A new technique to optimize the functions of fuzzy profit of queuing models: Application to a queuing model with publicity and renouncement

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\textbf{A B S T R A C T}

In this paper we model, using the Queuing Theory, the situation in which the manager of some facilities wishes to maximize the profit generated by the same. The manager tries to attract customers and the effectiveness of his publicity politic is measured in the fraction of customers who renounce. In this paper, unlike the classic models, we consider that the service time of the only server of the facilities behave with uncertainty, which may be represented by means of a fuzzy number. The objective is to determine the rate to be paid by every customer for his service and the level of publicity which the manager must utilize to maximize his profits. The consideration of service times as fuzzy numbers implies that the function to be optimized has fuzzy coefficients, this is why it is necessary to use fuzzy optimization for the solution of the problem and, in addition, a new technique is proposed in order to optimize the functions of fuzzy profit of queuing models. To demonstrate the validity of the proposed proceeding is resolved satisfactorily, and applied as an example, a system of fuzzy queues, which is often found in real situations. The fuzzy queuing systems are more realistic than the crisp systems commonly used in many practical situations. Besides, the extension of the queuing decision models to fuzzy environment allows these models to have further applications.

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1. Introduction

Queuing models are divided in two large groups. On the one hand we have the models that represent real situations, that is, the descriptive models which provide the mean values and the probabilities of the performance measures which describe the system when the patterns of arrivals and services are fixed, as well as the number of servers, the capacity of the system and the queue discipline.

On the other hand we have the normative models which report prescriptions of which the real situation should be, that is to say, the optimum point of view which the system should aim for, they are frequently denominated as models of design and control of queues (queues decision models) and they try to obtain what the parameters that optimize such models should value. They usually control the measures that refer to services, such as service distribution or service rate, number of servers, queue discipline or some of their combinations. Occasionally some control upon the arrival of customers to the system can be established so that they can be increased, decreased, they can be assigned to different types of servers or they can even be regulated with some kind of toll. In other occasions, the parameters of design (physical space for example) may limit or even obligate to establish other parameters which at first will not be controllable.

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In this paper, we put forward a model of design and control of queues in which the parameters that describe the distributions of the service time are fuzzy numbers, that is, in the fuzzy queuing theory frame. There is a wide literature that analyses design and control problems associated to the classical queuing models. Recent studies include Armstrong [1], Pekoz [2], Jain [3], Takagi et al. [4]. Besides Wang and Ke [5], Al-Seedy [6–8] and Drekin and Wooldford [9], among others, have studied queuing models with balk and reneging. The use of prices to regulate the number of clients in a queue was first studied by Naor [10]. His model has been generalized by numerous authors such as Mendelson [11], Dewan and Mendelson [12] and Van Ackere and Ninios [13].

Within the context of traditional queuing theory, the time that elapses between the arrivals and the service duration is determined by probability distributions. However, in many practical situations, the statistical information is obtained subjectively, so that the arrival pattern and the service pattern are more adequately described by linguistic terms, such as fast, slow or moderate, rather than by probability distributions. For this reason, fuzzy queuing models are more realistic than the classic models commonly used. If the classic queuing model with publicity and renouncement is extended to the fuzzy model, it can have a greater number of applications.

Although fuzzy queuing decision problems are more practical than the conventional ones, and fuzzy queuing models with publicity and renouncement are more realistic, few studies on these topics have been published: for example, on the basis of possibility theory, Buckley [14] discussed elementary multiple–server queuing systems with finite or infinite capacity, in which the arrivals and the departures follow a possibility pattern and Buckley et al. [15] extended the previous results [14] to fuzzy queuing decision problems. Chen [16] proposed a parametric programming method to construct the membership function of the fuzzy objective value for a queuing decision problem in which the cost coefficients and the arrival rate are fuzzy numbers; Zhang and Phillips [17] determined the policy for optimal client appointment for a queuing system in parallel with two heterogeneous servers using fuzzy control, Pardo and de la Fuente [18] optimize a priority-discipline queuing model using fuzzy set theory and in [19] they have calculated the optimal selection of the service rate for a finite input source fuzzy queuing system, and de la Fuente and Pardo [20] analyze a finite capacity queuing model with fuzzy data.

In the papers by Naor [10], Mendelson [11], Dewan and Mendelson [12] and Van Ackere and Ninios [13], these authors study control models for facilities with one only server, keeping in mind the cost of waiting of the users and the profits that this service provides. Dewan and Mendelson [12], Mendelson [11] and Naor [10] study the model from the client’s point of view. Dewan and Mendelson [12] consider that the waiting costs are non lineal and this means that the customers decision to use or not to use the facilities is based on the number of customers in the queue in the steady state. This last characteristic is the one which makes this model different from the models of Naor [10] and Van Ackere and Ninios [13] which mean that the decision of the user is based in the longitude of the queue that is observed on arrival.

