The arbitrage pricing theorem with incomplete preferences☆

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Abstract

This paper proves existence of equilibrium and the arbitrage pricing theorem for an asset exchange economy, where individuals’ preferences may be incomplete or intransitive. This extends existing results to more general preferences. We also prove the arbitrage pricing theorem for a theory of choice under uncertainty by Bewley [Bewley, T. F. (2002), Knightian decision theory: part I, Decisions in Economics and Finance 25, 79–110]. These preferences model Knightian uncertainty by preferences which may be incomplete but satisfy independence.

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1. Introduction

1.1. Background

The arbitrage pricing theorem (APT) is a central result in finance. It underlies most modern asset pricing theories such as CAPM and the Modigliani–Miller theorem in corporate finance. Derivative securities are priced using an inter-temporal version of arbitrage pricing. This paper proves a version of the APT for incomplete or intransitive preferences.

In recent years there has been much research in behavioural finance, see for instance Barberis and Thaler (2002). A key issue has been whether psychological factors such as regret or ambiguity-aversion affect behaviour in financial markets. Such biases may well give rise to preferences, which are either incomplete or intransitive.

The study of incomplete preferences may be motivated by arguing that is not evident whether completeness is a fundamental rationality principle in the same way as transitivity. Aumann (1962) defended this position strongly from both normative and positive standpoints. He suggested that a preference relation should be considered as a potentially incomplete pre-order, therefore allowing for the occasional “indecisiveness” of the individual.

Many agents in asset markets are not single investors but rather corporate bodies. Hence, investment decisions are often collective decisions. If markets were complete, then all group members would have the same preferences over investments. If markets are incomplete, then it is not possible to evaluate market values of all feasible investment decisions from the available price system. We study economies where markets are possibly incomplete, see Arrow (1963). In such economies, typically investors will disagree over the choice of investment plans, even if the competitive conditions prevail (see Duffie and Shafer, 1985; Haller, 1991). Thus firms’ decisions will be the outcome of a collective choice process. Social choice theory implies that the firm’s preference will be incomplete or intransitive, if it is non-dictatorial, see Sen (1970).

Another reason why preferences may be intransitive is regret. If a decision-maker suffers a utility loss when the course of action (s)he chose yielded a lower pay-off than a possible alternative course of action, (s)he is said to be influenced by regret. Regret is likely to lead to preferences which are intransitive, see Loomes and Sugden (1987).

We analyse the implications of incomplete or intransitive preferences for an asset exchange economy, in which unrestricted short selling is allowed and hence the portfolio space is not necessarily bounded below (see Milne, 1976; Page and Wooders, 1996; Werner, 1987, 1989). Our analysis allows both incompleteness and unboundedness at the same time. We show that any equilibrium must be in the constrained core and hence is a constrained Pareto optimum. This establishes a version of the first fundamental theorem of welfare economics. The value added of the present paper consists in bringing together the previously diverse literatures on arbitrage pricing, incomplete markets and general equilibrium with incomplete or intransitive preferences.

1.2. Bewley preferences

Knightian uncertainty can give rise to incomplete preferences. Knight (1921) made a distinction between risk, where the probabilities are known, and ambiguity, where probabilities are not known. This distinction does seem to have some intuitive appeal since, in many economic decisions, it is difficult or impossible to assign subjective probabilities. Experimental evidence shows that individuals behave differently when probabilities are unknown, see Camerer and Weber (1992).
Some models of Knightian uncertainty (also known as ambiguity) relax the independence axiom see, for instance, Schmeidler (1989). However, even with Knightian uncertainty, this axiom may be motivated by observing that consumption in mutually exclusive events cannot be complementary. If an individual is faced with a complex and unfamiliar kind of uncertainty she may well find some decisions difficult or impossible. Thus, there is a case for modelling Knightian uncertainty by incompleteness rather than modifying the independence axiom.

A model of Knightian uncertainty based on these principles has been proposed by Bewley (2002). In this theory, the decision-maker considers a number of probability distributions to be possible and adopts a cautious decision rule. An option $a$ is only preferred to $b$ if $a$ yields higher expected utility with respect to all possible probability distributions. If $a$ yields higher expected utility for some distributions and lower expected utility for others, then $a$ and $b$ are not comparable. Ambiguity is captured by the incompleteness of preferences.

1.3. Arbitrage pricing

The APT (see Ross, 1976) is based upon three assumptions: the market does not permit arbitrage opportunities, asset returns can be modelled by a factor structure and it is possible to eliminate risk by holding a diversified portfolio. Therefore, the basic result of APT is that under these assumptions, idiosyncratic risk can be diversified away and hence asset prices can be expressed in terms of the prices of a small number of factors.

After Ross’ seminal paper, there have been a number of studies characterising the notion of the equilibrium arbitrage pricing theorem (EAPT). With either a finite or infinite number of assets, Connor (1984) presented a general theory of equilibrium asset pricing unifying several models. Using weak assumptions on induced preferences over assets, Milne (1988) proved an equilibrium APT for a finite number of assets. This was extended by Kelsey and Milne (1995) to most of the leading non-expected utility theories such as Gateaux differentiable preferences (see Machina, 1982), rank dependent expected utility (see Quiggin, 1982), and Choquet expected utility (see Schmeidler, 1989). They also extended the APT to a larger class of expected utility preferences, namely, the APT was generalised to allow for state dependent utility and the possibility that different investors might have different subjective probabilities.

The present paper extends the APT in a different direction. Namely, we allow for preferences to be incomplete or intransitive. The assumption that preferences are completely ordered has been questioned in the previous literature (e.g., Aumann, 1962; Suppes, 1974). This is reasonable when individuals have only partial information or there are many decision-makers whose preferences may disagree.

