Local Distributed Decision

Pierre Fraigniaud † Amos Korman † David Peleg ‡

Abstract

A central theme in distributed network algorithms concerns understanding and coping with the issue of locality. Despite considerable progress, research efforts in this direction have not yet resulted in a solid basis in the form of a fundamental computational complexity theory. Inspired by sequential complexity theory, we focus on a complexity theory for distributed decision problems. In the context of locality, solving a decision problem requires the processors to independently inspect their local neighborhoods and then collectively decide whether a given global input instance belongs to some specified language. Decision problems also provide a natural framework for tackling fault-tolerance, as the processors have to collectively check whether the network is fault-free, and a node detecting a fault raises an alarm. This paper introduces several classes of distributed decision problems, proves separation among them and presents some complete problems.

More specifically, we consider the standard LOCAL model of computation and define LD (for local decision) as the class of decision problems that can be solved in constant number of communication rounds. We first study the intriguing question of whether randomization helps in local distributed computing, and to what extent. Specifically, we define the corresponding randomized class BPLD, and ask whether LD = BPLD. We provide a partial answer to this question by showing that in many cases, randomization does not help for deciding hereditary languages. In addition, we define the notion of local many-one reductions, and introduce the (nondeterministic) class NLD of decision problems for which there exists a certificate that can be verified in constant number of communication rounds. We show that LD ⊊ NLD and prove that there exists an NLD-complete problem. We also show that there exist problems not in NLD. On the other hand, we prove that the class NLD_{#n}, which is NLD assuming that each processor can access an oracle that provides the number of nodes in the network, contains all (decidable) languages. For this class we provide a natural complete problem as well.

Keywords: Local distributed algorithms, local decision, nondeterminism, randomized algorithms.

*Supported by a France-Israel cooperation grant (“Mutli-Computing” project) from the France Ministry of Science and Israel Ministry of Science.
†CNRS and University Paris Diderot, France. E-mail: {pierre.fraigniaud,amos.korman}@liafa.jussieu.fr. Supported by the ANR projects ALADDIN and PROSE and by the INRIA project GANG.
‡The Weizmann Institute of Science, Rehovot, Israel. E-mail: david.peleg@weizmann.ac.il.
1 Introduction

Motivation: Distributed computing concerns a collection of processors which collaborate in order to achieve some global task. With time, two main disciplines have evolved in the field. One discipline deals with timing issues, namely, uncertainties due to asynchrony (the fact that processors run at their own speed, and possibly crash), and the other concerns topology issues, namely, uncertainties due to locality constraints (the lack of knowledge about far away processors). Studies carried out by the distributed computing community within these two disciplines were to a large extent problem-driven. Indeed, several major problems considered in the literature concern coping with one of the two uncertainties. For instance, in the asynchrony-discipline, Fischer, Lynch and Paterson [14] proved that consensus cannot be achieved in the asynchronous model, even in the presence of a single fault, and in the locality-discipline, Linial [28] proved that $(\Delta+1)$-coloring cannot be achieved locally (i.e., in a constant number of communication rounds), even in the ring network.

One of the significant achievements of the asynchrony-discipline was its success in establishing unifying theories in the flavor of computational complexity theory. Some central examples of such theories are failure detectors [6, 7] and the wait-free hierarchy (including Herlihy’s hierarchy) [18]. In contrast, despite considerable progress, the locality-discipline still suffers from the absence of a solid basis in the form of a fundamental computational complexity theory. Obviously, defining some common cost measures (e.g., time, message, memory, etc.) enables us to compare problems in terms of their relative cost. Still, from a computational complexity point of view, it is not clear how to relate the difficulty of problems in the locality-discipline. Specifically, if two problems have different kinds of outputs, it is not clear how to reduce one to the other, even if they cost the same.

Inspired by sequential complexity theory, we focus on decision problems, in which one is aiming at deciding whether a given global input instance belongs to some specified language. In the context of distributed computing, each processor must produce a boolean output, and the decision is defined by the conjunction of the processors’ outputs, i.e., if the instance belongs to the language, then all processors must output “yes”, and otherwise, at least one processor must output “no”. Observe that decision problems provide a natural framework for tackling fault-tolerance: the processors have to collectively check whether the network is fault-free, and a node detecting a fault raises an alarm. In fact, many natural problems can be phrased as decision problems, like “is there a unique leader in the network?” or “is the network planar?”. Moreover, decision problems occur naturally when one is aiming at checking the validity of the output of a computational task, such as “is the produced coloring legal?”, or “is the constructed subgraph an MST?”. Construction tasks such as exact or approximated solutions to problems like coloring, MST, spanner, MIS, maximum matching, etc., received enormous attention in the literature (see, e.g., [5, 25, 26, 28, 29, 30, 31, 37]), yet the corresponding decision problems have hardly been considered.

Investigating the connections between construction and decision in the context of locality is beyond the scope of this paper. Still, we point out some interesting links between the two. For
example, one may think that local construction is harder than local decision. It turns out that this is not always true. Nevertheless, for problems for which it is true, an efficient decision algorithm can help the design of an efficient construction algorithm. For instance, the decision algorithm can be used as a black box for increasing the success probability of a Monte-Carlo construction algorithm, or for turning it into a Las-Vegas algorithm. The purpose of this paper is to investigate the nature of local decision problems.

We consider the \textsc{Local} model \cite{35}, which is a standard distributed computing model capturing the essence of locality. In this model, processors are woken up simultaneously, and computation proceeds in fault-free synchronous rounds during which every processor exchanges messages of unlimited size with its neighbors, and performs arbitrary computations on its data. Informally, let us define LD (for local decision) as the class of decision problems that can be solved in constant number of communication rounds in the \textsc{Local} model. (We focus on constant number of rounds because we are aiming at identifying problems that can be decided in number of communication rounds that is independent of the network size; We note that in the \textsc{Local} model, every decidable decision problem can be solved in $D$ communication rounds, where $D$ is the diameter of the network.)

Some decision problems are trivially in LD (e.g., “is the given coloring a $(\Delta+1)$-coloring?”, “do the selected nodes form an MIS?”, etc.), while some others can easily be shown to be outside LD (e.g., “is the network planar?”, “is there a unique leader?”, etc.). In contrast to the above examples, there are some languages for which it is not clear whether they belong to LD. To elaborate on this, consider the particular case where it is required to decide whether the network belongs to some specified family $F$ of graphs. If this question can be decided in a constant number of communication rounds, then this means, informally, that the family $F$ can somehow be characterized by relatively simple conditions. For example, a family $F$ of graphs that can be characterized as consisting of all graphs having no subgraph from $C$, for some specified finite set $C$ of finite subgraphs, is obviously in LD. However, the question of whether a family of graphs can be characterized as above is often non-trivial. For example, characterizing cographs as precisely the graphs with no induced $P_4$, attributed to Seinsche \cite{38}, is not easy, and requires nontrivial usage of modular decomposition.

Decision problems seem to provide a promising approach to building up a distributed computational theory for the locality-discipline. Indeed, as we will show, one can define local reductions in the framework of decision problems, thus enabling the introduction of complexity classes and notions of completeness. Inspired by standard complexity theory, we consider LD as the analogue of the class P, and introduce the classes NLD and BPLD, which are the respective analogues of the classes NP and BPP. This opens up some interesting research directions. Particularly, note that many natural problems (e.g., “is the network planar?”,”is it a forest?”) can be trivially shown to be outside LD, yet it is often much harder to prove that they are not in BPLD. The fundamental question of whether randomization increases the power of algorithmic procedures, and to what extent, has been investigated a lot in the locality-discipline, cf. \cite{32, 33}. Here we ask this question...
in the context of local decision, addressing questions such as whether LD = BPLD.

**Our contributions:** We first study the intriguing question of whether randomization helps for distributed decision problems. For \( p, q \in (0, 1] \), define \( \text{BPLD}(p, q) \) as the class of all distributed languages that can be decided by a randomized distributed algorithm that runs in a constant number of communication rounds and produces correct answers on legal (respectively, illegal) instances with probability at least \( p \) (resp., \( q \)), and let \( \text{BPLD} = \bigcup_{p,q \in [0,1]} \text{BPLD}(p,q) \). The question of whether randomization helps for local decision thus amounts to the question of whether LD = BPLD. We provide a partial answer to this question, by showing that \( \text{LD} = \bigcup_{q>1-p^2} \text{BPLD}(p,q) \) in the case of hereditary decision problems, i.e., problems that are closed under inclusion. This implies, in particular, that the problems of locally deciding whether the network is planar, or whether the network is a forest, are not in \( \bigcup_{q>1-p^2} \text{BPLD}(p,q) \).

Observe that the class \( \bigcup_{q>1-p^2} \text{BPLD}(p,q) \) contains \( \bigcup_p \text{BPLD}(p,1) \), which corresponds to the set of languages that can be decided by a one-sided randomized algorithm that may err (with at most a constant probability < 1) only on legal instances. Regarding the set of languages that can be decided by a one-sided randomized algorithm that may err only on illegal instances, we obtain an analogous result using a much simpler proof. Specifically, we show that this set coincides with LD under the assumption that the randomized algorithm is restricted to use a constant number of random bits per processor (i.e., per node of the network).