The model presented by us maintains the hypothesis of Van Ackere and Ninios’s model [13] which are basically:

- A facilities with one only server.
- The customers arrive, observe the longitude of the queue upon arrival and decide whether they remain or not in the system.
- A fixed rate is paid by the customer who goes through the system.
- A waiting cost is produced for the user per unit of time in the system.
- The manager makes an effort in attracting clients through publicity. The arrival rate of the user is therefore, variable and depends on the established level of publicity and the predetermined price. In this case, in turn, two different hypothesis can be established:
  1. The arrival rate depends only on the publicity level (basic model).
  2. The effectiveness of the expenses on publicity is a function of the quality of the service which may be measured trough the number of customers who join the queue. If the publicity is effective, many clients will be attracted and this may cause congestion in the system which in turn will carry out a large renounce of customers to incorporate it and a bad reputation of the service (word of mouth effect or word of mouth model).

Van Ackere and Ninios [13] have obtained through simulation techniques, the price of maximum profit for the manager of the facilities as well as the level of publicity taking as a start models M/Er /1/ k, that is, models with limited capacity (decision variable) whose time of arrival keeps variables independent identically distributed according to the exponential distribution and service time Erlang. Atkinson [21] reanalyses this model, using analytical and numerical techniques of queuing theory. Due to this characteristic, the paper of Atkinson [21] is the starting point for the analysis and development of our model: fuzzy queuing system with publicity and renouncement.

The objective of our analysis is to establish optimum strategies for this model in case the times of service of the server follow an exponential distribution whose mean value is only known vaguely, and therefore it is described in the form of a triangular fuzzy number (T.F.N.).

The handling of the models with fuzzy data follows the work carried out by Prade [22], Li and Lee [23], Kao, Li and Chen [24], Negi and Lee [25], Ke and Lin [26] and Chen [27], who analyzed fuzzy queuing models through Zadeh’s Extension Principle [28], and the works presented by Pardo and de la Fuente [18,19], de la Fuente and Pardo [20] and Chen [16] which have developed fuzzy queuing decision problems.

The rest of this paper is organized as follows: In Section 2, we describe a classic queuing model with publicity and renouncement: the characteristics of the model and the description of the optimization problem. In Section 3, we describe a fuzzy queuing model with publicity and renouncement: the characteristics of the model, the fuzzy probability of entry
into system in a model $M / \bar{M} / 1 / k$, the problem of fuzzy optimization with two methods to solve it, the second method is proposed by us. In Section 4, we show the process of optimization with an example. Finally, in Section 5, we conclude this paper.

2. Classic queuing model with publicity and renouncement (Atkinson [21])

2.1. Characteristics of the model

In the rest of the paper we denote by:

- $\lambda$: average arrival rate to the system
- $t$: average service time
- $\rho$: traffic intensity
- $R$: valuation of each user for the service provided by the system
- $F$: payment of each user by the service itself (cost of service)
- $k$: capacity of the system
- $M_b$: basic model
- $M_w$: word of mouth model
- $P_{Mb}$: level of expense in publicity in model basic
- $P_{Mw}$: level of expense in publicity in word of mouth model
- $P_{ad}$: probability of admittance into the system
- $S_{Mb}$: function of standardized profit in model basic
- $S_{Mw}$: function of standardized profit in word of mouth model.

The paper of Atkinson [21] develops a situation where a manager decides on the optimum use of a service. In this way, Atkinson considers some facilities with only one service to which the clients get to apply for a service according to a Poisson process with rate $\lambda$. Each user values in $R$ monetary units the service provided by the system and pays $F$ monetary units by the service itself (it is supposed that all clients are identical). Concerning the facilities, Atkinson considers that the times of service of the server are aleatorily independent variables and identically distributed according to the exponential distribution with average service time $t$.

The customer arrives, watches the number of customers in the system and the price $F$ of the service and decides whether he stays in the system or not. From the customer’s point of view, this will join the queue if he obtains some profit from the service.

The manager wishes to influence the use of the system in two different ways: on the one hand, the arrival rate depends on the level of publicity that the system carries out, on the other hand the decision of a customer joining the queue depends on the longitude observed in it and of the price $F$ that the manager stipulates for his service. These will be his decision variables if he tries to maximize his profit, which will be expressed as total income by the customers who access to the system except for publicity cost.

