A BROWNIAN NEEDLE IN DIRE STRAITS

D. Holcman1,2, and Z. Schuss2
1Ecole Normale Supérieure, Département de Mathématiques et de Biologie, 46 rue d’Ulm 75005 Paris, France.*
2Department of Applied Mathematics, Tel-Aviv University, Tel-Aviv 69978, Israel.
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We study the mean turnaround time of a Brownian needle in a narrow planar strip. When the needle is only slightly shorter than the width of the strip the computation becomes a nonstandard narrow escape problem. We develop a new boundary layer method, based on a conformal mapping of cusp-like narrow straits, to obtain an explicit asymptotic approximation to the mean turnaround time.

Brownian motion of anisotropic objects such as an ellipsoid or a needle in two-dimensions is anisotropic and governed by longitudinal, transversal, and rotational diffusion constants. [1–5]. In a planar strip, when the needle is only slightly shorter than the width of the strip its turning around becomes a rare event, because there is not much room in configuration space for the vertical position of the needle in the strip. Thus the computation of the mean time to turn around becomes a narrow escape problem. It does not fall, however, under any of the previously studied narrow escape time (NET) in planar or solid geometries [6–16] and many more, because the passage of the needle through a vertical position in a narrow strip is equivalent to the diffusion of anisotropic Brownian motion through a narrow neck cut from a boundary cusp. The problem of reaching an absorbing arc near a boundary cusp was reduced in [12] by a conformal map to the standard NET problem in a half plane and solved by extracting the singularity of the planar Neumann function. This method, as well as the boundary layer method of [6], fail in the problem at hand and the solution requires different mapping, as used in [17], and altogether different boundary layer analysis.

More specifically, we consider the motion of a Brownian needle of length $l$ in a narrow strip of width $l_0 > l$ such that $\varepsilon = (l_0 - l)/l_0 \ll 1$ (Fig. 1). The NET problem is to find the mean time for the needle to turn $180^\circ$. This is about twice the mean time $\bar{\tau}$ to reach vertical position. We find that $\bar{\tau}$ is to leading order independent of the initial position of the needle and

\[
\bar{\tau} = \frac{\pi}{\sqrt{l_0(l_0-l)}} \sqrt{\frac{D_x}{D_r}} \left(1 + O\left(\sqrt{\frac{l_0-l}{l_0}}\right)\right),
\]

where $D_x$ and $D_r$ are the longitudinal and rotational diffusion constants, respectively. This result $\bar{\tau} = O(\varepsilon^{-1/2})$ is more in line with [17] than the result of [12], where $\bar{\tau} = O(\varepsilon^{-1})$.

**Geometrical formulation.** The planar motion of the needle in Fig. 1 is described by two coordinates of the centroid and the rotational angle $\theta$ between the axes of the strip and the rod. The $y$-coordinate of the center of the needle is measured from the axis of the strip. The motion is confined to the domain $\Omega$ shown in Fig. 2. The needle turns across the vertical position if it goes from the domain on the left to that on the right or in the reverse direction. The mean time for such a transition is to leading order independent of the indicated domains, as long as their radii are much larger than the narrow neck and they do not extend into the narrow neck. In this case the mean transition time is to leading order twice the mean first passage time (MFPT) to $\theta = \pi/2$ [18].

The turnaround time is invariant to translations along the strip (the $x$-axis), therefore it suffices to describe the movement of the needle by its angle $\theta$ and the $y$-coordinate of its center. The position of the needle is defined for $\theta \mod \pi$, therefore the motion in the invariant strip can be mapped into that in the $(\theta, y)$ planar domain $\Omega$ (Fig.2)

\[
\Omega = \left\{(\theta, y) : |y| < \frac{l_0 - l \sin \theta}{2}, 0 < \theta < \pi\right\}.
\]

**Equations of motion of a Brownian needle in a planar strip.** In a rotating system of coordinates $(X, Y, \theta)$, where the instantaneous $X$-axis is parallel to the long axis of the rod and the $Y$-axis is perpendicular to it, the diffusive motion of the rod is an anisotropic