1.4. Organisation of the paper

In the following section, we present our model of an asset exchange economy. Then we derive a representation for an incomplete preference relation. This section also proves the existence of a competitive equilibrium for a class of asset exchange economies and shows that the equilibrium is in the constrained core. In Section 3, we derive the APT for the underlying preference relation. In Section 4, we shall extend the earlier result to include Bewley preferences. Some related literature is discussed in Section 5 and the concluding section discusses some implications of these results.
2. The asset exchange economy

2.1. Framework

We consider a competitive two-period asset exchange economy. The set of individuals is denoted by \( H = \{1, \ldots, H\} \), \((H \geq 1)\). There are two time periods. In the first, there is no uncertainty. Agents trade assets, which pay-off in the second period. Uncertainty is described by \( S \) possible states, one of which is realized at the beginning of the second period. Let \( \Omega = \{\omega_1, \ldots, \omega_S\} \) be the set of possible states of the world. We shall assume that there is only one physical commodity. Hence the first period commodity space is \( \mathbb{R} \) and the second period contingent commodity space is \( \mathbb{R}^{S+1} \), making the total commodity space \( \mathbb{R}^{S+1} \). Second period consumption is restricted to that which can be achieved by trades in assets.

Each consumer \( h \in H \) is characterised by a consumption set \( X^h \), an initial endowment \( e^h \in \mathbb{R}^{S+1} \), and a preference relation \( \succ^h \) defined over \( X^h \).

**Assumption 2.1.** For every \( h \in H \), the feasible set, \( X^h \) is non-empty, closed, convex, and bounded below.

2.2. Preferences

The subject matter of this section is the representation of preferences which may be incomplete or intransitive. One reason why it is useful to have a numerical representation of preferences is that it allows us to use calculus to find optima. Consider an individual who has a strict preference relation \( \succ \) on a set \( X \). If preferences are represented by a conventional single argument utility function, then it is easy to show that they must be complete and transitive. Thus no such representation exists for preferences which are incomplete or intransitive. However \( \succ \) can be represented by a two-argument function \( \phi : X \times X \rightarrow \mathbb{R} \) for which \( x \succ y \) if and only if \( \phi(x, y) > 0 \) for all \( x, y \in X \).

Given a strict preference relation \( \succ^h \) defined on \( X^h \times X^h \), let \( P^h(x^h) = \{y \in X^h : y \succ^h x^h\} \) and \( P^{-1}_h(x^h) = \{y \in X^h : x^h \succ^h y\} \) be the strict upper and lower contour sets, respectively.

**Assumption 2.2.** We make the following assumptions on preferences\(^2\):

1. Continuity. The strict upper and lower contour sets are open subsets of \( X^h \), \( \forall x^h \in X^h \);
2. Irreflexivity. \( x^h \notin P^h(x^h) \), \( \forall x^h \in X^h \);
3. Convexity. \( x^h \in \text{con}(P^h(x^h)) \), \( \forall x^h \in X^h \), where \( \text{con}(P) \) denotes convex hull of the set \( P \);
4. Non-satiation. For each \( \forall x^h \in X^h \), \( P^h(x^h) \neq \emptyset \).

**Definition 2.1.** Let \( \succ^h \) be a preference relation defined on \( X^h \), then \( \succ^h \) has an open graph if \( \Gamma(\succ^h) \) is an open subset of \( X^h \times X^h \), where the graph of \( \succ^h \) is given by

\[
\Gamma(\succ^h) = \{(x, y) \in X^h \times X^h : y \succ^h x\}.
\]

Our first result is due to Bergstrom et al. (1976), Theorem 4, and is stated without proof.

\(^2\)Notice that we do not use transitivity or strong convexity assumptions. However, a mild convexity assumption is imposed on strict upper contour sets (see Mas-Colell, 1974).
Proposition 2.1. If the preference relation $\succ^h$ is asymmetric, then it has open graph if and only if there exists a real continuous function $\phi : X^h \times X^h \to \mathbb{R}$ such that $\phi(x, y) > 0$ if and only if $x \succ^h y$.

2.3. Induced preferences

Agents do not trade goods directly. Instead goods are exchanged by trading financial securities. It is therefore more convenient to work with induced preferences over assets.

We shall define the commodity space in the asset economy to be the space $\mathbb{R}^{J+1}$, that is, $J$ assets and the first period commodity (asset 0), indexed by $j \in J = \{1, \ldots, J\}$. In order to obtain consumption, consumer $h$ holds assets $a^h \in \mathbb{R}^J$ which generate returns $\sum_{j \in J} Z_j a^h_j$, where $Z_j \in \mathbb{R}^S$ is the asset return matrix i.e.

$$
\begin{bmatrix}
Z_{11} & \ldots & Z_{1J} \\
\vdots & \ddots & \vdots \\
Z_{S1} & \ldots & Z_{SJ}
\end{bmatrix}
\begin{bmatrix}
a^h_1 \\
\vdots \\
a^h_J
\end{bmatrix} =
\begin{bmatrix}
\sum_{j \in J} Z_{1j} a^h_j \\
\vdots \\
\sum_{j \in J} Z_{Sj} a^h_j
\end{bmatrix}.
$$

In order to derive consumer preferences over assets, define a function $K : \mathbb{R}_+ \times \mathbb{R}^J \to \mathbb{R}_+ \times \mathbb{R}^S$ by $\zeta \beta = \alpha$, where $\beta \in \mathbb{R}^{J+1}$, $\alpha \in \mathbb{R}^{S+1}$, and $\zeta$ is the $(S+1) (J+1)$ semi-positive matrix, that is,

$$
\begin{bmatrix}
1 & 0 \\
0 & Z
\end{bmatrix}.
$$

The correspondence $A : \mathbb{R}_+ \times \mathbb{R}^J \to \mathbb{R}_+ \times \mathbb{R}^S$ by $\zeta \beta = \alpha$, where $\beta \in \mathbb{R}^{J+1}$, $\alpha \in \mathbb{R}^{S+1}$, and $\zeta$ is the $(S+1) (J+1)$ semi-positive matrix, that is,

$$
\begin{bmatrix}
1 & 0 \\
0 & Z
\end{bmatrix}.
$$

The correspondence $A$ is linear and onto the range $Q$, which is a vector subspace of $\mathbb{R}^{S+1}$. Define consumer $h$’s feasible asset trade set $A^h$ by $A^h = A^{-1}(Q \cap X^h)$. Let $A = x_{h \in H} A^h$. We may define induced preferences over assets as follows.