In addition, we establish some interesting structural results. We define the notion of local many-one reduction \( \preceq \) between distributed decision problems. In short, \( \mathcal{L} \preceq \mathcal{L}' \) if and only if there exists an algorithm that runs in a constant number of communication rounds, by which any instance for \( \mathcal{L} \) is mapped to an instance for \( \mathcal{L}' \) in a way that preserves the membership predicate. We further define NLD as the class of decision problems for which there exists a proof that can be verified in constant number of communication rounds. We show that \( \text{LD} \subset \text{NLD} \) and prove that there exists a problem, called Map Cover (MC), which is NLD-complete. We also show that there exist problems not in NLD. On the other hand, we prove that the class NLD\#n, which is NLD assuming that each node can access an oracle that provides the number of nodes in the graph, contains all (decidable) languages. Finally, we prove that there exists a natural problem, called Unique Set Cover (USC), which is NLD\#n-complete.

**Related work:** Locality issues have been thoroughly studied in the literature, via the analysis of various construction problems, including \((\Delta + 1)\)-coloring and Maximal Independent Set (MIS) \[1, 5, 23, 26, 28, 29, 34\], Minimum Spanning Tree (MST) \[12, 25, 36\], Maximal Matching \[19\], Maximum Weighted Matching \[30, 31, 40\], Minimum Dominating Set \[24, 27\], Spanners \[9, 13, 37\], etc. For some problems (e.g., coloring \[5, 23, 34\]), there are still large gaps between the best known results on specific families of graphs (e.g., bounded degree graphs) and on arbitrary graphs.
The question of what can be computed in a constant number of communication rounds was investigated in the seminal work of Naor and Stockmeyer [33]. In particular, that paper considers a subclass of LD, called LCL, which is essentially LD restricted to languages involving graphs of constant maximum degree, and involving processor inputs taken from a set of constant size, and studies the question of how to compute in $O(1)$ rounds the constructive versions of decision problems in LCL. The paper provides some beautiful general results. In particular, the authors show that if there exists a randomized algorithm that constructs a solution for a problem in LCL in $O(1)$ rounds, then there is also a deterministic algorithm constructing a solution for this problem in $O(1)$ rounds. Unfortunately, the proof of this result relies heavily on the definition of LCL. Indeed, the constant bound constraints on the degrees and input sizes allow the authors to cleverly use Ramsey theory. It is thus not clear whether it is possible to extend this result to all languages in LD.

The question of whether randomization helps in decreasing the locality parameter of construction problems has been the focus of numerous studies. To date, there exists evidence that, for some problems at least, randomization does not help. For instance, [32] proves this for 3-coloring the ring. In fact, for low degree graphs, the gaps between the efficiencies of the best known randomized and deterministic algorithms for problems like MIS, $(\Delta + 1)$-coloring, and Maximal Matching are very small. On the other hand, for graphs of arbitrarily large degrees, there seem to be indications that randomization does help, at least in some cases. For instance, $(\Delta + 1)$-coloring can be randomly computed in expected $O(\log n)$ communication rounds on $n$-node graphs [1, 29], whereas the best known deterministic algorithm for this problem performs in $2^{O(\sqrt{\log n})}$ rounds [34].

Decision problems were studied very recently in the CONGEST model. In contrast to the LOCAL model, this model assumes that the message size is bounded, hence dealing with congestion is the main issue. Specifically, tight bounds are established in [20] for the time and message complexities of the problem of deciding whether a subgraph is an MST, and time lower bounds for many other subgraph-decision problems (e.g., spanning tree, connectivity) are established in [8]. It is interesting to note that some of these lower bounds imply strong unconditional time lower bounds on the hardness of distributed approximation for many classical problems in the CONGEST model. Decision problems have received recent attention from the asynchrony-discipline too, in the framework of wait-free computing [17]. In this framework, the focus is on task checkability. Wait-free checkable tasks have been characterized in term of covering spaces, a fundamental tool in algebraic topology.

The theory of proof-labeling schemes [21, 22] was designed to tackle the issue of locally verifying (with the aid of a proof, i.e., a certificate, at each node) solutions to problems that cannot be decided locally (e.g., “is the given subgraph a spanning tree of the network?”, or, “is it an MST?”). In fact, the model of proof-labeling schemes has some resemblance to our definition of the class NLD. Investigations in the framework of proof-labeling schemes mostly focus on the minimum size of the certificate necessary so that verification can be performed in a single round. The notion of
proof-labeling schemes also has interesting similarities with the notions of local detection [2], local checking [3], or silent stabilization [11], which were introduced in the context of self-stabilization [10]. The use of oracles that provide information to nodes was studied also in the context of distributed construction tasks. For instance, this framework was studied in [15] for MST construction and in [15] for 3-coloring a cycle.

Finally, we note that our proof for Theorem 4.6 seems to be somewhat related to Ulam’s Conjecture, which states the following: let \( G \) be a graph with at least three nodes and let \( G_v = G - \{v\} \). Then there is a unique (up to isomorphism) graph \( G \) with a given (multi)set \( \{G_v, v \in V(G)\} \). This conjecture is known to be true for trees, and false for directed graphs. It is open for undirected graphs. See [39].

2 Decision problems and complexity classes

Let us first recall some basic notions in distributed computing. We consider the \( \text{LOCAL} \) model [35], which is a standard model capturing the essence of locality. In this model, processors are assumed to be nodes of a network \( G \), provided with arbitrary distinct identities, and computation proceeds in fault-free synchronous rounds. At each round, every processor \( v \in V(G) \) exchanges messages of unrestricted size with its neighbors in \( G \), and performs arbitrary computations on its data. More precisely, during the execution of a distributed algorithm, all processors are woken up simultaneously, and, initially, a processor is solely aware of its own identity, and possibly to some local input too. Then, in each round, every processor

(1) sends messages to its neighbors,
(2) receives messages from its neighbors, and
(3) performs arbitrary computations.

After a number of rounds (that may depend on the network \( G \) and may vary among the processors, simply because nodes have different identities, potentially different inputs, and are typically located at non-isomorphic positions in the network), every processor \( v \) terminates and outputs some value \( \text{out}(v) \). The running time of an algorithm executed in a network \( G \) is the maximum, taken over all possible identity assignments of the nodes and all possible inputs, of the number of rounds until all processors terminate. A distributed algorithm is called local if its running time is upper bounded by a universal constant, independent of the network \( G \). Note that, w.l.o.g., one can always assume that a local algorithm running in time at most \( t \) operates at each node \( v \) in two stages: (A) collect all information available in \( B_G(v, t) \), the \( t \)-neighborhood, or ball of radius \( t \) of \( v \) in \( G \), including inputs, identities and adjacencies, and (B) compute the output based on this information.

We now refine some of the above concepts, in order to formally define our objects of interest. Obviously, a distributed algorithm that runs on a graph \( G \) operates separately on each connected component of \( G \), and nodes of a component \( G' \) of \( G \) cannot distinguish the underlying graph \( G \).
from $G'$. For this reason, we consider connected graphs only.

**Definition 2.1** A configuration is a pair $(G, w)$ where $G$ is a connected graph, and every node $v \in V(G)$ is assigned as its local input a binary string $w(v) \in \{0, 1\}^*$.

In some problems, the local input of every node is empty, i.e., $w(v) = \epsilon$ for every $v \in V(G)$, where $\epsilon$ denotes the empty binary string. Since an undecidable collection of configurations remains undecidable in the distributed setting too, we consider only decidable collections of configurations. Formally, we define the following.

**Definition 2.2** A distributed language is a decidable collection $L$ of configurations.

In general, there are several possible ways of representing a configuration of a distributed language corresponding to standard distributed computing problems. Some examples considered in this paper are the following.

- **Leader** = $\{ (G, w) \mid \|w\|_1 = 1 \}$ consists of all configurations such that there exists a unique node with local input 1, with all the others having local input 0.
- **Consensus** = $\{ (G, (w_1, w_2)) \mid \exists u \in V(G), \forall v \in V(G), w_2(v) = w_1(u) \}$ consists of all configurations such that all nodes agree on the value proposed by some node.
- **Coloring** = $\{ (G, w) \mid \forall v \in V(G), \forall w \in N(v), w(v) \neq w(w) \}$ where $N(v)$ denotes the (open) neighborhood of $v$, that is, all nodes at distance 1 from $v$.
- **MIS** = $\{ (G, w) \mid S = \{ v \in V(G) \mid w(v) = 1 \} \text{ forms a MIS} \}$.
- **SpanningTree** = $\{ (G, (name, head)) \mid T = \{ e_v = (v, v^+) \mid v \in V(G), head(v) = name(v^+) \} \text{ is a spanning tree of } G \}$ consists of all configurations such that the set $T$ of edges $e_v$ between every node $v$ and its neighbor $v^+$ satisfying $name(v^+) = head(v)$ forms a spanning tree of $G$.

(The language MST, for minimum spanning tree, can be defined similarly).

An identity assignment $Id$ for a graph $G$ is an assignment of distinct integers to the nodes of $G$. A node $v \in V(G)$ executing a distributed algorithm in a configuration $(G, w)$ initially knows only its own identity $Id(v)$ and its own input $w(v)$, and is unaware of the graph $G$. After $t$ rounds, $v$ acquires knowledge only of its $t$-neighborhood $B_G(v, t)$. A distributed algorithm that runs in $t$ rounds consists of a prescribed sequence of instructions executed at each node, such that for every configuration $(G, w)$ and identity assignment $Id$ for $V(G)$, the algorithm produces a local output $out(v) \in \{0, 1\}^*$ at each node $v \in V(G)$ based on the $t$-neighborhood of $v$ in $G$.