Joining the two points of view, that of the customer and that of the manager of the facilities, Atkinson determines the maximum longitude of the queue in the system and the maximum capacity, and he demonstrates that this model can be considered, in practical effects, a finite capacity queuing model, $M / M / 1 / k$ or $M / M / 1 / \lfloor R - F \rfloor$, with capacity equal to $\lfloor R - F \rfloor$. The capacity $k = \lfloor R - F \rfloor$ is the maximum capacity that should have the system if the manager wants the customers to join to the queue.

2.2. Description of the optimization problem

For both models the problem of maximization has as variables to optimize the arrival rate of the customers $\lambda$ (and implicitly the publicity rate) and the price $F$ which must be fixed by the manager for the service that he offers. Atkinson considers that there are two different situations to model the level of publicity, basic model and word of mouth model, and in both he calculates the function of standardized profit:

1. **Basic model ($M_b$):** The level of expenditure in publicity affects the arrival rate directly, so that the arrival rate $\lambda$ is a lineal function of the expenditure in publicity, $P_{Mb}$, to be precise:

   \[ \lambda = P_{Mb}. \]  

   The function of standardized profit to be maximized is:

   \[ S_{Mb} (\rho, k) = \rho P_{ad} \left( 1 - \frac{k}{R} \right) - \frac{\rho}{R} \]  

   (1)

---

$\lfloor R - F \rfloor$ denotes the largest integer less than or equal to a number $[R - F]$ (floor function).
with \( P_{\text{ad}} \) the probability of admittance into the system. Given that the queuing model can be considered a \( M/M/1/k \) model, the probability that a customer that arrives at the system will remain in it has the expression (Winston [29]):

\[
P_{\text{ad}} = \frac{1 - \rho^{k}}{1 - \rho^{k+1}}.
\]  

(3)

Then, the function of standardized profit to be maximized is:

\[
S_{\text{Mb}} (\rho, k) = \rho \left( 1 - \frac{k}{R} \right) \frac{1 - \rho^{k}}{1 - \rho^{k+1}} - \frac{\rho}{R}
\]  

which is a function of the traffic intensity \( \rho = \lambda t \) and of the maximum capacity of the system \( k \). If we can optimize \( \rho \), we can calculate \( \lambda = \rho / t \) and \( P_{\text{Mb}} \), and with \( k \) we can calculate \( F(k = [R - F]) \).

2. Word of mouth model \((M_{W})\): For an established price of service, high levels of publicity will mean that a big quantity of customers will balk at incorporating into the system due to the long distance of the customers in queue observed on arrival, this implies a bad reputation for the system and consequently future efforts in publicity will have less impact. Thus high cost in publicity affects negatively the arrival rate, so that if the probability of admittance into the system is \( P_{\text{ad}} \) (that is, the probability that the client will no balk) and \( P_{\text{Mw}} \) is the level of expense in publicity, then:

\[
\lambda = P_{\text{Mw}} \cdot P_{\text{ad}}.
\]  

(5)

And using (3) in the word of mouth model the expense level in publicity is:

\[
P_{\text{Mw}} = \frac{\lambda}{P_{\text{ad}}} = \lambda \frac{1 - \rho^{k+1}}{1 - \rho^{k}}.
\]  

(6)

The function of standardized profit to maximize in this model is:

\[
S_{\text{Mw}} (\rho, k) = \rho P_{\text{ad}} \left( 1 - \frac{k}{R} \right) - \frac{\rho}{RP_{\text{ad}}}
\]  

and with (3) is:

\[
S_{\text{Mw}} (\rho, k) = \rho \left( 1 - \frac{k}{R} \right) \frac{1 - \rho^{k}}{1 - \rho^{k+1}} - \frac{\rho}{R} \left( 1 - \rho^{k+1} \right).
\]  

(8)

