bounded above by $OPT$ and bounded below by $\frac{2}{3} OPT$. □

A similar approach has recently been applied to the asymmetric variant of the traveling salesman problem with the triangle inequality. The techniques used here can be extended to show that in this case, the natural generalization of the Held-Karp bound has the analogous monotonicity property, and yields a value that is at least $OPT / \lceil \log n \rceil$ [9].

References