Let $L$ be a distributed language. We say that a distributed algorithm $A$ decides $L$ if and only if for every configuration $(G, w)$, and for every identity assignment $Id$ for the nodes of $G$, every node of $G$ eventually terminates and outputs “yes” or “no”, satisfying the following decision rules:

- If $(G, w) \in L$, then $out(v) = \text{“yes”}$ for every node $v \in V(G)$;
If \((G, w) \notin L\), then there exists at least one node \(v \in V(G)\) such that \(\text{out}(v) = \text{"no"}\).

We are now ready to define one of our main subjects of interest, the class \(LD\), for local decision.

**Definition 2.3** \(LD\) is the class of all distributed languages that can be decided by a local distributed algorithm.

For instance, \(\text{Coloring} \in LD\) and \(\text{MIS} \in LD\). On the other hand, it is not hard to see that languages such as \(\text{Leader}, \text{Consensus},\) and \(\text{SpanningTree}\) are not in \(LD\).

A randomized distributed algorithm is a distributed algorithm that, at its start, enables every node \(v\) to toss an arbitrary number of random bits obtaining a string \(r(v) \in \{0, 1\}^*\). This number may depend on the identity of \(v\) as well as on its local input. For \(p, q \in (0, 1]\), we say that a randomized distributed algorithm \(\mathcal{A}\) is a \((p, q)\) decider for \(L\), or, that it decides \(L\) with “yes” success probability \(p\) and “no” success probability \(q\), if and only if for every configuration \((G, w)\), and for every identity assignment \(\text{Id}\) for the nodes of \(G\), every node of \(G\) eventually terminates and outputs “yes” or “no”, and the following properties are satisfied:

- If \((G, w) \in L\), then \(\Pr[\text{out}(v) = \text{"yes"} \text{ for every node } v \in V(G)] \geq p\),
- If \((G, w) \notin L\), then \(\Pr[\text{out}(v) = \text{"no"} \text{ for at least one node } v \in V(G)] \geq q\),

where the probabilities in the above definition are taken over all possible coin tosses performed by nodes. We define the class \(BPLD\), for “Bounded-error Probabilistic Local Decision”, as follows.

**Definition 2.4** For \(p, q \in (0, 1]\), \(BPLD(p, q)\) is the class of all distributed languages that have a local randomized distributed \((p, q)\)-decider, (i.e., can be decided by a local randomized distributed algorithm with “yes” success probability \(p\) and “no” success probability \(q\)), and \(BPLD = \bigcup_{p, q \in (0, 1]} BPLD(p, q)\).

Note that, as opposed to the class \(BPP\), it is not clear how to “boost” the success probabilities by performing several runs of a distributed randomized algorithm solving a problem in \(BPLD(p, q)\) for some \(p, q\). For instance, performing \(k\) runs, and having each node individually decide its output according to the majority of its outputs obtained during the \(k\) runs, may increase the “yes” success probability, but at the cost of a potential collapse of the “no” success probability. Indeed, the “no” outputs may occur at different nodes at each run. Hence, taking majority at each node potentially leads all nodes to eventually outputting “yes” in a scenario where in each of the runs at least one node outputs “no”. That is, a majority based policy may yield a situation in which \(k\) runs each ending with a “no” decision eventually results in a “yes” decision. It is not clear whether it is guaranteed that such a situation occurs with low probability. An alternative strategy for modifying the success probabilities by performing \(k\) runs involves requiring each node to individually output “no” if it decided “no” on at least one of the runs. In this case, the “no” success probability increases from \(q\) to at least \(1 - (1 - q)^k\). However, the “yes” success probability then decreases from \(p\) to \(p^k\).
3 Does randomization help?

Consider some graph $G$, and a subset $U$ of the nodes of $G$, i.e., $U \subseteq V(G)$. Let $G[U]$ denote the vertex-induced subgraph of $G$ defined by the nodes in $U$. Given a configuration $(G, w)$, let $w[U]$ denote the input $w$ restricted to the nodes in $U$. For simplicity of presentation, if $H$ is a subgraph of $G$, we denote $w[V(H)]$ by $w[H]$. A prefix of a configuration $(G, w)$ is a configuration $(G[U], w[U])$, where $U \subseteq V(G)$ (note that in particular, $G[U]$ is connected). We say that a language $\mathcal{L}$ is hereditary if every prefix of every configuration $(G, w) \in \mathcal{L}$ is also in $\mathcal{L}$. A typical example of an hereditary language is Coloring. As another example of an hereditary language, consider a family $\mathcal{G}$ of hereditary graphs, i.e., that is closed under vertex deletion; then the language $\{(G, \epsilon) \mid G \in \mathcal{G}\}$ is hereditary. Examples of hereditary graph families are planar graphs, interval graphs, forests, chordal graphs, cographs, perfect graphs, etc.

Theorem 3.1 below asserts that, for hereditary languages, randomization does not help if one imposes that $q > 1 - p^2$, the "no" success probability distribution to be at least as large as one minus the square of the "yes" success probability. Somewhat more formally, we prove that $\bigcup_{q > 1 - p^2} \text{BPLD}(p, q) = \text{LD}$ for hereditary languages. Recall that [33] investigates the question of whether randomization helps for constructing a solution for a problem in LCL in constant time. We stress that the technique used in [33] for tackling this question relies heavily on the definition of LCL, specifically, that only graphs of constant degree and of constant input size are considered. Hence it is not clear whether the technique of [33] can be useful for our purposes, as we impose no such assumptions on the degrees or input sizes. Instead, we use a completely different approach.

**Theorem 3.1** Let $\mathcal{L}$ be an hereditary language. If $\mathcal{L} \in \text{BPLD}(p, q)$ for $p \in (0, 1]$ and $q > 1 - p^2$, then $\mathcal{L} \in \text{LD}$.

**Proof.** Let us start with some definitions. Let $G = (V, E)$ be a connected graph. The distance $\text{dist}_G(u, v)$ between two nodes of $G$ is the minimum number of edges in a path connecting $u$ and $v$ in $G$. The distance between two subsets $U_1, U_2 \subseteq V$ is defined as

$$\text{dist}_G(U_1, U_2) = \min\{\text{dist}_G(u, v) \mid u \in U_1, v \in U_2\}.$$ 

Fix an integer $d > 1$. A $d$-splitter of $G$ is a triplet $(S, U_1, U_2)$ of pairwise disjoint subsets of nodes such that $S \cup U_1 \cup U_2 = V$, and $\text{dist}_G(U_1, U_2) \geq d$. Given an $d$-splitter $(S, U_1, U_2)$ of $G$, let $G_i = G[U_i \cup S]$, for $i = 1, 2$.

The following structural claim holds for every language $\mathcal{L} \in \text{BPLD}(p, q)$ where $p, q \in (0, 1]$ and $q > 1 - p^2$, and does not require the hereditary property.

**Lemma 3.2** There exists an integer $d > 1$ such that for every configuration $(G, w)$, and every $d$-splitter $(S, U_1, U_2)$ of $G$, we have

$$((G_1, w_1) \in \mathcal{L} \text{ and } (G_2, w_2) \in \mathcal{L}) \Rightarrow (G, w) \in \mathcal{L}$$
where $w_i$ is the input $w$ restricted to nodes in $G_i$, $i = 1, 2$.

To establish the lemma, let $A$ be a randomized algorithm deciding $L$, with constant running time $t$, "yes" success probability $p$, and "no" success probability $q > 1 - p^2$. Fix $0 < r < p^2 + q - 1$, and define $k = \left\lceil \frac{(4t+1)\log p}{\log(1-r)} \right\rceil$ and $d = 4t + k + 2$. Assume, towards contradiction, that there exists a $d$-splitter $(S, U_1, U_2)$ of $G$, such that $(G_1, w_1) \in L$ and $(G_2, w_2) \in L$, yet $(G, w) \notin L$. (The fact that $(G_1, w_1) \in L$ and $(G_2, w_2) \in L$ implies that both $G_1$ and $G_2$ are connected, however, we note, that for the claim to be true, it is not required that $G[U_1]$, $G[U_2]$ or $G[S]$ are connected.)

Given a vertex $u \in S$, we define the level of $u$ by $\ell(u) = \text{dist}_G(U_1, \{u\})$. For an integer $i \in [1, d]$, let $L_i$ denote the set of nodes in $S$ of level $i$. For an integer $i \in (t, d - t)$, let $S_i = \bigcup_{j=i-t}^{i+t} L_j$, and finally, for a set $J \subseteq (t, d - t)$ of integers, let $S_J = \bigcup_{i \in J} S_i$.