Definition 2.2. The set of assets preferred to $\tilde{a}^h$ is defined by

$$
\tilde{P}_h(\tilde{a}^h) = \{ a^h : Z a^h \in P_h(Z \tilde{a}^h) \}.
$$

Assets are desired solely for their returns in the second period. The connection between basic preferences defined over commodity space and induced preferences defined over portfolio space can be summarised in the following result due to Milne (1976), in Lemma 2.1. The original result was stated for the weak preference relation, while we work with strict preferences. However since the changes are minor, the proof is omitted.

Lemma 2.1. Let $X^h$ satisfy Assumption 2.1, associated preferences satisfy Assumption 2.2, and for any $\tilde{a}^h \in A^h$, there exists $\tilde{a}^h \in A^h$ such that $\tilde{a}^h \in \tilde{P}_h(\tilde{a}^h)$. Then

1. $A^h$ is closed, convex and $0 \in A^h$;
2. The graph of $P_h$ i.e. $\{ (a^h, \tilde{a}^h) : \tilde{a}^h \in P_h(\tilde{a}^h) \}$ is open in $A^h \times A^h$;
3. For every $a^h \in A^h$, $a^h \in \text{con}(P_h(a^h))$;
4. For every $a^h \in A^h$, $P_h(a^h) \neq \emptyset$ (non-satiation);
5. For every $a^h \in A^h$, $a^h \in P_h(a^h)$ and $a^h \in \partial (P_h(a^h))$, where $\partial (K)$ denotes the boundary of the set $K$.

\footnote{Note that $Q \cap X^h \neq \emptyset$ since $\{0\} \subset Q \cap X^h$.}
2.4. Equilibria in unbounded economies

In this section we present our definition of equilibrium and prove existence. If there are no restrictions on short sales, then the asset trade sets will not be bounded below. As before, a portfolio of assets will be described by a vector \( a^h \in \mathbb{R}^J \). The amount of the \( j \)th asset held by consumer \( h \) is denoted by \( x^h_j \), which may be positive or negative. Each consumer \( h \) is assumed to have an endowment of assets \( \bar{a}^h \in \mathbb{R}^J \). For each \( h \in \mathcal{H} \), let \( \text{sp}(Z_j) \) be the span of \( Z_j \).

When an asset market is incomplete, the relevant consumption set of each consumer is the set of allocations, which are attainable by way of exchange of assets, i.e.,

\[
\mathcal{X}_h = \{ x^h : x^h - e^h \in \text{sp}(Z_j) \}.
\]

Asset markets so constructed may be incomplete in the sense that the available assets do not span \( \mathcal{X}^h \).

**Assumption 2.3.** For each \( h \in \mathcal{H} \), \( \bar{a}^h = \text{int}(A^h) \).

Let \( \mathcal{E} = (A^h, \hat{P}_h, \bar{a}^h, \bar{x}^h) \) denote the unbounded asset exchange economy. Let \( \mathcal{P} = \{ q \in \mathbb{R}^J : |q| \leq 1 \} \) and \( B^h(q) = \{ a^h \in A^h : q \cdot a^h \leq q \cdot \bar{a}^h \} \) be the set of relative prices and the consumer \( h \)'s “budget set” in commodity space for a given price system \( q \in \mathbb{R}^J \), respectively.

**Definition 2.3.** An equilibrium for the asset exchange economy, \( \mathcal{E} \), is an \((H + 1)\) tuple of vectors \((a^*_1, \ldots, a^*_H, q^*)\) such that:

1. \( a^*_h \in B^h(q^*) \), for \( 1 \leq h \leq H \);
2. \( q^* \in \mathcal{P} \);
3. \( \hat{P}_h(a^*_h) \cap B^h(q^*) = \emptyset \), for \( 1 \leq h \leq H \);
4. \( \sum_{h \in \mathcal{H}} a^*_h = \sum_{h \in \mathcal{H}} a^h \).

Condition 3 requires that budget set does not contain a bundle which is preferred to the equilibrium consumption. Since preferences may be incomplete, this is not equivalent to requiring that the equilibrium consumption be preferred to any other bundle in the budget set. We believe that this is a reasonable notion of equilibrium since it describes a state of the economy in which nobody has an incentive to change their asset holdings.

For equilibrium with incomplete preferences, the choice of a status quo point is important. In common with much of the previous literature, we use the equilibrium allocation as the status quo point. In contrast to Rigotti and Shannon (2005), in their equilibrium with inertia, assume the endowment is the status quo point. Which of these assumptions is more compelling depends on the circumstances. Consider the case where an individual’s endowment consists mainly of the ability to supply labour, they would be accustomed to trade this for consumption goods. Such an individual may regard his/her usual consumption as the relevant status quo point. It is not implausible that such an individual would perceive any restriction on his/her ability to trade labour for consumption goods as a loss. The other conditions are standard.

**Theorem 2.1.** Assume that \( \mathcal{X}^h \) satisfies Assumption 2.1, \( A^h \) and associated preferences satisfy the conditions of Lemma 2.1 and Assumption 2.3, then the economy \( \mathcal{E} \) has an equilibrium \((q^*, a^*, x^*)\).

**Proof.** Let \( \mathcal{E}_r = (A^h, \hat{P}_h, \bar{a}^h, \bar{x}^h) \) be an \( r \)-bounded economy constructed in the following way. Consumer \( h \) has a consumption set

\[
A^h_r = \{ a^h \in A^h : ||a^h|| \leq r \},
\]
Remark 2.1. Shafer’s existence theorem yields equilibrium price systems where \( r > 0 \) is large enough so that \( a^h \in \text{int} \ (A^h_r) \). Given Assumption 2.2 and the fact that \( a^h \in \text{int} \ (A^h_r) \), \( \mathcal{E}_r \) has an equilibrium. Since the graph of \( P_{h,r} \) is open in \( A^h_r \times A^h_r \) and \( a^h \in \partial (P_{h,r}(a^h)) \), the economy satisfies the assumptions of Theorem 2 in Shafer (1976). This implies that an equilibrium exists for the bounded economy.