![FIG. 1: Rod in strip. The strip width is $l_0$ and the rod length is $l < l_0$. The position of the rod is characterized by the angle $\theta$ and the fixed coordinates $x, y$ of the center or in a rotating system of coordinates $(X, Y, \theta)$.](image-url)
Brownian motion, and can be described by the stochastic equations
\[
\begin{align*}
\dot{X} &= \sqrt{2D_X} \dot{w}_1, \\
\dot{Y} &= \sqrt{2D_Y} \dot{w}_2, \\
\dot{\theta} &= \sqrt{2D_r} \dot{w}_3,
\end{align*}
\]
where \(w_1, w_2, w_3\) are standard independent Brownian motions [19], \(D_X\) is the longitudinal diffusion coefficient along the axis, \(D_Y\) the transversal diffusion constant, and \(D_r\) the rotational diffusion coefficient. The diffusion constant \(D_r\) has unit of 1/seconds contrary to \(D_X, D_Y\). Due to the anisotropy, the rod makes in general larger excursions in the \(X\)-direction than in the \(Y\)-direction and this usually characterized by the ratio \(D_Y / D_X\). In a fixed system of Cartesian coordinates \((x, y)\), the translational and rotational motion of the centroid \((x(t), y(t))\) and the angle of rotation \(\theta(t)\) of the rod is governed by the Itô stochastic equations [19]
\[
\begin{align*}
\dot{x} &= \cos \theta \sqrt{2D_X} \dot{w}_1 - \sin \theta \sqrt{2D_Y} \dot{w}_2 \\
\dot{y} &= \sin \theta \sqrt{2D_X} \dot{w}_1 + \cos \theta \sqrt{2D_Y} \dot{w}_2 \\
\dot{\theta} &= \sqrt{2D_r} \dot{w}_3.
\end{align*}
\]
The trajectories of (3) are reflected at the boundary in the co-normal direction [19, 20] (see below). The transition probability density function of the needle in the product space \(\Omega \times \mathbb{R}\),
\[
p(t, x, y, \theta) \, dx = \Pr\{\{x(t), y(t), \theta(t)\} \in x + dx\},
\]
satisfies the Fokker-Planck equation
\[
\frac{\partial p(t, x)}{\partial t} = -\nabla \cdot J(t, x),
\]
where the flux is given with \(D_1(\theta) = D_X \cos^2 \theta + D_Y \sin^2 \theta,\)
\[
J(t, x) = -\left(\begin{array}{c}
D(\theta) \frac{\partial p}{\partial x} + \frac{[D_X - D_Y(\sin 2\theta)] \partial p}{2} \\
D(\theta + \pi/2) \frac{\partial p}{\partial y} + \frac{[D_X-D_Y(\sin 2\theta)] \partial p}{2} \\
D_r \frac{\partial p}{\partial \theta}
\end{array}\right), \tag{5}
\]
with no flux boundary conditions. Due to the symmetry and periodicity of \(p(t, x, y, \theta)\) with respect to \(\theta\), it suffices to consider the fundamental domain \(\Omega_1 = \Omega \cap \{\theta < \pi/2\}\).

**The NET of a Brownian needle.** The NET, (twice the MFPT \(\tau\) from \(x = (x, y, \theta) \in \Omega\) to \(\theta = \pi/2\)) is translation-invariant with respect to \(x\) and is thus the solution \(\hat{\tau} = u(\theta, y)\) of the boundary value problem [19]
\[
D_r \frac{\partial^2 u(\theta, y)}{\partial \theta^2} + D_Y(\theta) \frac{\partial^2 u(\theta, y)}{\partial y^2} = -1 \quad \text{for} \quad (\theta, y) \in \Omega_1, \tag{6}
\]
where \(D_Y(\theta) = D_X \sin^2 \theta + D_Y \cos^2 \theta\) and with the boundary conditions
\[
\frac{\partial u}{\partial n} = 0 \text{ on the curved boundary and at } \theta = 0, \quad u\left(\frac{\pi}{2}, y\right) = 0 \text{ for } |y| < l_0 - l, \tag{7}
\]
where the co-normal vector \(\hat{n}(\theta)\) is given by
\[
\hat{n}(\theta) = \left(\begin{array}{c} D_r \\ 0 \\ D_Y(\theta) \end{array}\right) n(\theta) \tag{8}
\]
with \(n(\theta)\) – the unit outer normal vector at the boundary, and the co-normal derivative of \(u(\theta, y)\) is given by
\[
\frac{\partial u}{\partial n} = \hat{n}(\theta) \cdot \nabla u(\theta, y). \tag{9}
\]