For an input $w$, an id-assignment $I$, and a set $U \subseteq V$, let $\mathcal{E}(G, w, I, U)$ denote the event that when running $A$ on $(G, w)$ with id-assignment $I$, all nodes in $U$ output "yes". Define

$$ I = \{i \in (2t, d - 2t) | \Pr[\mathcal{E}(G, w, I, S_i)] < 1 - r \text{ for every id-assignments } I \}. $$

**Claim 3.3** There exists $i \in (2t, d - 2t)$ such that $i \notin I$.

**Proof.** For proving Claim 3.3, we upper bound the size of $I$ by $k$. This is done by covering the integers in $(2t, d - 2t)$ by at most $4t + 1$ sets, such that each one is $(4t + 1)$-independent, that is, for every two integers in the same set, they are at least $4t + 1$ apart. Specifically, for $s \in [1, 4t + 1]$ and $m = \lceil (d - 8t)/(4t + 1) \rceil$, we define $M_s = \{s + 2t + j(4t + 1) | j \in [0, m]\}$. Observe that, as desired, $(2t, d - 2t) \subseteq \bigcup_{s \in [1, 4t + 1]} M_s$, and for each $s \in [1, 4t + 1], M_s$ is $(4t + 1)$-independent. In what follows, fix $s \in [1, 4t + 1]$ and let $M = M_s$. Since $(G_1, w_1) \in L$, we know that for any id-assignment $I$,

$$ \Pr[\mathcal{E}(G_1, w_1, I, S_{M \cap I})] > p. $$

Observe that for $i \in (2t, d - 2t)$, the $t$-neighborhood in $G$ of every node in $S_i$ is contained in $G_1$. Therefore, as algorithm $A$ runs in $t$ rounds, we have

$$ \Pr[\mathcal{E}(G, w, I, S_{M \cap I})] = \Pr[\mathcal{E}(G_1, w_1, I, S_{M \cap I})] > p. \quad (1) $$

Consider two integers $a$ and $b$ in $M$. We know that $|a - b| \geq 4t + 1$. Hence, the distance in $G$ between any two nodes $u \in S_a$ and $v \in S_b$ is at least $2t + 1$. Thus, the events $\mathcal{E}(G, w, I, S_a)$ and $\mathcal{E}(G, w, I, S_b)$ are independent for any id-assignment $I$. It follows by the definition of $I$, that for every id-assignment $I$,

$$ \Pr[\mathcal{E}(G, w, I, S_{M \cap I})] < (1 - r)^{|M \cap I|}. \quad (2) $$

By (1) and (2), we have that $p < (1 - r)^{|M \cap I|}$ and thus $|M \cap I| < \log p / \log(1 - r)$. Since $(2t, d - 2t)$ can be covered by the sets $M_s$, $s = 1, \ldots, 4t + 1$, each of which is $(4t + 1)$-independent, we get that

$$ |I| = \sum_{s=1}^{4t+1} |M_s \cap I| < (4t + 1)(\log p / \log(1 - r)) \leq k. $$

Combining this bound with the fact that $d - 4t - 1 > k$, it follows by the pigeonhole principle that there exists some $i \in (2t, d - 2t)$ such that $i \notin I$, as desired. \qed
Fix \( i \in (2t, d - 2t) \) such that \( i \notin I \), and let \( \mathcal{F} = \mathcal{E}(G, w, \text{Id}, S_i) \). By definition, there must exist an id-assignment \( \text{Id} \) for which \( \Pr[\mathcal{F}] \geq 1 - r > 2 - q - p^2 \), that is
\[
\Pr[\mathcal{F}] < p^2 + q - 1. \tag{3}
\]

Let \( H_1 \) denote the subgraph of \( G \) induced by the nodes in \( (\bigcup_{j=1}^{i-t-1} L_j) \cup U_1 \). We similarly define \( H_2 \) as the subgraph of \( G \) induced by the nodes in \( (\bigcup_{j=i+t} L_j) \cup U_2 \). Note that \( S_i \cup V(H_1) \cup V(H_2) = V \).

Observe that the \( t \)-neighborhood in \( G \) of each node \( u \in V(H_1) \) equals the \( t \)-neighborhood in \( G_1 \) of \( u \), and similarly, the \( t \)-neighborhood in \( G \) of each node \( u \in V(H_2) \) equals the \( t \)-neighborhood in \( G_2 \) of \( u \).

Thus, for \( i = 1, 2 \), since \( (G_i, w_i) \in \mathcal{L} \), we get
\[
\Pr[\mathcal{E}(G, w, \text{Id}, V(H_i))] = \Pr[\mathcal{E}(G_i, w_i, \text{Id}, V(H_i))] > p.
\]

Let \( \mathcal{F}' = \mathcal{E}(G, w, \text{Id}, V(H_1) \cup V(H_2)) \). As the events \( \mathcal{E}(G, w, \text{Id}, V(H_1)) \) and \( \mathcal{E}(G, w, \text{Id}, V(H_2)) \) are independent, it follows that \( \Pr[\mathcal{F}'] > p^2 \), that is
\[
\Pr[\mathcal{F}'] < 1 - p^2 \tag{4}
\]

By Eqs. (3) and (4), and using union bound, it follows that \( \Pr[\mathcal{F} \cup \mathcal{F}'] < q \). Thus
\[
\Pr[\mathcal{E}(G, w, \text{Id}, V(G))] = \Pr[\mathcal{E}(G, w, \text{Id}, S_i \cup V(H_1) \cup V(H_2))] = \Pr[\mathcal{F} \cap \mathcal{F}'] > 1 - q.
\]

This is in contradiction to the assumption that \( (G, w) \notin \mathcal{L} \). This concludes the proof of Lemma 3.2.

Consider now an hereditary language \( \mathcal{L} \) such that \( \mathcal{L} \in \text{BPLD}(p, q) \) for some \( p \in (0, 1] \) and \( q > 1 - p^2 \). Let \( d \) be such that Lemma 3.2 is satisfied. Let \( A \) be a randomized algorithm deciding \( \mathcal{L} \), with constant running time \( t \), "yes" success probability \( p \), and "no" success probability \( q > 1 - p^2 \).

We show that \( \mathcal{L} \in \text{LD} \) by proving the existence of a deterministic local algorithm \( D \) that recognizes \( \mathcal{L} \). Given a configuration \( (G, w) \), and an id-assignment \( \text{Id} \), Algorithm \( D \), applied at a node \( u \) outputs "yes" if and only if the \( 2d \)-neighborhood of \( u \) in \( (G, w) \) belongs to \( \mathcal{L} \). More formally, Algorithm \( D \) is specified by the following:
\[
\text{out}(u) = \text{"yes"} \iff (B_G(u, 2d), w[\text{B}_G(u, 2d)]) \in \mathcal{L}.
\]

Obviously, Algorithm \( D \) is a deterministic algorithm that runs in time \( 2d \). We claim that Algorithm \( D \) decides \( \mathcal{L} \). Indeed, since \( \mathcal{L} \) is hereditary, if \((G, w) \in \mathcal{L} \), then every prefix of \((G, w) \) is also in \( \mathcal{L} \), and thus, every node \( u \) outputs \( \text{out}(u) = \text{"yes"} \). Now consider a configuration \( (G, w) \notin \mathcal{L} \), and assume by contradiction that there exists an id-assignment \( \text{Id} \) such that, by applying \( D \) on \((G, w) \) with id-assignment \( \text{Id} \), every node \( u \) outputs \( \text{out}(u) = \text{"yes"} \). Let \( U \subseteq V(G) \) be maximal by inclusion, such that \( G[U] \) is connected and \( (G[U], w[U]) \in \mathcal{L} \). Obviously, \( U \) is not empty, as \( (B_G(u, 2d), w[B_G(u, 2d)]) \in \mathcal{L} \) for every node \( u \). On the other hand, we have \( |U| < |V(G)| \), because \( (G, w) \notin \mathcal{L} \). Let \( u \in V(G) \setminus U \) be such that its neighborhood in \( G \) intersects \( U \), i.e., \( B_G(u, 1) \cap U \neq \emptyset \). See Figure 1 in the Appendix. Define \( G' \) as the subgraph of \( G \) induced by \( U \cup V(B_G(u, d)) \). Observe that \( G' \) is connected. Towards contradiction, our goal is to show that \((G', w[G']) \in \mathcal{L} \).
Let $H$ denote the graph which is maximal by inclusion such that $H$ is connected and

$$B_G(u,d) \subset H \subseteq B_G(u,d) \cup (U \cap B_G(u,2d)) .$$

Let $W^1, W^2, \cdots, W^\ell$ be the $\ell$ connected components of $G[U] \setminus B_G(u,d)$, ordered arbitrarily. Let $W^0$ be the empty graph, and for $k = 0, 1, 2, \cdots, \ell$, define the graph $Z^k = H \cup W^0 \cup W^1 \cup W^2 \cup \cdots \cup W^k$. Observe that $Z^k$ is connected for each $k = 0, 1, 2, \cdots, \ell$. We prove by induction on $k$ that $(Z^k, w[Z^k]) \in \mathcal{L}$ for every $k = 0, 1, 2, \cdots, \ell$. This will establish the contradiction since $Z^\ell = G'$. For the basis of the induction, the case $k = 0$, we need to show that $(H, w[H]) \in \mathcal{L}$. However, this is immediate by the facts that $H$ is a connected subgraph of $B_G(u,2d)$, the configuration $(B_G(u,2d), w[B_G(u,2d)]) \in \mathcal{L}$, and $\mathcal{L}$ is hereditary. Assume now that we have $(Z^k, w[Z^k]) \in \mathcal{L}$ for $0 \leq k < \ell$, and consider the graph $Z^{k+1} = Z^k \cup W^{k+1}$. Define the sets of nodes

$$S = V(Z^k) \cap V(W^{k+1}), \quad U_1 = V(Z^k) \setminus S, \quad \text{and} \quad U_2 = V(W^{k+1}) \setminus S .$$