We shall now prove the existence of equilibrium for the unbounded economy. Let \( q_r, a_r, \) and \( x_r \) denote sequences of the prices, allocation of assets and allocation of consumption goods in the \( r \)-bounded economy, respectively. Notice first that \( q_r \) is an element of a compact set, i.e. \( q_r \in \Delta \). Since the total amount of consumption good is finite, \( x_r \) is also an element of a compact set. Consider the limit as \( r \to \infty \). We may assume, without loss of generality, that \( x_r \to x^* \) and \( q_r \to q^* \). Let \( a^h \in Z^{-1} x^* h \) for \( 1 \leq h \leq H \). Let \( a^h - \tilde{a}^h \) for \( 1 \leq h \leq H - 1 \) and \( a^H = \sum_{h=1}^{H} a^h - \sum_{h=1}^{H-1} a^h \). Then

\[
Z a^* H = Z \left( \sum_{h=1}^{H} \tilde{a}^h - \sum_{h=1}^{H-1} \tilde{a}^h \right) = Z \left( \sum_{h=1}^{H} \tilde{a}^h - \sum_{h=1}^{H} \tilde{a}^h + \tilde{a}^H \right) = Z \tilde{a}^H = x^* H,
\]

since by construction \( Z \sum_{h=1}^{H} \tilde{a}^h = Z \sum_{h=1}^{H} a^h \).

Now suppose, if possible, that there exists \( b^h \in \bar{P}_h(a^* h) \cap B^h(q^*) \). Since the graph of \( \bar{P}_h \) is open, there exists \( \epsilon > 0 \) such that \( b^h - \epsilon e \in \bar{P}_h(a^* h) \cap B^h(q^*) \) where \( e \) denotes the unit vector in \( \mathbb{R}^L \). Since \( a^h \in \bar{P}_h(a^* h) \) for sufficiently large \( r \), \( \tilde{a}^h \in \bar{P}_h(a^* h) \). Moreover

\[
q_r \cdot \tilde{a}^h = q_r \cdot b^h - \epsilon q_r \cdot e < q_r \cdot b^h = q_r \cdot \tilde{a}^h,
\]

for sufficiently large \( r \). But this is a contradiction since \( a_r \) is an equilibrium for the \( r \)-bounded economy. Thus we must have \( \bar{P}_h(a^* h) \cap B^h(q^*) = \emptyset \). This completes the proof. \( \Box \)

**Remark 2.1.** Shafer’s existence theorem yields equilibrium price systems \( q_r \neq 0 \), but not necessarily \( q_r \geq 0 \). In our model, monotonicity of preferences implies that equilibrium security prices must be positive, \( q_r > 0 \).

### 2.5. Optimality of competitive allocations

In general, an equilibrium allocation is Pareto optimal only if the market structure is essentially complete. If markets are incomplete there is no reason to expect an equilibrium allocation to be efficient. However, we can show that the equilibrium is constrained Pareto optimal. This means that it is not possible to make all individuals better off by reallocating the existing assets. More generally, we can show that the equilibrium will be in the constrained core.

**Definition 2.4.** The constrained core is defined to be the set of allocations \( (a^1 h, \ldots, a^H h) \) of one portfolio for each consumer such that there does not exist an allocation \( (a^1 h, \ldots, a^H h) \) and a non-empty subset \( G \subset H \) for which \( a^h \in \bar{P}_h(a^* h), \forall h \in G \) and \( \sum_{h \in G} a^h = \sum_{h \in G} \bar{a}^h \).

Hence, an asset allocation is in the constrained core if for any given coalition \( G \) there does not exist a way to re-allocate assets, which makes each individual in the coalition better off. Incompleteness of preferences reduces the number of potential Pareto improvements. Thus Pareto optimality and the core are weaker concepts with incomplete preferences. How much weaker than the usual concepts they are will depend on the degree of incompleteness of preferences.
The following result is a version of the first fundamental theorem of welfare economics. It shows that every equilibrium of the asset exchange economy is in the constrained core and hence is constrained Pareto optimal.

**Theorem 2.2.** In an asset exchange economy \( E = (A^h, \hat{P}_h, \bar{a}^h, \tilde{x}^h)_{h \in H} \) every competitive equilibrium \((q^*, a^*, x^*)\) is in the constrained core.

**Proof.** Let \((q^*, a^*, x^*)\) be the equilibrium allocations and price system. Suppose that there exists a non-empty subset \( G \subset H \) and \( a^h \in A^h \) for \( h \in G \) such that \( \sum_{h \in G} a^h = \sum_{h \in \bar{G}} \bar{a}^h \) and \( a^h \in \hat{P}_h \) \((a^*)\). Since \( \hat{P}_h(a^*) \cap B^h(q^*) = \emptyset \), one obtains \( q^* a^h > q^* \bar{a}^h, \forall h \in G \). Summing over \( h \in G \), one has \( q^* \sum_{h \in G} a^h > q^* \sum_{h \in \bar{G}} \bar{a}^h \). This contradicts group feasibility, that is, \( \sum_{h \in G} a^h = \sum_{h \in \bar{G}} \bar{a}^h \).

This completes the proof. \( \square \)

### 3. Arbitrage pricing

In this section we shall give a proof of the arbitrage pricing theorem. The APT illustrates the implications of diversification for equilibrium prices of securities. Diversification is the strategy of investing small fractions of wealth in each of a large number of securities and thereby reducing risk. The basic argument in support of diversification is related to risk aversion, that is, diversifiable risks can be eliminated from equilibrium allocations and asset prices can be determined by factor (economy-wide) risks, see for instance Milne (1988).

Asset returns depend on a set of factors which reflect economy-wide risks such as exchange rate risks or commodity price risks. In addition the returns of any given security are affected by idiosyncratic risk, which is risk uncorrelated with any of the factors. The intuition behind the APT is that idiosyncratic risk may be diversified away. Hence, in equilibrium, the price of an asset only depends on its relation to factors which affect the whole economy.

**Definition 3.1.** A factor structure on \( \mathbb{R}^J \) is a linear transformation \( F : \mathbb{R}^J \rightarrow M \), where \( M \subset \mathbb{R}^J \).

We shall assume that, by deleting redundant factors, if necessary, \( F \) is onto. Let \( \ker(F) \) denote the set of \( a \in \mathbb{R}^J \) such that \( F(a) = 0 \). The elements of \( \ker(F) \) will be called *idiosyncratic risk*.