**Asymptotic approximation of the NET.** To construct the solution of the mixed Dirichlet-Neumann boundary value problem (6)-(8) we introduce the dimensionless variables \(X' = \frac{x}{l}, \quad Y' = \frac{y}{l}, \quad \xi(t) = \frac{x(t)}{l}, \quad \eta(t) = \frac{y(t)}{l}, \quad \xi(t) = \frac{x(t)}{l}, \eta(t) = \frac{\xi(t)}{\cos \theta}\), and normalized diffusion coefficients \(D'_X = \frac{D_X}{l^2}, \quad D'_Y = \frac{D_Y}{l^2}, \quad D_r(\theta) = \frac{D_r(\theta)}{l^2}\). This maps the domain \(\Omega\) in (2) into
\[
\Omega' = \left\{ (\theta, \eta) : |\eta| < 1 - (1 - \varepsilon) \sin \theta, \quad 0 < \theta < \pi \right\}. \tag{11}
\]
To convert (6) to canonical form, we introduce the variable
\[
\varphi(\theta) = \int_{0}^{\theta} \sqrt{\frac{D_r(\theta')}{D_r}} \, d\theta', \tag{12}
\]
which defines the inverse function $\theta = \theta(\varphi)$, and set $u(\theta, y) = U(\varphi, \eta)$ to obtain

$$U_{\varphi\varphi}(\varphi, \eta) + U_{\eta\eta}(\varphi, \eta) = U_{\varphi}(\varphi, \eta)\sqrt{D_r} \frac{dD_{\eta}^{-1/2}(\theta)}{d\theta} = \frac{1}{D_{\eta}(\theta)}. \tag{13}$$

The domain $\Omega'$, defined in (11), is mapped into the similar domain

$$\Omega'' = \left\{ (\varphi, \eta) : |\eta| < 1 - \frac{(1 - \varepsilon) \sin \theta(\varphi)}{2}, 0 < \varphi < \varphi(\pi) \right\}. \tag{14}$$

in the $(\varphi, \eta)$ plane. Because the co-normal direction at the boundary becomes normal, so does the co-normal derivative. It follows that the no-flux boundary condition (7) and the absorbing condition (8) become respectively

$$\frac{\partial U(\varphi, \eta)}{\partial n} = 0 \text{ for } (\theta(\varphi), \eta) \text{ on } \partial \Omega'' \quad \frac{\partial U(0, \eta)}{\partial \varphi} = 0 \text{ for } |\eta| < \frac{\varepsilon}{2}, \quad U \left( \phi \left( \frac{\pi}{2} \right), \eta \right) = 0 \text{ for } |\eta| < \frac{\varepsilon}{2}, \tag{15}$$

where $\partial \Omega''$ is the curved boundary in the scaled Figure 1. The gap at $\theta = \pi/2$ is preserved and the (dimensionless) radius of curvature of the boundary at the gap is

$$R' = \frac{2D_{\eta} \left( \frac{\pi}{2} \right)}{(1 - \varepsilon)D_r} = \frac{2D_X}{(1 - \varepsilon)l_{D}^{2}D_r}. \tag{16}$$