A crucial observation is that $(S, U_1, U_2)$ is a $d$-splitter of $Z^{k+1}$. This follows from the following arguments. Consider a simple directed path $P$ in $Z^{k+1}$ going from a node $x \in U_1$ to a node $y \in U_2$. Since $x \notin V(W^{k+1})$ and $y \in V(W^{k+1})$, we get that $P$ must pass through a vertex in $B_G(u,d)$. Let $z$ be the last vertex in $P$ such that $z \in B_G(u,d)$, and consider the directed subpath $P_z$ of $P$ going from $z$ to $y$. Now, let $P' = P_{[z,y]} \setminus \{z\}$. The first $d' = \min\{d, |P'|\}$ vertices in the directed subpath $P'$ must belong to $V(H) \subseteq V(Z^k)$. In addition, observe that all nodes in $P'$ must be in $V(W^{k+1})$. It follows that the first $d'$ nodes of $P'$ are in $S$. Since $y \notin S$, we get that $|P'| \geq d' = d$, and thus $|P| > d$. Consequently, $d_{Z^{k+1}}(U_1, U_2) \geq d$, as desired. Now, by the induction hypothesis, we have $(G_1, w[G_1]) \in \mathcal{L}$, because $G_1 = G[U_1 \cup S] = Z^k$. In addition, we have $(G_2, w[G_2]) \in \mathcal{L}$, because $G_2 = G[U_2 \cup S] = W^{k+1}$, and $W^{k+1}$ is a prefix of $G[U]$. We can now apply Corollary 3.4 and conclude that $(Z^{k+1}, w[Z^{k+1}]) \in \mathcal{L}$. This concludes the induction proof. The theorem follows.

Let Planar = \{(G, \epsilon) : G is planar\}, Interval = \{(G, \epsilon) : G is an interval graph\} and CycleFree = \{(G, \epsilon) : G has no cycle\}. One can easily check that neither of these three languages is in LD. It appears intuitive that they are not in BPLD as well, however, a formal proof of that does not seem to be easy. Theorem 3.1 yields the following.

**Corollary 3.4** Let $p \in (0, 1]$ and $1 \geq q > 1 - p^2$, then Planar, Interval and CycleFree are not in BPLD($p,q$).

Theorem 3.1 applies to "yes" and "no" success probabilities $p$ and $q$ satisfying $p^2 + q > 1$. As discussed before, it is possible to obtain different success probabilities by performing $k$ runs and having each node individually outputts “no” if it decided “no” in at least one of the runs. The "yes" success probability becomes $p' = p^k$ while the "no" success probability becomes $q' = 1 - (1 - q)^k$. Nevertheless, for $p^2 + q \leq 1$, we get $p'^2 + q' \leq 1$ too. It is thus unclear how to enlarge the domain of success probabilities $p, q$ for which Theorem 3.1 holds.
Observe that $\bigcup_{0 < p \leq 1} \text{BPLD}(p, 1)$ corresponds to the class of languages that can be decided by a one-sided randomized algorithm which can err (with at most constant probability $< 1$) only on legal configurations but must detect a non-legal configuration with probability 1. This class of languages is discussed in Theorem 3.1. Regarding the analogous class of languages that can be decided by a one-sided randomized algorithm that must answer correctly on legal configurations, we obtain the following theorem.

**Theorem 3.5** Let $\mathcal{L} \in \text{BPLD}(1, q)$ for some $q \in (0, 1]$. Assume that there exists a randomized local algorithm deciding $\mathcal{L}$ that generates at most some constant number of random bits at each node. Then $\mathcal{L} \in \text{LD}$. 

**Proof.** Let $q \in (0, 1]$, and $\mathcal{L} \in \text{BPLD}(1, q)$. Let $A$ be a local randomized algorithm deciding $\mathcal{L}$ that generates at most some constant number $k$ of random bits at each node. Let $t$ be the running time of $A$. We define a local deterministic algorithm $D$ deciding $\mathcal{L}$ as follows. Let $(G, w)$ be a configuration, and let $\text{Id}$ be a given id-assignment. Every node $v$ collects all information from $B_G(v, t)$, the ball of radius $t$ centered at $v$ in $(G, w)$. Then $v$ individually simulates every possible execution of $A$, by generating all possible sequences of $k$ random bits that could be generated in the nodes of $B_G(v, t)$. More precisely, for each possible trial of $k |B_G(v, t)|$ random bits, $v$ reconstructs the actions of the nodes in $B_G(v, t)$. Specifically, for a vertex $w$ at distance $d \leq t$ from $v$, it suffices to reconstruct the actions of $w$ up to round $t - k$. Subsequently, $v$ decides as follows. If all the simulated executions result in the output “yes” at $v$, then $v$ outputs “yes”. Otherwise, $v$ outputs “no”. To show that $D$ is correct, note that if $(G, w) \in \mathcal{L}$, then for every possible trial, every node outputs “yes” in $A$, and it therefore does the same in $D$. If, on the other hand, $(G, w) \notin \mathcal{L}$, then since $q > 0$, there exists a trial in which at least one node outputs “no” in $A$. This node outputs “no” in $D$ as well. \(\square\)

## 4 Nondeterminism and complete problems

In this section, we focus on languages that belong to classes beyond LD. These classes are defined using oracles providing the nodes with information about the configuration in which they are operating. A typical example of an oracle is one providing the nodes with the size of the graph. This specific oracle is motivated by the numerous examples in the literature for which the knowledge of the size of the network is required to efficiently compute solutions of distributed computing problems (cf., e.g., [28, 30, 34]). As we show later, this oracle indeed gives immense power to the nodes, and is at the core of local distributed computational complexity.

Let us first define a notion of reduction that fits the class LD. For two languages $\mathcal{L}_1, \mathcal{L}_2$, we say that $\mathcal{L}_1$ is *locally reducible* to $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \preceq \mathcal{L}_2$, if there exists a local algorithm $A$ such that, for every configuration $(G, w)$ and every id-assignment $\text{Id}$, $A$ produces $\text{out}(v) \in \{0, 1\}^*$ as output...
at every node $v \in V(G)$ so that

$$(G, w) \in L_1 \iff (G, \text{out}) \in L_2.$$ 

By definition, LD is closed under local reductions, that is, for every two languages $L_1, L_2$ satisfying $L_1 \preceq L_2$, if $L_2 \in LD$ then $L_1 \in LD$. Using local reductions, we now define classes with oracles.

**Definition 4.1** Let $C$ be a complexity class, and let $L$ be a parametrized language. Then $C^L$ is the class of languages that are locally reducible to a language in $C$, assuming an oracle deciding $L$ is available at every node.

Hereafter, we focus on the oracle deciding the following language: $\#n = \{(G, n) \mid |V(G)| = n\}$. Providing the nodes with an oracle for $\#n$, is equivalent to informing them of the number of nodes in the network for which they belong. Indeed, each node can individually query the oracle with increasing values of $n$, starting from 1, until it gets a “yes” answer, yielding the actual size of the network. The significance of knowing the number of nodes in the context of decision problems is exemplified in the framework of nondeterministic local decision.

A distributed verification algorithm is a distributed algorithm $A$ that gets as input, in addition to a configuration $(G, w)$, a global certificate vector $c$, i.e., every node $v$ of a graph $G$ gets as input a binary string $w(v) \in \{0, 1\}^*$, and a certificate $c(v) \in \{0, 1\}^*$. A verification algorithm $A$ verifies $L$ if and only if for every configuration $(G, w)$, the following hold:

- If $(G, w) \in L$, then there exists a certificate $c$ such that for every id-assignment $\text{Id}$, algorithm $A$ applied on $(G, w)$ with certificate $c$ and id-assignment $\text{Id}$ outputs $\text{out}(v) = \text{"yes"}$ for all $v \in V(G)$;

- If $(G, w) \notin L$, then for every certificate $c$ and for every id-assignment $\text{Id}$, algorithm $A$ applied on $(G, w)$ with certificate $c$ and id-assignment $\text{Id}$ outputs $\text{out}(v) = \text{"no"}$ for at least one node $v \in V(G)$.

One motivation for studying the nondeterministic verification framework comes from settings in which one must perform local verifications repeatedly. In such cases, one can afford to have a relatively “wasteful” preliminary step in which a certificate is computed for each node. Using these certificates, local verifications can then be performed very fast. See [21] [22] for more details regarding such applications. Indeed, the definition of a verification algorithm finds similarities with the notion of proof-labeling schemes discussed in [21] [22]. Informally, in a proof-labeling scheme, the construction of a “good” certificate $c$ for a configuration $(G, w) \in L$ may depend also on the given id-assignment. Since the question of whether a configuration $(G, w)$ belongs to a language $L$ is independent from the particular id-assignment, we prefer to let the “good” certificate $c$ depend only on the configuration. In other words, as defined above, a verification algorithm operating on a configuration $(G, w) \in L$ and a “good” certificate $c$ must say “yes” at every node regardless of the id-assignment.
We now define the class NLD, for *nondeterministic local decision*. (our terminology is by direct analogy to the class NP in sequential computational complexity).

**Definition 4.2** NLD is the class of all distributed languages that can be verified by a local distributed algorithm.

### 4.1 Separation results

The following theorem illustrates the power gained by assuming that each node knows the total number of nodes.