In the functions \( S_{\text{Mb}} (\rho, k) \) and \( S_{\text{Mw}} (\rho, k) \) variable \( k \) is discrete, taking only values \( k = 1, 2, \ldots, [R] \), for this reason finding the pair of values \( (\rho, k) \) which supply the maximum value of the standardized profit functions \( S_{\text{Mb}} (\rho, k) \) and \( S_{\text{Mw}} (\rho, k) \) consists in fixing the value of \( k \) and finding the respective value of \( \rho_{i} \) and \( \rho_{j} \) which optimize the functions of the standardized profit \( S_{\text{Mb}} (\rho_{i}, 1) \) and \( S_{\text{Mw}} (\rho_{j}, j) \). The functions \( S_{\text{Mb}} (\rho, k) \) and \( S_{\text{Mw}} (\rho, k) \) are unimodal functions for \( \rho \geq 0 \) (a function is unimodal if it has only one maximum or minimum in the region to be searched), and we can optimize with respect to this variable using methods of non linear optimization of bisection interval. With the pairs \((\rho_{1}, 1), (\rho_{2}, 2), \ldots, (\rho_{[R]}, [R])\) we calculate the standardized profits \( S_{\text{Mb}} (\rho_{i}, 1) \) and \( S_{\text{Mw}} (\rho_{j}, j) \), reaching optimum in the pair that supply max \( S_{\text{Mb}} \) and max \( S_{\text{Mw}} \).

3. Fuzzy queuing model with publicity and renouncement

3.1. Characteristics of the model

The fuzzy model which is introduced maintains the characteristics of classic model with respect to the behavior of the customers in the system, that’s to say, the arrival of clients to the facilities takes place according to a Poisson process with rate \( \lambda \), the price \( F \) is charged to the user for the services, and \( R \) the valuation of each user for the service provided by the system.

Contrary to the classic model, it is now considered that the only server that offers service in the facilities does it according to exponential times of services, whose mean \( t \) is only known in a vague manner and is established through of a fuzzy number. The queuing model is now adapted to a fuzzy model which can be expressed in a similar notation to the Kendall notation as \( M/M/1/k \).

The object of the study consists in calculating the arrival rate \( \lambda \) and the price \( F \) which must be established by the manager to maximize his profit. The fuzzy standardized profit is:

1. Basic model: In the fuzzy basic model is supposed, as in the classic model, that \( P_{\text{Mb}} \) is a crisp number with \( P_{\text{Mb}} = \lambda \).

If \( \tilde{S}_{\text{Mb}} (\tilde{\rho}, k) \) represents the fuzzy standardized profit, then:

\[
\tilde{S}_{\text{Mb}} (\tilde{\rho}, k) = \tilde{\rho} \tilde{P}_{\text{ad}} \left( 1 - \frac{k}{R} \right) - \frac{\tilde{\rho}}{R}
\]  

with \( \tilde{\rho} = \lambda \tilde{t} \) and \( F = R - k \).
2. Word of mouth model: In the fuzzy word of mouth model is supposed that the publicity is a fuzzy number of the type:

\[ \tilde{P}_{\text{Mw}} = \frac{\lambda}{\tilde{P}_{\text{ad}}} \]

and the fuzzy standardized profit is:

\[ \tilde{S}_{\text{Mw}} (\tilde{\rho}, k) = \tilde{\rho} \tilde{P}_{\text{ad}} \left( 1 - \frac{k}{R} \right) - \frac{\tilde{\rho}}{R \tilde{P}_{\text{ad}}} . \]

(11)

3.2. Fuzzy probability of entry into the system \( \tilde{P}_{\text{ad}} \) in a model \( M/\tilde{M}/1/k \)

On the basis of Zadeh’s extension principle [28], the concept of possibility and fuzzy Markov chains [Stanford [30]], Li and Lee [23] have proposed a general approximation to analyze fuzzy queuing models. So, they consider that each fuzzy queuing model can be considered as a perception of a usual queuing system, which can be denominated as the original of fuzzy queuing model. In the model of the paper the original of fuzzy model \( M/\tilde{M}/1/k \) is the finite capacity queuing classic model \( M/M/1/k \), with the membership function:

\[ \mu_{M/M/1/k} (M/M/1/k) = \mu_f (t) \]

(12)

where \( \mu_f (t) \) represents the membership function of the fuzzy number \( \tilde{t} \).

For this reason, they obtain the possibility distribution of the performance measures of fuzzy queuing models applying Zadeh’s extension principle [28] to the solutions of the original models with the parameters known in a precise way. In general, all the functions with fuzzy parameters \( \tilde{t} \) can be defined by:

\[ \mu_f (\tilde{t}) (z) = \sup_{x \in \mathbb{R}} \{ \mu_f (x) / z = f (x) \} . \]

(13)

In the case of the probability of entry into the system, is \( \mu_{\tilde{P}_{\text{ad}}} (p) = \sup \{ \mu_{\tilde{\rho}} (\rho) / p = \frac{1 - \rho^k}{1 - \rho^{k+1}} \} \), with \( \rho = \lambda t \) and \( \mu_{\tilde{\rho}} (\rho) = \mu_{\tilde{\rho}_{\rho = \lambda t}} (t) \). Thus:

\[ \tilde{P}_{\text{ad}} = \frac{1 - \tilde{\rho}^k}{1 - \tilde{\rho}^{k+1}} \]