**Definition 3.2.** Two portfolios \( a \) and \( b \) are factor equivalent if \( F(a) = F(b) \). Similarly, two allocations are factor equivalent if consumer \( h \)'s portfolios in the two allocations are factor equivalent for \( 1 \leq h \leq H \).

**Definition 3.3.** Let \( D \subset \mathbb{R}^J \) denote the subspace of diversified portfolios. Consumer \( h \) is a diversifier if whenever \( F(a) = F(b) \) with \( a \in D \) and \( b \in \mathbb{R}^J \setminus D \), then \( a \in \hat{P}_h(b) \).

Definition 3.3 says that any diversified portfolio must be strictly preferred to a factor equivalent portfolio which is not diversified. Less technically, if consumer \( h \) is a diversifier then idiosyncratic risk is a “bad” for him/her.

Diversification has been introduced in connection with factor pricing (see, for instance, Ross, 1976; Chamberlain, 1983). One can think of a diversified portfolio as containing small holdings of each of a large number of assets. When asset returns have a factor structure, diversification can be used to reduce idiosyncratic risk. If the number of assets is finite, idiosyncratic risk can be completely eliminated only if the asset returns are linearly dependent. With an infinite number of assets, the law of large numbers implies that a well-diversified portfolio is possible even when the returns of any finite set of assets are linearly independent. We have deliberately not specified in detail which
portfolios are diversified. Whether or not a portfolio is perceived to be diversified will depend on both preferences and the returns of assets. We shall return to the structure of the set of diversified portfolios in Section 4, where we make specific assumptions about preferences and returns.

In the following, we shall provide some conditions under which all consumers in a factor economy will be able to diversify away idiosyncratic risk.

**Definition 3.4.** An allocation is insured if it consists entirely of diversified portfolios.

**Assumption 3.1.** An asset economy is said to be insurable if \( F(D) = M \setminus \{0\} \) and the aggregate endowment lies in \( D \), i.e., \( \bar{a} \in D \).

Insurability allows a consumer to eliminate idiosyncratic risk by holding a diversified portfolio. The following proposition asserts that in a competitive equilibrium, all consumers choose diversified portfolios.

**Proposition 3.1.** In an insurable asset exchange economy in which all consumers are diversifiers, any competitive allocation is diversified.

**Proof.** Apply Proposition 2.1 in Kelsey and Milne (1995). □

Let \( \phi^h : A_h \times A_h \rightarrow \mathbb{R} \) be a function of two arguments, which represents preferences as in Lemma 1.

**Assumption 3.2.** (Differentiability)

For all \( a \in A_h \), \( \phi^h(\cdot,a) : A_h \times A_h \rightarrow \mathbb{R} \) is differentiable.

Since \( \phi^h \) is differentiable, if a consumer is satiated in idiosyncratic risk with a diversified portfolio, then in equilibrium, the price of idiosyncratic risk is zero. The following proposition demonstrates this formally.

**Proposition 3.2.** In an insurable asset exchange economy in which all consumers are diversifiers and at least one consumer has differentiable preferences, let \( q \in \mathbb{R}^q \) be an equilibrium price system and let \( a' \) be a portfolio which consists only of idiosyncratic risk i.e. \( a' \in \ker(F) \), then \( q \cdot a' = 0 \).

**Proof.** Since \( a' \in \ker(F) \), \( a^* \) is factor equivalent to \( a^* + ta' \). Consider the function

\[
\zeta(t) = \phi(a^* + ta', a^*).
\]

Since \( a^* + ta' \in A \setminus D, \forall t \neq 0 \), and since consumer \( h \) is a diversifier, \( t=0 \) maximises \( \zeta(t) \), so \( \zeta'(0) = 0 \). Now \( \zeta'(t) = d_{a^* + ta'} \phi \cdot a' \), hence\(^4\)

\[
\zeta'(0) = d_{a^*} \phi \cdot a^* = 0.
\] (1)

Suppose, if possible, that \( q \cdot a' < 0 \). Define a portfolio \( \tilde{a}(t) \) by \( \tilde{a}(t) = (1 + \delta t) a^* + ta' \), where \( \delta = -\frac{q \cdot a'}{q \cdot a^*} > 0 \). Note that \( \forall t, q \cdot \tilde{a}'(t) = (1 + \delta t) q \cdot a^* + tq \cdot a' = q \cdot a^* + \frac{q \cdot a'}{q \cdot a^*} tq \cdot a^* + tq \cdot a' = q \cdot a^* \).

Hence \( \tilde{a}(t) \) is in the equilibrium budget set.

Now define \( \zeta(t) = \phi(\tilde{a}(t), a^*) \). Then \( \zeta'(t) = d\phi_{\tilde{a}(t)} \cdot (\delta a^* + a') \). Hence \( \zeta'(0) = d_{a^*} \phi \cdot (\delta a^* + a^*) = \delta d_{a^*} \phi \cdot a^* + d_{a^*} \phi \cdot a^* = \delta d_{a^*} \phi \cdot a^* > 0 \). Since \( d_{a^*} \phi \cdot a^* = 0 \) by Eq. (1) and \( \delta d_{a^*} \phi \cdot a^* > 0 \) by

\(^4\)The notation \( d_{a^*} \phi \) denotes the usual derivative on \( \mathbb{R}^n \) evaluated at \( a^* \). When taking the derivative, \( \phi \) is viewed as a function of the first argument only. If the state space is infinite dimensional, the derivative should be interpreted as a Frechet derivative.
monotonicity. However this contradicts the fact that \( a^* \) is an equilibrium consumption bundle, since \( \zeta'(0) > 0 \) implies that for small \( t \), \( \hat{a}(t) \) is preferred to \( a^* \).

The case where \( q \cdot a' > 0 \) can be handled similarly. □

**Remark 3.1.** If the preferences of all consumers are not differentiable, then it may be the case that \( q \) is non-unique and contains non-zero prices. However, it will still be the case that there is an equilibrium in which the price of idiosyncratic risk is zero. In this case there will exist an equilibrium in which the arbitrage pricing theorem holds, however we would not be able to ensure that the APT holds in all equilibria.

Theorem 3.1 implies that the asset prices may be written as a linear combination of the factor prices.