**Theorem 4.3** For every language \( L \), we have \( L \in \text{NLD}^{\#n} \).

**Proof.** Let \( L \) be a language. The certificate of a configuration \((G, w) \in L\) is a map of \( G \), with nodes labeled with distinct integers in \( \{1, ..., n\} \), where \( n = |V(G)| \), together with the inputs of all nodes in \( G \). In addition, every node \( v \) receives the label \( \lambda(v) \) of the corresponding vertex in the map. Precisely, the certificate at node \( v \) is \( c(v) = (G', w', i) \) where \( G' \) is an isomorphic copy of \( G \) with nodes labeled from 1 to \( n \), \( w' \) is an \( n \)-dimensional vector such that \( w'[\lambda(u)] = w(u) \) for every node \( u \), and \( i = \lambda(v) \). The verification algorithm involves checking that the configuration \((G', w')\) is identical to \((G, w)\). This is sufficient because distributed languages are sequentially decidable, hence every node can individually decide whether \((G', w')\) belongs to \( L \) or not, once it has secured the fact that \((G', w')\) is the actual configuration. It remains to show that verifying that the configuration \((G', w')\) is identical to \((G, w)\) can be done locally, assuming that every node has access to an oracle deciding \( \#n \).

The verifying algorithm operates as follows. First, every node queries the oracle to get the number of nodes \( n \) of the actual configuration, and a node noticing \( n \neq |V(G')| \) outputs “no”. Second, every node \( v \) checks that it has received the input as specified by \( w' \), i.e., \( v \) checks whether \( w'[\lambda(v)] = w(v) \), and outputs “no” if this does not hold. Third, each node \( v \) communicates with its neighbors to check that (1) they all got the same map \( G' \) and the same input vector \( w' \), and (2) they are labeled the way they should be according to the map \( G' \). If some inconsistency is detected by a node, then this node outputs “no”. We claim that if all nodes pass these three phases without outputting “no”, then \((G', w')\) is identical to \((G, w)\). Indeed, if all nodes pass the three phases without outputting “no”, then they all agree on the map \( G' \) and on the input vector \( w' \), they know that the map has the right size, and they know that their respective neighborhood fits with what is indicated on the map. Let \( L_i \) be the \( n \)-dimensional boolean array such that \( L_i[j] = 1 \) if and only if node labeled \( i \) has a neighboring node labeled \( j \). Observe that all nodes have received pairwise distinct labels in their certificate, and that these labels are in \([1, n]\). Indeed, otherwise, by the fact that \( |V(G)| = n \), there would be at least one label not assigned to any node, and this fact would be detected during the third phase of the above algorithm. So \( L_i \) is defined for every \( i = 1, \ldots, n \).

By construction, the \( n \times n \) matrix \( M \) whose \( i \)th row is \( L_i \) is the adjacency matrix of both \( G \) and
Thus, $G$ and $G'$ are isomorphic. This completes the proof of Theorem 4.3.

**Theorem 4.4** $\text{LD} \subset \text{NLD}$.

**Proof.** To establish the theorem it is sufficient to show that there exists a language $\mathcal{L}$ such that $\mathcal{L} \notin \text{LD}$ and $\mathcal{L} \in \text{NLD}$. Let $\text{tree} = \{(G, \epsilon) \mid G \text{ is a tree}\}$. We have $\text{tree} \notin \text{LD}$. To see why, consider a cycle $C$ with nodes labeled consecutively from 1 to $4n$, and the path $P_1$ (resp., $P_2$) with nodes labeled consecutively $1, \ldots, 4n$ (resp., $2n + 1, \ldots, 4n, 1, \ldots, 2n$), from one extremity to the other. For any algorithm $A$ deciding $\text{tree}$, all nodes $n + 1, \ldots, 3n$ output “yes” in configuration $(P_1, \epsilon)$ for any identity assignment for the nodes in $P_1$, while all nodes $3n + 1, \ldots, 4n, 1, \ldots, n$ output “yes” in configuration $(P_2, \epsilon)$ for any identity assignment or the nodes in $P_2$. Thus if $A$ is local, then all nodes output “yes” in configuration $(C, \epsilon)$, a contradiction. In contrast, we next show that $\text{tree} \in \text{NLD}$. The (nondeterministic) local algorithm $A$ verifying $\text{tree}$ operates as follows. Given a configuration $(G, \epsilon)$, the certificate given at node $v$ is $c(v) = \text{dist}(v, r)$ where $r \in V(G)$ is an arbitrary fixed node. The verification procedure is then as follows. At each node $v$, $A$ inspects every neighbor (with its certificates), and verifies the following:

- $c(v)$ is a non-negative integer,
- if $c(v) = 0$, then $c(w) = 1$ for every neighbor $w$ of $v$, and
- if $c(v) > 0$, then there exists a neighbor $w$ of $v$ such that $c(w) = c(v) - 1$, and, for all other neighbors $w'$ of $v$, we have $c(w') = c(v) + 1$.

If $G$ is a tree, then applying Algorithm $A$ on $G$ with the certificate yields the answer “yes” at all nodes regardless of the given id-assignment. On the other hand, if $G$ is not a tree, then we claim that for every certificate, and every id-assignment $\text{Id}$, Algorithm $A$ outputs “no” at some node. Indeed, consider some certificate $c$ given to the nodes of $G$, and let $C$ be a simple cycle in $G$. Assume, for the sake of contradiction, that all nodes in $C$ output “yes”. In this case, each node in $C$ has at least one neighbor in $C$ with a larger certificate. This creates an infinite sequence of strictly increasing certificates, in contradiction with the finiteness of $C$.

**Theorem 4.5** $\text{NLD} \subset \text{NLD}^\#n$.

**Proof.** Let $\text{InpEqSize} = \{(G, w) \mid \forall v \in V(G), w(v) = |V(G)|\}$. By Theorem 4.3, $\text{InpEqSize} \in \text{NLD}^\#n$. We show that $\text{InpEqSize} \notin \text{NLD}$. Assume, for the sake of contradiction, that there exists a local nondeterministic algorithm $A$ deciding $\text{InpEqSize}$. Let $t$ be the running time of $A$. Consider the cycle $C$ with $2t + 1$ nodes $u_1, u_2, \ldots, u_{2t+1}$, enumerated clockwise. Assume that the input at each node $u_i$ of $C$ satisfies $w(u_i) = 2t + 1$. Then, there exists a certificate $c$ such that, for any identity assignment $\text{Id}$, algorithm $A$ outputs “yes” at each node of $C$. Now, consider the configuration $(C', w')$ where the cycle $C'$ has $4t+2$ nodes, and for each node $v_i$ of $C'$, $w'(v_i) = 2t + 1$. 

---

15
We have \((C', w') \notin \text{InpEqSize}\). To fool Algorithm \(A\), we enumerate the nodes in \(C'\) clockwise, i.e., \(C = (v_1, v_2, \ldots, v_{2t+2})\). We then define the certificate \(c'\) as follows:

\[
c'(v_i) = c'(v_{i+2t+1}) = c(u_i) \quad \text{for } i = 1, 2, \ldots, 2t + 1.
\]

Fix an id-assignment \(\text{Id}'\) for the nodes in \(V(C')\), and fix \(i \in \{1, 2, \ldots, 2t + 1\}\). There exists an id-assignment \(\text{Id}_1\) for the nodes in \(V(C)\), such that the output of \(A\) at node \(v_i\) in \((C', w')\) with certificate \(c'\) and id-assignment \(\text{Id}'\) is identical to the output of \(A\) at node \(u_i\) in \((C, w)\) with certificate \(c\) and id-assignment \(\text{Id}_1\). Similarly, there exists an id-assignment \(\text{Id}_2\) for the nodes in \(V(C)\) such that the output of \(A\) at node \(v_{i+2t+1}\) in \((C', w')\) with certificate \(c'\) and id-assignment \(\text{Id}'\) is identical to the output of \(A\) at node \(u_i\) in \((C, w)\) with certificate \(c\) and id-assignment \(\text{Id}_2\). Thus, Algorithm \(A\) at both \(v_i\) and \(v_{i+2t+1}\) outputs “yes” in \((C', w')\) with certificate \(c'\) and id-assignment \(\text{Id}'\). Hence, since \(i\) was arbitrary, all nodes output “yes” for this configuration, certificate and id-assignment, contradicting the fact that \((C', w') \notin \text{InpEqSize}\). \(\square\)

### 4.2 Completeness results

We now prove the existence of a complete problem for NLD called \(\text{Map Cover (MC)}\). In \(\text{MC}\), every node \(v\) is given as input a pair \(w(v) = ((L_v, w_v), S_v)\). The first element of the pair, \((L_v, w_v)\), is called a lifted configuration. Every lifted configuration is centered, that is, one identifies a specific node \(c_v \in V(L_v)\), called the center of \(L_v\), for every \(v \in V(G)\). The second element of the pair, \(S_v\), is a finite collection of configurations. A configuration \((G, w)\) is in \(\text{MC}\) if and only if there exist a node \(v^* \in V(G)\), a configuration \((G^*, w^*) \in S_{v^*}\), and a collection of maps \(\{\pi_v \mid V(L_v) \rightarrow V(G^*), v \in V(G)\}\), such that the following two properties hold:

1. For each \(v \in V(G)\), \(\pi_v\) is an isomorphism between \(L_v\) and \(B_{G^*}(\pi_v(c_v), r)\) preserving the respective inputs \(w_v\) and \(w^*\), for some integer \(r > 0\);
2. The map \(\pi : V(G) \rightarrow V(G^*)\) defined by \(\pi(v) = \pi_v(c_v)\) is an isomorphism between \(B_G(v, 1)\) and \(B_{G^*}(\pi(v), 1)\) for each \(v \in V(G)\).