(14)

and the fuzzy standardized profit functions turn out to be:

\[ \tilde{S}_{\text{Mb}} (\tilde{\rho}, k) = \tilde{\rho} \left( 1 - \frac{k}{R} \right) \frac{1 - \tilde{\rho}^k}{1 - \tilde{\rho}^{k+1}} - \frac{\tilde{\rho}}{R} \]

(15)

\[ \tilde{S}_{\text{Mw}} (\tilde{\rho}, k) = \tilde{\rho} \left( 1 - \frac{k}{R} \right) \frac{1 - \tilde{\rho}^k}{1 - \tilde{\rho}^{k+1}} - \frac{\tilde{\rho} (1 - \tilde{\rho}^{k+1})}{R (1 - \tilde{\rho}^{k})} \]

(16)

but \( \tilde{\rho} = \lambda \tilde{t} \) then:

\[ \tilde{S}_{\text{Mb}} (\lambda, \tilde{t}, k) = \lambda \tilde{t} \left( 1 - \frac{k}{R} \right) \frac{1 - (\lambda \tilde{t})^k}{1 - (\lambda \tilde{t})^{k+1}} - \frac{\lambda \tilde{t}}{R} \]

(17)

\[ \tilde{S}_{\text{Mw}} (\lambda, \tilde{t}, k) = \lambda \tilde{t} \left( 1 - \frac{k}{R} \right) \frac{1 - (\lambda \tilde{t})^k}{1 - (\lambda \tilde{t})^{k+1}} - \frac{\lambda \tilde{t} (1 - (\lambda \tilde{t})^{k+1})}{R (1 - (\lambda \tilde{t})^{k})} . \]

(18)

On the other hand, given that the probability of entry into the system decreases as the traffic intensity in the model increases, if it is denoted the \( \alpha \)-cut of \( \tilde{\rho} \) as \( \rho_\alpha = \left[ \rho_{\alpha}^- , \rho_{\alpha}^+ \right] \), then applying Bucley and Qu’s method [31] the \( \alpha \)-cut of \( \tilde{P}_{\text{ad}} \) is:

\[ P_{\text{ada}} = \left[ \frac{1 - \rho_{\alpha}^-}{1 - \rho_{\alpha}^{k+1}} , \frac{1 - \rho_{\alpha}^+}{1 - \rho_{\alpha}^{k+1}} \right] . \]

(19)

If we keep in mind that \( \rho = \lambda t \) is an increasing monotonic function in \( t \), then \( \rho_{\alpha} = \lambda \tilde{t}_{\alpha} \) and \( \tilde{\rho}_{\alpha} = \lambda \tilde{t}_{\alpha} \) (Bucley and Qu’s method [31]) and the \( \alpha \)-cut \( P_{\text{ada}} \) is:

\[ P_{\text{ada}} = \left[ \frac{1 - (\lambda \tilde{t}_{\alpha})^k}{1 - (\lambda \tilde{t}_{\alpha})^{k+1}} , \frac{1 - (\lambda \tilde{t}_{\alpha})^k}{1 - (\lambda \tilde{t}_{\alpha})^{k+1}} \right] . \]

(20)
3.3. The problem of fuzzy optimization

We have analyzed several methods to find the optimum in functions with fuzzy parameters, such as proposed in Zimmermann [32], Luhandjula [33], Rommelfanger et al. [34], Lai and Hwang [35] and Bazaraa et al. [36], between others. Of among the proposed methods we think that the most appropriate method to optimize the fuzzy standardized profit functions is the method proposed by Rommelfanger et al. [34]. This method supposes to carry out a great number of operations and calculations, just as we will describe next. For that reason, in this work we propose a new method to optimize functions with fuzzy parameters. To show the efficiency of the approaches and to prove the validity of this new method, the theoretical results obtained in this section will be applied to an example in the Section 4.

Therefore, we carry out the optimization of the fuzzy standardized profit functions $S_{mb}(\lambda, \tilde{t}, k)$ and $S_{muv}(\lambda, \tilde{t}, k)$ from two methods: (i) we use the approximation of Rommelfanger et al. [34] (in Lai and Hwang [35]) to the problem of optimization with imprecise coefficients, and (ii) we will carry out a new optimization by a method proposed by us for the optimization of functions of fuzzy profit in queuing models, method that can be applied for the optimization of other types of functions with fuzzy parameters.