**Theorem 3.1 (APT).** Let \( q \) be the competitive equilibrium price system of an insurable asset exchange economy. Assume that all consumers are diversifiers and at least one consumer has differentiable preferences, then there exists a linear functional \( \mu : M \to \mathbb{R} \) on \( M \) such that for all \( a \in \mathbb{R}^J \),

\[
q \cdot a = \mu \cdot F(a).
\]

**Proof.** Define \( \mu : M \to \mathbb{R} \) by \( \mu(f) = q \cdot a \), where \( a \) has a factor structure \( f = F(a) \). Assume that \( F(a) = F(a') \), then \( a - a' \in \ker(F) \) consists only of idiosyncratic risk. Hence, by Proposition 3.2, we must have \( q \cdot (a - a') = 0 \). This implies that \( q \cdot a = q \cdot a' \) and therefore \( \mu : M \to \mathbb{R} \) is well defined. Since \( \mu \) is a linear function defined on a vector space \( M \) with \( \mu(f) \in \mathbb{R} \), it must satisfy the following condition:

\[
\mu(\eta f_1 + \kappa f_2) = \eta \mu(f_1) + \kappa \mu(f_2).
\]

Thus, \( q \cdot a = \mu \cdot F(a) \). This completes the proof. □

The intuition for Theorem 3.1 is that the APT depends mainly on the linear structure of asset returns. Hence it is still possible to prove a version of the APT even when preferences are incomplete and/or intransitive.

### 4. Bewley preferences

In this section, we consider a model of decision-making under uncertainty in which the distinction between risk and ambiguity is formalised by assuming that individuals have incomplete preferences as in Bewley (2002). In this model, Knightian uncertainty (or ambiguity) is represented by preferences, which satisfy the axioms of Anscombe and Aumann (1963) except that they are not required to be complete. These axioms are shown to imply that a decision-maker has a set of subjective probabilities. According to Bewley, an option \( a \) is preferred to \( b \) if its expected value is higher with respect to all the probability distributions in this set. This may be explained intuitively as follows. If a previously unavailable alternative appears, an individual will use it only if doing so would put her in an unambiguously preferred position. We show that the APT also applies to Bewley preferences. Since these preferences are not differentiable, we cannot simply apply our previous analysis.\(^5\)

When faced with a decision problem involving uncertainty, individuals are sometimes unable to assign probabilities to relevant events. In particular, if a preference ordering does not satisfy

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\(^5\)We shall note that the social choice justification for incompleteness is not appropriate for the application of Bewley preferences, because neither independence nor monotonicity is necessarily preserved at a social level.
completeness, individuals are not necessarily able to compute a unique expected utility for each act. It has been argued that, in such circumstances, individuals have beliefs, which take the form of ranges or intervals for an event. However, this approach is questionable since it is no longer the case that the probabilities of a number of mutually exclusive events will sum to one. This problem can be circumvented by assuming that individuals’ beliefs should be represented by a convex set of probability distributions. The probabilities in a given distribution from that set will sum to one, while for any given event, the probabilities assigned to that event will form an interval.

Let \( \Omega = \{ \omega_1, \ldots, \omega_S \} \) be a finite set of states of nature and \( \Pi \) be a closed convex set of subjective probability distributions on \( \Omega \). Let \( a = (a_1, \ldots, a_S) \) and \( b = (b_1, \ldots, b_S) \) be two portfolios, where \( a_S \) denotes the pay-off of portfolio \( a \) in state \( \omega_S \).

Bewley (2002) showed that his axioms imply that there exists a closed convex set \( \Pi \) of subjective probabilities such that,

\[
\forall a \succ b \iff \forall \pi \in \Pi, E_\pi[u(a)] > E_\pi[u(b)] \iff \sum_{i=1}^{S} \pi_i u(a_S) > \sum_{i=1}^{S} \pi_i u(b_S),
\]

where \( E_\pi \) denotes expectation with respect to the additive probability distribution \( \pi \) and \( u(\cdot) \) is the decision-maker’s Von Neumann–Morgenstern (VNM) utility function. He argues that the above expression is a possible formulation of Knight’s (1921) distinction between risk and uncertainty. The size of \( \Pi \) is a measure of the Knightian uncertainty, or equivalently the decision-maker’s ignorance about the probabilities. Pay-offs are risky if \( \Pi \) has only one element and uncertain otherwise.

**Assumption 4.1.** The VNM utility function \( u \) is strictly increasing, concave and continuously differentiable.

Let \( Z \) be the space of all functions from the set of all subsets \( \sum \) of \( \Omega \) to \( \mathbb{R} \). Define \( \{ f^i : \Omega \to \mathbb{R} \} \) to be \( I \) linearly independent members of \( Z \). Hence, returns on asset \( j \in J \) can be written as

\[
Z_j = \sum_{i \in J} \beta^i_j f^i + \epsilon_j,
\]

where \( \epsilon_j \) is the idiosyncratic risk. We assume that the idiosyncratic risk is un-correlated with the factors, i.e. \( E[\epsilon_j | f^i] = 0 \) for \( 1 \leq i \leq I, 1 \leq j \leq J \).

**Assumption 4.2.** The idiosyncratic risk on security \( j, \epsilon_j \), is assumed to be state independent.

Assumption 4.2 implies that the idiosyncratic risk is defined only by objective probabilities. There are two ramifications of this assertion. First, all individuals will have the same subjective probabilities for the idiosyncratic risk. This is important because if individuals do not agree on the probability distribution of the idiosyncratic risk, then they would not agree which portfolios are diversified. Hence, the requirement that the price of a portfolio consisting of solely idiosyncratic risk is zero cannot be obtained. Second, each individual’s beliefs over idiosyncratic risk may be represented by a single subjective probability distribution.

With the above construction, \( M \) is the vector subspace of \( Z \) spanned by \( \{ f^i \}_{i \in I} \) and \( F \) is the function defined by

\[
F(Z_j) = \sum_{i \in I} \beta^i_j f^i.
\]

The space \( D \) of diversified portfolios is defined to be the linear span of \( \{ f^i \}_{i \in I} \).
Definition 4.1. Define the derived utility function $V^h$ on the choice set $A_b$ by

$$V^h(a, b) = \min_{\pi \in \pi} \left[ \sum_{s=1}^{S} \pi_s u(Za) - \sum_{s=1}^{S} \pi_s u(Zb) \right]$$

such that $V^h(a, b) > 0$ iff $a > b$.