**Theorem 4.6** \(\text{MC}\) is NLD-complete.

**Proof.** We now show that \(\text{MC} \in \text{NLD}\). For this purpose, we design a nondeterministic local algorithm \(A\) that decides whether a configuration \((G, w)\) is in \(\text{MC}\). Such an algorithm \(A\) is designed to operate on \((G, w, c)\), where \(c\) is a certificate. The configuration \((G, w)\) satisfies that \(w(v) = ((L_v, w_v), S_v)\) for every node \(v\), with \(c_v\) the center of \(L_v\). Algorithm \(A\) aims at verifying whether there exists a node \(v^*\) with a configuration \((G^*, w^*) \in S_{v^*}\) for which there exists a collection of maps \(\{\pi_v\}\) satisfying the specification of \(\text{MC}\). The certificate \(c(v)\) at a node \(v\) is interpreted as a pair of two sub-certificates. The first sub-certificate is a lifted configuration \((L'_v, w'_v)\) centered at some specified node \(c'_v \in V(L'_v)\), together with an identity assignment \(\text{Id}'_v\) for the nodes in \(V(L'_v)\).
The second sub-certificate of $c(v)$ is an identity assignment $\text{Id}_v''$ for the nodes in $V(L_v)$. Algorithm $A$ operates in three phases, as follows.

**Phase 1:** Algorithm $A$ verifies that all nodes agree on their first sub-certificate. If some inconsistency is detected between neighbors, $A$ outputs “no”. Therefore, we can assume from now on that the first sub-certificate of all nodes is the same lifted configuration $(L',w')$ centered at some specified node $c' \in V(L')$ together with a specific identity assignment $\text{Id}'$ for the nodes of $L'$. In addition, each node verifies that $L'$ is connected, and outputs “no” if it is not the case. So we can assume from now on that $L'$ is connected. Informally, the common lifted configuration $(L',w')$ is the candidate for being the desired configuration $(G^*,w^*)$ in the specification of $\text{MC}$. We thus address $(L',w')$ as the candidate configuration.

**Phase 2:** Algorithm $A$ inspects the input of each node $v$ individually, and verifies that the following three conditions hold.

1. the id-assignment $\text{Id}_v''$ is a sub-id-assignment of $\text{Id}'$ (i.e., every identity in $\text{Id}_v''$ is present in $\text{Id}'$), and thus we can view $\text{Id}_v''$ as a mapping $\pi_v : V(L_v) \rightarrow V(L')$ that maps every node in $L_v''$ to the node in $L$ with the same identity;

2. $\pi_v$ induces an isomorphism between $L_v$ and $B_{L'}(\pi_v(c_v),r)$, for some $r > 0$, preserving the inputs $w_v$ and $w^*$;

3. $(L',w') \in S_v$ whenever $\pi_v(c_v) = c'$.

If any of the above three conditions is not satisfied, then Algorithm $A$ outputs “no” at $v$. Otherwise, $v$ proceeds to the next phase.

**Phase 3:** Define $\pi : V(G) \rightarrow V(L')$ by $\pi(u) = \pi_u(c_u)$ for every $u \in V(G)$. Algorithm $A$ now verifies that $\pi$ is an isomorphism between $B_G(u,1)$ and $B_{L'}(\pi(u),1)$. Again, if this is not the case, Algorithm $A$ outputs “no” at $v$.

If a node $v$ passes all three phases without outputting “no”, then it finally outputs “yes”.

This completes the description of the verification algorithm $A$. Observe that Algorithm $A$ runs in a single round. We prove that $A$ is correct.

We first prove that if $(G,w) \notin \text{MC}$, then there are no certificates $c$ for which $A$ outputs “yes” at all node. For this purpose, we show that if there exists a certificate $c$ such that all nodes output “yes” when running Algorithm $A$ on $(G,w,c)$ then necessarily $(G,w) \in \text{MC}$. By following the instructions of Algorithm $A$, if follows that all nodes agree on a candidate configuration $(L',w')$, and that the collection of maps $\{\pi_u\}$ defined in Phase 2 are as prescribed in the definition of $\text{MC}$.

It remains to show that there exists a node $v' \in V(G)$, such that $(L',w') \in S_{v'}$. We claim that, for every node $w \in V(L')$, there exists a node $v \in V(G)$ such that $\pi(v) = w$. This follows from
Phase 3 and from the fact that, by Phase 1, $G'$ is connected. To see why, assume, for the purpose of contradiction, that there exists $w \in V(L')$ such that there is no $v \in V(G)$ satisfying $\pi(v) = w$. Then, fix a node $u \in V(G)$, and choose $w$ minimizing its distance to $\pi(u)$. Since $L'$ is connected, there exists a node $w'$ which is a neighbor of $w$ in $L'$ closer to $\pi(u)$. There exists a node $v' \in V(G)$ satisfying $\pi(v') = w'$. This contradicts the fact that $v$ passes Phase 3 without outputting “no”. Thus for every node $w \in V(L')$, there exists a node $v \in V(G)$ such that $\pi(v) = w$, as claimed. It follows that there exists a node $v \in V(G)$ such that $\pi_v(c_v) = \pi(v) = c'$. Since $v$ passes the third test in Phase 2 without outputting “no”, it follows that $(L', w') \in S_v$. This guarantees that $(G, w) \in MC$, as required.

We finally show that if $(G, w) \in MC$ then there exists a certificate $c$ such that, for any identity assignment, Algorithm $A$ running on $(G, w, c)$ insures that all nodes output “yes”. Let $(G, w) \in MC$, then, by definition, there exists a node $u^* \in V(G)$ with a configuration $(G^*, w^*) \in S_{u^*}$ for which the maps $\{\pi_u\}$ are as prescribed in the definition of $MC$. The first sub-certificate at each node is simply a copy of the configuration $(G^*, w^*)$ together with the specified node $\pi(c_{u^*})$ and some identity assignment $Id'$ for the nodes in $V(G^*)$. The first sub-certificate is thus the same at each node. The second sub-certificate at each node $v$ is the identity assignment $Id''_v$ for the nodes in $V(L_v)$ defined by the identity assignment $Id'$, and the image $\pi_v(u) \in V(G^*)$, for $u \in V(L_v)$. By following the instructions of Algorithm $A$, we get that, for any identity assignment to the nodes in $V(G)$, Algorithm $A$ operating on $(G, w, c)$ outputs “yes” at each node $v$. This completes the proof that $MC \leq NLD$.

We now show that $MC$ is NLD-hard. Let $L \in NLD$. We show that $L \leq MC$. For this purpose, we describe a local distributed algorithm $D$ transforming any configuration for $L$ into a configuration for $MC$, preserving the memberships to these languages. Let $t \geq 0$ be some integer such that there exists a nondeterministic algorithm $A_L$ deciding $L$ in time $t$. Let $(G, w)$ be a configuration for $L$, and let $Id$ be an identity assignment. Algorithm $D$ operating at a node $v \in V(G)$ outputs a pair $((L_v, w_v), S_v)$, where $L_v = B_G(v, t)$, and $w_v$ is the input $w$ restricted to $B_G(v, t)$. The set $S_v$ is the collection of configurations defined as follows. For a binary string $x$, let $|x|$ denote the length of $x$, i.e., the number of bits in $x$. For every $v \in V(G)$, let $\psi(v) = 2^{|Id(v)|+|w(v)|}$. Node $v$ generates all configurations $(G', w') \in L$ where $G'$ is a graph with $k \leq \psi(v)$ vertices, and $w'$ is a collection of $k$ input strings of length at most $\psi(v)$. We show that $(G, w) \in L$ if and only if $D(G, w) \in MC$.

First, consider the case $(G, w) \in L$, and let $v^*$ be a node maximizing $\psi(v)$. We have $\psi(v^*) \geq \max\{|Id(u) | u \in V(G)\} \geq n$ where $n = |V(G)|$, and $\psi(v^*) \geq \max\{|w(u) | u \in V(G)\}$. Therefore, node $v^*$ has constructed the original configuration $(G, w)$, and thus $(G, w) \in S_{v^*}$. Using the identity map, it follows that there exists a map $\pi_u : V(L_u) \to V(G^*)$ for every $u \in V(G)$ satisfying the required properties of (P1) and (P2). This guarantees that $D(G, w) \in MC$, as desired.