3.3.1. Method 1: Approximation of Rommelfanger et al. [34]

1. Basic model

In the basic model the fuzzy standardized profit function is:

$$S_{mb}(\lambda, \tilde{t}, k) = \lambda \tilde{t} \left(1 - \frac{k}{R}\right) \frac{1 - (\lambda \tilde{t})^k}{1 - (\lambda \tilde{t})^{k+1}} - \frac{\lambda \tilde{t}}{R}.$$  \hspace{1cm} (21)

Then, the problem of optimization can be expressed as:

Maximize \quad $S_{mb}(\lambda, \tilde{t}, k)$

subject to \quad $\lambda \geq 0$  \hspace{1cm} (22)

where $S_{mb}(\lambda, \tilde{t}, k)$ is a non lineal function that depends on $\lambda$ and $k$, with the parameter $\tilde{t}$, which is known.

But in the function $S_{mb}(\lambda, \tilde{t}, k)$ variable $k$ is discrete, therefore the function to be optimized only depends on variable $\lambda$:

Maximize \quad $\tilde{S}_{mb}(\lambda)$

subject to \quad $\lambda \geq 0$  \hspace{1cm} (23)

and the process of optimization is carried out fixing the variable $k$ and calculating $\max \tilde{S}_{mb}(\lambda) \; \forall k = 1, 2, \ldots, [R]$.

According to the expression of $\alpha$-cut $P_{oda}$ (20), can be expressed the $\alpha$-cut of the function of the fuzzy standardized profit, depending on the variable of the model of optimization $\lambda$, as (Rommelfanger et al. [34]):

$$S_{mb}(\lambda) = \left[\tilde{S}_{mb}(\lambda), S_{mb}(\lambda)\right]$$  \hspace{1cm} (24)

with:

$$S_{mb}(\lambda) = \lambda \tilde{t}_{\alpha} \left(1 - \frac{k}{R}\right) \frac{1 - (\lambda \tilde{t}_{\alpha})^k}{1 - (\lambda \tilde{t}_{\alpha})^{k+1}} - \frac{\lambda \tilde{t}_{\alpha}}{R}$$  \hspace{1cm} (25)

Given that the coefficients of the objective function take the form of a fuzzy set the problem of optimization must contain the additional information which is supplies by the membership function $\tilde{t}$. For this reason, Rommelfanger et al. [34] consider a finite group of degree of $\alpha$-cuts, $\{\beta_1, \beta_2, \ldots, \beta_i\} \subset [0, 1)$ and the function to be optimized will be made up of $i$ objectives. For each degree $\beta_j, j = 1, \ldots, i$ the interval of variable $t$ is:

$$t_{\beta_j} = \left[t_{\beta_j}, \tilde{t}_{\beta_j}\right]$$  \hspace{1cm} (26)

and the optimization problem (23) can be reduced to the multi-objective non-lineal optimization problem consisting in:

Maximize $$\left[\tilde{S}_{mb}(\lambda), S_{mb}(\lambda), \tilde{S}_{mb}(\lambda), S_{mb}(\lambda), \ldots, \tilde{S}_{mb}(\lambda), S_{mb}(\lambda)\right]$$

s.t. \quad $\lambda \geq 0$  \hspace{1cm} (27)

with:

$$S_{mb}(\beta_j)(\lambda) = \lambda \tilde{t}_{\beta_j} \left(1 - \frac{k}{R}\right) \frac{1 - (\lambda \tilde{t}_{\beta_j})^k}{1 - (\lambda \tilde{t}_{\beta_j})^{k+1}} - \frac{\lambda \tilde{t}_{\beta_j}}{R}$$  \hspace{1cm} (28)

$$\tilde{S}_{mb}(\beta_j)(\lambda) = \lambda \tilde{t}_{\beta_j} \left(1 - \frac{k}{R}\right) \frac{1 - (\lambda \tilde{t}_{\beta_j})^k}{1 - (\lambda \tilde{t}_{\beta_j})^{k+1}} - \frac{\lambda \tilde{t}_{\beta_j}}{R}$$  \hspace{1cm} (29)

for all $j = 1, \ldots, i$. 