Definition 4.1 implies that the decision rule is incomplete. That is, if $a$ gives higher expected utility with respect to some probability distribution, while at other times $b$ gives higher expected utility then, $a$ and $b$ will not be comparable.

Remark 1. We only use the representation in Definition 4.1 when differentiability becomes an issue, and the more fundamental requirement is smoothness of indifference surfaces (or boundaries of upper contour sets). The problem with Bewley preferences for EAPT is that the boundary of upper contour sets is non-smooth in general. Hence there is a possibility that equilibrium prices will be non-unique. To establish that the equilibrium price of idiosyncratic risk is zero we need to show that the boundary of the upper contour set does not have any kinks in the direction of the idiosyncratic risk.

Proposition 1. An individual with Bewley preferences will strictly prefer a portfolio which contains zero idiosyncratic risk to a factor equivalent portfolio which contains non-zero idiosyncratic risk.

Proof. Suppose that $a$ and $b$ are factor equivalent portfolios, where $a$ is diversified and $b$ is not. Then, by Lemma 3.1 in Kelsey and Milne (1995), $(b-a)$ is a “mean preserving increase in risk” on $a$. Let $\hat{\pi} \in \Pi$, then by Rothschild and Stiglitz (1970)

$$E_{\hat{\pi}}[u(Zb)] = E_{\hat{\pi}}[u(Za + (Zb - Za))] < E_{\hat{\pi}}[u(Za)].$$

Since $\hat{\pi}$ is an arbitrary element of $\Pi$, it follows that

$$\forall \pi \in \Pi, \quad E_{\pi}[u(Zb)] < E_{\pi}[u(Za)]$$

and hence $a \in \hat{P}(b)$. □

We have assumed that all idiosyncratic risk is state independent and hence objective. Moreover, in Proposition 4.1, we use the result that idiosyncratic risk involves a mean preserving increase in risk. Hence, if $a'$ is a portfolio, which contains only idiosyncratic risk, $u(Z(a + \lambda a')) - u(Za) \leq 0$, for all $\lambda$. We can use these two facts to show that the directional derivative in the direction of the idiosyncratic risk exists.

Lemma 1. Let $f_i : \mathbb{R} \rightarrow \mathbb{R}, \quad 1 \leq i \leq m$ be a family of functions such that

1. $f_i(0) = 0$, for $1 \leq i \leq m$;
2. $f_i'(0) = 0$, for $1 \leq i \leq m$;
3. $0$ maximises $f_i(\alpha)$.

Define $\xi(x) = \min_{1 \leq i \leq m} f_i(x)$. Then, $\xi$ is differentiable at 0 and $\xi'(0) = 0$. 


Proof. If \( h > 0 \), then \( 0 \geq \frac{\tilde{\xi}(h) - \tilde{\xi}(0)}{h} \leq \min_{1 \leq i \leq m} \left\{ \frac{f_i(h) - f_i(0)}{h} \right\} \). Since \( \lim_{h \to 0} \frac{f_i(h) - f_i(0)}{h} = 0 \), for \( 1 \leq i \leq m \), this establishes that
\[
\lim_{h \to 0^+} \frac{\xi(h) - \xi(0)}{h} = 0.
\]

If \( h < 0 \), then
\[
0 \leq \frac{\xi(0) - \xi(h)}{-h} = \frac{\xi(h) - \xi(0)}{h}.
\]

Thus \( 0 \geq \frac{\xi(h) - \xi(0)}{h} = \min_{1 \leq i \leq m} \frac{f_i(h) - f_i(0)}{h} \). Since \( \lim_{h \to 0} \frac{f_i(h) - f_i(0)}{h} = 0 \), for \( 1 \leq i \leq m \), this establishes that \( \lim_{h \to 0} \frac{\xi(h) - \xi(0)}{h} = 0 \) and hence \( \xi \) is differentiable at 0. This completes the proof. \( \square \)

Lemma 2. Assume that \( f_s(\alpha) \) is a family of functions such that

1. \( f_s'(0) = 0 \), for all \( s \in \Omega \);
2. \( 0 \) maximises \( f_s(\alpha) \).

Define \( \xi(\alpha) = \min_{\pi \in \Pi} \sum \pi_s f_s(\alpha) \), where \( \Pi \) is a closed convex set of probability distributions. Then \( \xi \) is differentiable at 0 and \( \xi'(0) = 0 \).

Proof. If \( \Pi \) is the convex hull of a finite set of probabilities, the result follows from Lemma 4.1, since only the extremal points of the set matter for determining the minimum.

Now, let \( \Pi \) be an arbitrary closed convex subset of \( \Delta(S) \) where \( \Delta(S) \) denotes the set of all probability distributions over \( S \). Let \( \tilde{\pi} \in \arg\min_{\pi \in \Pi} \sum \pi_s f_s(0) \). Since we are minimising a linear function over a compact convex set, we may choose \( \tilde{\pi} \) to lie in the boundary of \( \Pi \). By the separating hyperplane theorem, there exists a half space \( \tilde{H} \) such that \( \Pi \subseteq \tilde{H} \) and \( \tilde{\pi} \in \arg\min_{\pi \in \tilde{H}} \sum \pi_s f_s(0) \). Define \( H = \tilde{H} \cap \Delta(S) \), and \( \psi(\alpha) = \min_{\pi \in H} \sum \pi_s f_s(\alpha) \). Then for all \( \alpha \), \( \psi(\alpha) \leq \xi(\alpha) \), since the minimum is taken over a larger set. The set \( H \) has a finite number of extremal points, hence by Lemma 4.1, \( \psi \) is differentiable at 0 and \( \psi'(0) = 0 \). Define \( \phi(\alpha) = \sum \tilde{\pi}_s f_s(\alpha) \). Then for all \( \alpha \), \( \xi(\alpha) \leq \phi(\alpha) \).

Now, if \( h > 0 \), \( 0 \geq \frac{\xi(h) - \xi(0)}{h} = \frac{\xi(h) - \psi(0)}{h} - \frac{\psi(h) - \psi(0)}{h} \). Since \( \lim_{h \to 0^+} \frac{\psi(h) - \psi(0)}{h} = 0 \), this establishes that \( \lim_{h \to 0^+} \frac{\xi(h) - \xi(0)}{h} = 0 \).