Conversely, consider the case $(G, w) \notin L$. Assume, for the purpose of contradiction, that $D(G, w) \in MC$. This means that there exists a node $v^* \in V(G)$, a configuration $(G^*, w^*) \in S_{v^*}$, and
a collection of maps \( \{ \pi_v \} \) as specified in the definition of \( \mathcal{MC} \). By construction, the transformation \( \mathcal{D} \) guarantees that \( S_{x^*} \) contains only configurations that belong to \( \mathcal{L} \). Thus, in particular, \((G^*, w^*) \in \mathcal{L} \). Therefore, there exists a certificate \( c^* \) for this configuration such that, for any identity assignment \( \text{Id}^* \), Algorithm \( A_L \) applied on \((G^*, w^*, c^*) \) with id-assignment \( \text{Id}^* \) outputs “yes” at each node of \( G^* \). We now define a certificate \( c \) for the configuration \((G, w) \) as follows: for every \( v \in V(G) \), we set \( c(v) = c^*(\pi(v)) \). Fix a node \( v \in V(G) \). A crucial observation is that:

\begin{itemize}
  \item \((*) \) the mapping \( \pi \), restricted to \( V(B_G(v, t)) \), induces an isomorphism between \( B_G(v, t) \) and \( B_{G^*}(\pi(v), t) \).
\end{itemize}

Note that \((P1) \) guarantees that there exists an isomorphism between \( B_G(v, t) \) and \( B_{G^*}(\pi(v), t) \) because, in our case, \( L_v = B_G(v, t) \). However, it is not obvious that \( \pi \) is such an isomorphism. We prove \((*) \) by induction on the radius \( s \) of the ball. Specifically, we prove that for any integer \( s \), \( 0 \leq s \leq t \), we have that \( \pi \) restricted to \( V(B_G(v, s)) \) induces an isomorphism between \( B_G(v, s) \) and \( B_{G^*}(\pi(v), s) \). The basis of the induction, the case \( s = 0 \), is trivial. Assume that the claim holds for \( s \), \( 0 \leq s < t \). For integer \( i \), let \( S_i \) (respectively, \( S^*_i \)) denote the sphere defined as the set of nodes at distance \( i \) from \( v \) in \( G \) (resp., from \( \pi(v) \) in \( G^* \)). Consider a node \( x^* \in S^*_{s+1} \) and let \( y^* \) be a node in \( S^*_{s} \) such that \( \{ x^*, y^* \} \in E(G^*) \). By the induction hypothesis, there exists a node \( y \in S_{s} \) such that \( \pi(y) = y^* \). By \((P2) \), there exists \( x \in V(G) \), such that \( \{ x, y \} \in E(G) \) and \( \pi(x) = x^* \). By the induction hypothesis, \( x \notin V(B_G(v, s)) \). It follows that \( x \in S_{s+1} \). Thus, \( \pi \) restricted to \( S_{s+1} \) covers the set \( S^*_{s+1} \). Now, since \( B_G(v, t) \) and \( B_{G^*}(\pi(v), t) \) are isomorphic, we have that \( |S_{s+1}| = |S^*_{s+1}| \). Thus, \( \pi \) restricted to \( S_{s+1} \) is a bijection between this sphere and the sphere \( S^*_{s+1} \). Now, by \((P2) \), for every two nodes \( z, z' \in V(B_G(v, s+1)) \) such that \( \{ z, z' \} \in E(G) \), we have \( \{ \pi(z), \pi(z') \} \in E(G^*) \). Again, by the fact that \( B_G(v, t) \) and \( B_{G^*}(\pi(v), t) \) are isomorphic, we have that \( |E(B_G(v, s+1))| = |E(B_{G^*}(\pi(v), s+1))| \). Thus, we have \( \{ z, z' \} \in E(B_G(v, s+1)) \) if and only if \( \{ \pi(z), \pi(z') \} \in E(B_{G^*}(\pi(v), s+1)) \). Consequently, \( \pi \) induces an isomorphism between \( B_G(v, s+1) \) and \( B_{G^*}(\pi(v), s+1) \), which concludes the induction step as well as the proof of \((*) \).

Now that \((*) \) is established, we Next, observe that, by \((P1) \), \( w(u) = w^*(\pi(u)) \) for every \( u \in B_G(v, t) \). It follows that \( \pi \) is actually an isomorphism between the balls \( B_G(v, t) \) and \( B_{G^*}(\pi(v), t) \) preserving the inputs \( w \) and \( w^* \). Furthermore, since the certificate of each node \( u \in V(B_G(v, t)) \) is \( c(u) = c^*(\pi(u)) \), it follows that \( \pi \) also preserves the certificates in these balls. Consider an identity assignment \( \text{Id} \) for the nodes in \( V(G) \). We define an identity assignment \( \text{Id}^* \) for \( V(G^*) \) as follows. We first give identities to the nodes in \( V(B_{G^*}(\pi(v), t)) \) using \( \pi \) and the identity assignment \( \text{Id} \) restricted to the nodes in \( V(B_G(v, t)) \), i.e., for every \( x^* \in V(B_{G^*}(\pi(v), t)) \), we set \( \text{Id}^*(x^*) = \text{Id}(\pi^{-1}(x^*)) \). We then extend \( \text{Id}^* \) to all the nodes in \( V(G^*) \) in an arbitrary manner. Algorithm \( A_L \) when executed at \( v \) in the configuration \((G, w) \) with certificate \( c \) and id-assignment \( \text{Id} \) behaves identically to how it behaves when executed at \( \pi(v) \) in the configuration \((G^*, w^*) \) with certificate \( c^* \) and id-assignment \( \text{Id}^* \). Now, we know that Algorithm \( A_L \) executed in configuration \((G^*, w^*) \) outputs “yes” at every node when using the certificate \( c^* \) and the id-assignment \( \text{Id}^* \). It follows that Algorithm \( A_L \) executed in configuration \((G, w) \) with certificate \( c \) and id-assignment \( \text{Id} \) outputs “yes” at node \( v \). Since \( v \)
has been fixed as an arbitrary node in \( V(G) \), we get that Algorithm \( A_{\mathcal{L}} \) executed on \((G, w)\) with certificate \( c \) and id-assignment \( \text{Id} \) outputs “yes” at every node. This contradicts the fact that \((G, w) \notin \mathcal{L}\), and completes the NLD-hardness part of the proof.

We show that \( \text{NLD}^{\#n} \) has a natural complete language, called \( \text{Unique Set Cover} (\text{USC}) \), defined as follows. Every node \( v \) is given as input an element \( \mathcal{E}(v) \), and a finite collection of sets \( S(v) \). The union of these inputs is in the language if there exists a node \( v \) such that one set in \( S(v) \) equals the union of all the elements given to the nodes. Formally, we define \( \text{USC} = \{(G, (\mathcal{E}, S)) \mid \exists v \in V(G), \exists S \in S(v) \text{ s.t. } S = \{ \mathcal{E}(v) \mid v \in V(G) \}\}. \)

**Theorem 4.7** \( \text{USC} \) is \( \text{NLD}^{\#n} \)-complete.

**Proof.** The fact that \( \text{USC} \in \text{NLD}^{\#n} \) follows from Theorem 4.3. To prove that \( \text{USC} \) is \( \text{NLD}^{\#n} \)-hard, we consider some \( \mathcal{L} \in \text{NLD}^{\#n} \) and show that \( \mathcal{L} \preceq \text{USC} \). For this purpose, we describe a local distributed algorithm \( A \) transforming any configuration for \( \mathcal{L} \) to a configuration for \( \text{USC} \) preserving the memberships to these languages. Let \((G, w)\) be a configuration for \( \mathcal{L} \) and let \( \text{Id} \) be an identity assignment. Algorithm \( A \) operating at a node \( v \) outputs a pair \((\mathcal{E}(v), S(v))\), where \( \mathcal{E}(v) \) is the “local view” at \( v \) in \((G, w)\), i.e., the star subgraph of \( G \) consisting of \( v \) and its neighbors, together with the inputs of these nodes and their identities, and \( S(v) \) is the collection of sets \( S \) defined as follows. For a binary string \( x \), let \(|x|\) denote the length of \( x \), i.e., the number of bits in \( x \). For every vertex \( v \), let \( \psi(v) = 2^{|\text{Id}(v)| + |w(v)|} \). Node \( v \) first generates all configurations \((G', w')\) where \( G' \) is a graph with \( k \leq \psi(v) \) vertices, and \( w' \) is a collection of \( k \) input strings of length at most \( \psi(v) \), such that \((G', w') \in \mathcal{L} \). For each such configuration \((G', w')\), node \( v \) generates all possible \( \text{Id}' \) assignments to \( V(G') \) such that for every node \( u \in V(G') \), \(|\text{Id}(u)| \leq \psi(v) \). Now, for each such pair of a graph \((G', w')\) and an \( \text{Id}' \) assignment, algorithm \( A \) associates a set \( S \in S(v) \) consisting of the \( k = |V(G')| \) local views of the nodes of \( G' \) in \((G', w')\). We show that \((G, w) \in \mathcal{L} \iff A(G, w) \in \text{USC}) \).

If \((G, w) \in \mathcal{L} \), then by the construction of Algorithm \( A \), there exists a set \( S \in S(v) \) such that \( S \) covers the collection of local views for \((G, w)\), i.e., \( S = \{ \mathcal{E}(u) \mid u \in G \} \). Indeed, the node \( v \) maximizing \( \psi(v) \) satisfies \( \psi(v) \geq \max\{|\text{Id}(u)| \mid u \in V(G)\} \geq n \) and \( \psi(v) \geq \max\{|w(u)| \mid u \in V(G)\} \). Therefore, that specific node has constructed a set \( S \) which contains all local views of the given configuration \((G, w)\) and Id assignment. Thus \((G, w) \in \text{USC}) \).

Now consider the case that \( A(G, w) \in \text{USC} \). In this case, there exists a node \( v \) and a set \( S \in S(v) \) such that \( S = \{ \mathcal{E}(u) \mid u \in G \} \). Such a set \( S \) is the collection of local views of nodes of some configuration \((G', w') \in \mathcal{L} \) and some \( \text{Id}' \) assignment. On the other hand, \( S \) is also the collection of local views of nodes of the given configuration \((G, w) \in \mathcal{L} \) and Id assignment. It follows that \((G, w) = (G', w') \in \mathcal{L} \).
Figure 1: Construction in the proof of Theorem 3.1.
References


[39] [http://mathworld.wolfram.com/UlamsConjecture.html]