First, we simplify (13) by setting

$$g(\varphi) = \sqrt{D_r} \frac{dD_{\eta}^{-1/2}(\theta)}{d\theta}, \quad U(\varphi, \eta) = f(\varphi)V(\varphi, \eta) \tag{17}$$

and choosing $f(\varphi)$ such that $f'(\varphi) = \frac{1}{2}f(\varphi)g(\varphi)$. Note that

$$\left. \frac{dD_{\eta}^{-1/2}(\theta)}{d\theta} \right|_{\theta = 0, \pi/2, \pi} = 0. \tag{18}$$

Equation (13) becomes

$$V_{\varphi\varphi} + V_{\eta\eta} = \frac{1}{f(\varphi)} \left\{ [g(\varphi)f'(\varphi) - f''(\varphi)]V - \frac{1}{D_{\eta}(\theta(\varphi))} \right\}. \tag{19}$$

Next, we move the origin to the center of curvature of the lower boundary by setting $\zeta = - (\eta - R' - \frac{\pi}{2}) + i \left[ \varphi - \varphi \left( \frac{\pi}{2} \right) \right]$ and use the conformal mapping,

$$\omega = \zeta - R'\alpha \frac{\zeta - R'\alpha}{R' - \alpha\zeta}, \tag{20}$$

with $\omega = re^{i\psi}$. We also have

$$w'(\zeta) = \frac{1}{R'} \left( 1 + \alpha w \right)^2 \tag{21}$$

$$|w'(\zeta)|^2 = |1 - w + \sqrt{\varepsilon} w|^4 \frac{1}{4\varepsilon R'^2} \tag{22}$$

The image $\Omega''$ of the domain $\Omega$ is given in Figure 3. Setting $V(\varphi, \eta) = W(\rho, \psi)$, fixing $\rho = 1$ in $\Omega''$, and abbreviating $W = W(\psi, 1)$, equation (19) becomes to leading order

$$W_{\psi\psi} + \frac{h(\psi)}{|\omega'|^{2}} W = - \frac{1}{|\omega'|^{2}} k(\psi), \tag{23}$$

where $h(\psi) = f'(\psi)g(\psi)f(\psi)|_{\rho=1}$, $k(\psi) = f(\psi)D_{\eta}(\theta(\psi))|_{\rho=1}$. Using (21) and (22) and neglecting terms of order $O(\varepsilon)$, we rewrite (23) as

$$W_{\psi\psi} + \frac{4\varepsilon R'^2 h(\psi)}{|e^{i\psi}(1 - \sqrt{\varepsilon}) - 1|^4} W = - \frac{4\varepsilon R'^2}{|e^{i\psi}(1 - \sqrt{\varepsilon}) - 1|^4} k(\psi) \tag{24}$$

In view of (18), the boundary conditions (15) become

$$W_{\psi}(e^{i\sqrt{\varepsilon}}) = 0, \quad W(\pi) = 0. \tag{25}$$

**Boundary layer analysis.** The outer solution of (24) is a linear function $W_{\text{outer}}(\psi) = a\psi + b$, where $a$ and $b$ are yet undetermined constants. The uniform approximation is constructed as $W_{\text{uniform}}(\psi) = W_{\text{outer}}(\psi) + W_{\text{bl}}(\psi)$, where the boundary layer $W_{\text{bl}}(\psi)$ is a function $Y(\xi)$ of the boundary layer variable $\xi = \psi/\sqrt{\varepsilon}$. The boundary layer equation is

$$Y''(\xi) + \frac{4R'^2 h(0)}{(1 + \xi^2)^2} Y(\xi) = - \frac{4R'^2}{(1 + \xi^2)^2} k(0). \tag{26}$$
which is simplified by the substitution $Y(\xi) = \tilde{Y}(\xi) + 1/h(0)k(0)$ to

$$\tilde{Y}''(\xi) + \frac{4R^2h(0)}{(1 + \xi^2)^2} \tilde{Y}(\xi) = 0. \quad (27)$$

The boundary conditions (25) become $\tilde{Y}'(c) = 0$ and $\tilde{Y}(\infty) = 1/h(0)k(0)$.