To solve the problem \( (27) \) they start by finding the optimum solution of each objective separately, that they denote \( \lambda_{\beta_j,\text{min}} \) and \( \lambda_{\beta_j,\text{max}} \), that is:

- \( \lambda_{\beta_j,\text{min}} \) is the optimum solution to the problem:

\[
\begin{align*}
\text{Maximize} & \quad \bar{S}_{\text{Mb}\beta_j}(\lambda) \\
\text{s.t.} & \quad \lambda \geq 0 \\
\end{align*}
\]  

(30)

- \( \lambda_{\beta_j,\text{max}} \) is the optimum solution to the problem:

\[
\begin{align*}
\text{Maximize} & \quad \tilde{S}_{\text{Mb}\beta_j}(\lambda) \\
\text{s.t.} & \quad \lambda \geq 0. \\
\end{align*}
\]  

(31)

In second place, with \( \lambda_{\beta_j,\text{min}} \) and \( \lambda_{\beta_j,\text{max}} \), \( \forall j = \{1, \ldots, i\} \) they calculate:

\[
\begin{align*}
S^+_{\beta_j,\text{max}} &= \tilde{S}_{\text{Mb}\beta_j}(\lambda_{\beta_j,\text{max}}) \\
S^+_{\beta_j,\text{min}} &= \bar{S}_{\text{Mb}\beta_j}(\lambda_{\beta_j,\text{min}}) \\
S^-_{\beta_j,\text{max}} &= \bar{S}_{\text{Mb}\beta_j}(\lambda_{\beta_j,\text{min}}) \\
S^-_{\beta_j,\text{min}} &= \tilde{S}_{\text{Mb}\beta_j}(\lambda_{\beta_j,\text{max}}). \\
\end{align*}
\]  

(32)

Next, they establish the membership functions of the objectives:

\[
\mu_{\tilde{S}_{\text{Mb}\beta_j}}(\lambda) = \begin{cases} 
1 & \text{if } S^+_{\beta_j,\text{max}} < \tilde{S}_{\text{Mb}\beta_j}(\lambda) \\
\frac{\bar{S}_{\text{Mb}\beta_j}(\lambda) - S^-_{\beta_j,\text{max}}}{S^+_{\beta_j,\text{max}} - S^-_{\beta_j,\text{max}}} & \text{if } S^-_{\beta_j,\text{max}} \leq \bar{S}_{\text{Mb}\beta_j}(\lambda) \leq S^+_{\beta_j,\text{max}} \\
0 & \text{if } \bar{S}_{\text{Mb}\beta_j}(\lambda) < S^-_{\beta_j,\text{max}}
\end{cases}
\]

(33)

\[
\mu_{\bar{S}_{\text{Mb}\beta_j}}(\lambda) = \begin{cases} 
1 & \text{if } S^+_{\beta_j,\text{min}} < \bar{S}_{\text{Mb}\beta_j}(\lambda) \\
\frac{\bar{S}_{\text{Mb}\beta_j}(\lambda) - S^-_{\beta_j,\text{min}}}{S^+_{\beta_j,\text{min}} - S^-_{\beta_j,\text{min}}} & \text{if } S^-_{\beta_j,\text{min}} \leq \bar{S}_{\text{Mb}\beta_j}(\lambda) \leq S^+_{\beta_j,\text{min}} \\
0 & \text{if } \bar{S}_{\text{Mb}\beta_j}(\lambda) < S^-_{\beta_j,\text{min}}
\end{cases}
\]

(34)

Finally they calculate the optimization problem:

\[
\begin{align*}
\text{Maximize} & \quad \alpha \\
\text{s.t.} & \quad \mu_{\tilde{S}_{\text{Mb}\beta_j}}(\lambda) \geq \alpha \\
& \quad \mu_{\bar{S}_{\text{Mb}\beta_j}}(\lambda) \geq \alpha \\
& \quad \forall j = \{1, \ldots, i\} \\
& \quad \lambda \geq 0
\end{align*}
\]  

(35)

or:

\[
\begin{align*}
\text{Maximize} & \quad \alpha \\
\text{s.t.} & \quad \frac{\bar{S}_{\text{Mb}\beta_j}(\lambda) - S^-_{\beta_j,\text{max}}}{S^+_{\beta_j,\text{max}} - S^-_{\beta_j,\text{max}}} \geq \alpha \\
& \quad \frac{\bar{S}_{\text{Mb}\beta_j}(\lambda) - S^-_{\beta_j,\text{min}}}{S^+_{\beta_j,\text{min}} - S^-_{\beta_j,\text{min}}} \geq \alpha \\
& \quad \forall j = \{1, \ldots, i\} \\
& \quad \lambda \geq 0
\end{align*}
\]  

(36)

The solution of the problem \( (27) \) (denoted \( \lambda^*_k \)) provides the value of \( \lambda \) that optimizes all objective functions \( S_{\text{Mb}\beta_j}(\lambda) \) and \( \tilde{S}_{\text{Mb}\beta_j}(\lambda) \), \( \forall j = \{1, \ldots, i\} \) simultaneously.