Now, if \( h < 0 \), \( 0 \geq \frac{\xi(h) - \xi(0)}{h} = \frac{\xi(0) - \xi(h)}{-h} - \frac{\phi(0) - \phi(h)}{-h} \). Since \( \lim_{h \to 0^-} \frac{\phi(0) - \phi(h)}{-h} = 0 \), this establishes that \( \lim_{h \to 0^-} \frac{\xi(h) - \xi(0)}{h} = 0 \). The result follows. \( \square \)

Corollary 4.1. Suppose that for all \( a \in \mathcal{D} \), \( u \) is differentiable at \( z_a \). Then, \( V_h \) has a directional derivative at \( a \) in the direction of idiosyncratic risk.

We shall note that the assumptions we imposed are sufficient to ensure smoothness in the direction of the idiosyncratic risk. Hence, Proposition 4.1 and Corollary 4.1 enable us to prove the APT for Bewley preferences.

Corollary 4.2. (APT) Let \( q \) be the competitive equilibrium price system for an insurable asset economy. Assume that all consumers have Bewley preferences and at least one
consumer’s preferences satisfy Corollary 4.1, then there exists a linear functional $\mu$ on $M$ such that

$$q_j = \sum_{i \in I} \beta^j_i \mu(f^i).$$

5. Related literature

Some related research can be found in Mukerji and Tallon (2001) who use the CEU model of ambiguity to explain incompleteness of financial markets. They show that if there is sufficient ambiguity, certain assets will not be traded, which reduces the market’s ability to share risks. Thus ambiguity can potentially explain why financial markets are incomplete. To show this, it is necessary to assume that the range of variation of the idiosyncratic component of financial assets is greater than the range of variation due to economic shocks. (This can be roughly translated into the model of the present paper as saying the loss due to a bad resolution of the idiosyncratic risk is greater than the potential loss due to a bad resolution of the factor risk.) In their model, unlike risk, ambiguity cannot be insured or diversified by financial markets or intermediaries.

A key step in their argument is to show that in the presence of ambiguity, individuals will not want to diversify their portfolios. In contrast, to the proof of the APT with Bewley preferences in Section 4, we show that individuals will want to hold diversified portfolios despite Knightian uncertainty. This difference can be explained because the two papers aim to explain different aspects of financial markets and hence have different models. They aim to explain which assets are traded, while we aim to explain the pricing of assets, taking the set of assets available in the market as given.

There are three differences between their model and ours. Firstly they use the CEU model of ambiguity while we use Bewley preferences. Secondly we assume that idiosyncratic risk is unambiguous, while they assume it is ambiguous. Finally they make different assumptions about the pay-off of assets. In particular they have $n$ assets. Their state space is a Cartesian product $S = S_0 \times S_1 \times \ldots \times S_n$. The idiosyncratic risk on asset $i$ only depends on $s_i \in S_i$. The set $S_0$ consists of ‘economic states’ these are factors which affect the pay-off of all securities. A consequence of this assumption is that the state space becomes larger as more assets are added.

We believe that the most important difference from our model is the different assumptions about the pay-off of securities. Their assumptions imply that the pay-off of each security is effectively determined by a different state space. In contrast we have a common state space, which determines the pay-off of all assets. Our model is more typical of the finance literature. The assumption that idiosyncratic risk is unambiguous is not crucial for our results. Consider the case where there are a finite number of assets with linearly dependent returns as in Connor (1984), then individuals may diversify by holding a finite number of assets in appropriate ratios. It is easy to show that, in this case, individuals will have a strict preference for diversification, even if the idiosyncratic risk is ambiguous. This holds both for the Bewley and CEU models of ambiguity. Strict preferences for diversification imply the APT as in Section 3.

In future research it would be desirable to develop a more general model to explain both which assets are traded and the pricing of those assets which are traded. This presumably could be done by combining the analysis of Mukerji and Tallon (2001) with that of the present paper. In such a model we conjecture that there would be partial diversification. Individuals would be able to diversify some but not all risks.
Like us, Rigotti and Shannon (2005) present a general equilibrium model of asset trading with Bewley preferences. They prove existence and optimality of equilibrium in a general equilibrium model, where consumers have Bewley preferences. They show that both no trade in equilibrium and indeterminacies in equilibrium price and allocations are robust features of markets with Knightian uncertainty. The present paper has a different focus since it is mainly concerned with asset pricing. Our results are not a special case of Rigotti and Shannon (2005), since the results in Sections 2 and 3 are proved for general incomplete preferences and do not require Bewley’s specific functional form.

6. Conclusion

In this paper we have shown that the arbitrage pricing theorem, a central result on asset pricing, will continue to hold when preferences are incomplete or intransitive. The focus of this paper is mainly directed at preferences, where beliefs cannot be represented as a single probability distribution, rather than non-expected utility preferences which are non-linear in probabilities. Throughout it has been assumed that individual preferences were given by an irreflexive binary relation with open graph and the asset trade set was unbounded. Our analysis does not suggest that behavioural issues such as Knightian uncertainty or regret do not affect asset prices. What it shows is that any effect on asset prices operates via the factor prices.

Experimental evidence from decision-making shows that complex decisions, regarding demand for and pricing of assets, are made in situations where preference orderings do not confirm to Savage’s expected utility theory. In such situations, it is important to examine whether equilibrium exists in markets, whether Pareto optimality holds, and finally, whether the APT is still robust. Therefore, our study generalises various results in the existing literature.

Possibly, the most noteworthy aspect of this paper is the part that concerns the APT. It was shown that the APT remains robust when preferences may be incomplete or intransitive, and when uncertainty cannot be represented by unique subjective probabilities. One of the results of incompleteness and ambiguity-aversion is that ambiguity can make simple plans of actions be un-dominated. It appears that simple economic behaviour becomes rational when it is seen from a Knightian point of view. Alternatively, incompleteness or intransitivity can be motivated by the difficulties arising from constructing an aggregate preference for a group of individuals. Therefore, we believe the extensions analysed in this paper have useful implications for the pure theory of finance.

References