The boundary layer equation (27) has two linearly independent solutions, $\tilde{Y}_1(\xi)$ and $\tilde{Y}_2(\xi)$, which are linear for sufficiently large $\xi$. Initial conditions for $\tilde{Y}_1(\xi)$ and $\tilde{Y}_2(\xi)$ can be chosen so that $\tilde{Y}_2(\xi) = const$ as $\xi \to \infty$ (e.g., $\tilde{Y}_2(0) = -4.7, \tilde{Y}_2'(0) = -1$). Thus the boundary layer function is given by

$$W_{\text{bl}}(\psi) = A\tilde{Y}_1 \left( \frac{\psi}{\sqrt{\xi}} \right) + B\tilde{Y}_2 \left( \frac{\psi}{\sqrt{\xi}} \right) + C, \quad (28)$$

where $A$ and $B$ are constants to be determined and $C$ is related to the constant $1/h(0)k(0)$ and is also determined below from the boundary and matching conditions. The matching condition is that $W_{\text{bl}}(\psi) = A\tilde{Y}_1 (\psi/\sqrt{\xi}) + B\tilde{Y}_2 (\psi/\sqrt{\xi}) + C$ remains bounded as $\xi \to \infty$, which implies $A = 0$. It follows that at the absorbing boundary $\psi = \pi$ we have

$$W_{\text{unif}}(\pi) = a\pi + b' = 0, \quad W_{\text{unif}}'(\pi) = a. \quad (29)$$

where the constant $b'$ incorporates all remaining constants. At the reflecting boundary we have to leading order

$$W_{\text{unif}}'(c\sqrt{\xi}) = W_{\text{outer}}'(c\sqrt{\xi}) + W_{\text{bl}}'(c\sqrt{\xi}) = a + B\frac{\tilde{Y}_2'(c)}{\sqrt{\xi}} = 0, \quad (30)$$

which gives $B = -\frac{a\sqrt{\xi}}{\tilde{Y}_2'(c)}$, $b' = -a\pi$. The uniform approximation to $W(\omega)$ is given by

$$W_{\text{unif}}(\rho\epsilon\psi) = a \left( \psi - \pi - \frac{\sqrt{\xi}}{\tilde{Y}_2'(c)} \right), \quad (31)$$

so that using (17), (18), and (21), we obtain from (31)

$$\frac{\partial u}{\partial n} \bigg|_{\xi = a\omega} = f \left( \frac{\psi}{\pi} \right) \frac{\partial W(\rho\epsilon\psi)}{\partial \psi} \bigg|_{\psi = \pi} \omega'(\xi)_{\xi = -1} \frac{\partial \phi}{\partial \theta} \bigg|_{\theta = \pi/2} = a\sqrt{2} \left( 1 + O(\sqrt{\xi}) \right). \quad (32)$$

Because $W(\omega)$ scales with $1/f(\varphi)$ relative to $V(\varphi, \eta)$, we may choose at the outset $f(\varphi(\pi/2)) = 1$.

To determine the value of $a$, we integrate (6) over $\Omega$, use (32), and the fact that $\int dy = l_0\epsilon$, to obtain $a = -|\Omega|\sqrt{\xi}/(l_0D_r\sqrt{2\xi})$. Now (31) gives the MFPT at any point $x$ in the head as

$$\tilde{\tau} = u(x) \sim W\left( \rho\epsilon\psi \right) \sim -a\pi = \frac{\pi|\Omega|\sqrt{\xi}}{l_0D_r\sqrt{2\xi}} \left( 1 + O(\sqrt{\xi}) \right) \quad (33)$$

for $\epsilon \ll 1$. Reverting to the original dimensional variables, we get (1).

The mean turnaround time of a Brownian needle in a narrow planar strip depends on the value of the parameter $\epsilon$. Obviously, $\tilde{\tau} = O(1)$ for $\epsilon = O(1)$, however, as shown above, $\tilde{\tau} = O(\epsilon^{-1/2})$ for $\epsilon \ll 1$. Our analysis exhibits difficulties inherent to the study of planar diffusion of shaped objects, e.g., in domains crowded with obstacles. The generalization to diffusion of a needle in a three-dimensional cylinder is pretty straightforward.