With the optimum solution \( \lambda^*_k \) the average of the objective function (average standardized profit) is calculated, given the value of \( k \) fixed initially:

\[
S^*_\text{Mb} (\lambda^*_k, k) = \frac{1}{2i} \sum_{j=1}^{i} \left[ S_{\text{Mb}\beta_j}(\lambda^*_k) + \tilde{S}_{\text{Mb}\beta_j}(\lambda^*_k) \right].
\]

(37)

And the process of optimization is resolved calculating \( S^*_\text{Mb} (\lambda^*_k, k) \) for all \( k, k = \{1, 2, \ldots, [R]\} \) and determining later the global optimum of \( S^*_\text{Mb} \) as:

\[
\text{Max } S^*_\text{Mb} = \text{max } \{ S^*_\text{Mb} (\lambda^*_1, 1), S^*_\text{Mb} (\lambda^*_2, 2), \ldots, S^*_\text{Mb} (\lambda^*_k, [R]) \}.
\]

(38)

Last, to carry out the optimization it must be kept in mind that the objective function of the queuing model under study fulfills (Atkinson [21]):

---
1. The functions $S_{M_{b_i}} (\lambda)$ and $\tilde{S}_{M_{b_i}} (\lambda)$ are unimodal functions for $\lambda \geq 0$, so they can be optimized with respect to this variable using methods of non linear optimization of bisection interval. Thus optimum values $\lambda_{b_i, \text{min}}$ and $\lambda_{b_i, \text{max}}$ are obtained, and with these values we can build the membership functions $\mu_{S_{M_{b_i}}}$ and $\mu_{\tilde{S}_{M_{b_i}}}$ defined for a given value $k$.

The optimization problem (35) with non linear restrictions can be solved using Sequential Unconstrained Minimization Techniques (Fiacco and McCormick [37]).

2. The discrete function of standardized profit depending on $k$, $S_{M_b} (\lambda, k)$ when $k = 1, 2, \ldots, [R]$, is also unimodal function and their optimum value is obtained in a $k$ value close to the unity. This way, an efficient orderliness of search consists in solving the previous procedure continually with growing values of $k$ (initiating in $k = 1$), until the optimum global value is obtained which will determine the optimum values of $k, \lambda^*_b, F^*$ and $S^*_M$. 

2. Word of mouth model

In the word of mouth model the fuzzy standardized profit function is:

$$\tilde{S}_{M_{w}} (\lambda, \bar{t}, k) = \lambda \bar{t} \left( 1 - \frac{k}{R} \right) \frac{1 - (\lambda \bar{t})^k}{1 - (\lambda \bar{t})^{k+1}} - \lambda \bar{t} \left( 1 - \frac{1 - (\lambda \bar{t})^{k+1}}{R (1 - (\lambda \bar{t})^{k+1})} \right).$$  \hspace{1cm} (39)

In this model, the problem of optimization is expressed as:

Maximize $\tilde{S}_{M_{w}} (\lambda, \bar{t}, k)$

subject to $\lambda \geq 0$  \hspace{1cm} (40)

where $\tilde{S}_{M_{w}} (\lambda, \bar{t}, k)$ is a non linear function that depends on $\lambda$ and $k$, with the parameter $\bar{t}$, which is known.

In a same way that in the basic model, in the function $\tilde{S}_{M_{w}} (\lambda, \bar{t}, k)$ variable $k$ is discrete, therefore the function to be optimized only depends on variable $\lambda$:

Maximize $\tilde{S}_{M_{w}} (\lambda)$

subject to $\lambda \geq 0$  \hspace{1cm} (41)

and the process of optimization is carried out fixing the variable $k$ and calculating Max $\tilde{S}_{M_{w}} (\lambda), \forall k = 1, 2, \ldots, [R]$.

According to the expression of $\alpha$-cut $P_{adu}$, the $\alpha$-cut of the fuzzy standardized profit function, depending on the variable of the model of optimization $\lambda$, is:

$$S_{M_{w \alpha}} (\lambda) = [S_{M_{w \alpha}} (\lambda), \tilde{S}_{M_{w \alpha}} (\lambda)]$$  \hspace{1cm} (42)

with:

$$S_{M_{w \alpha}} (\lambda) = \lambda t_\alpha \left( 1 - \frac{k}{R} \right) \frac{1 - (\lambda t_\alpha)^k}{1 - (\lambda t_\alpha)^{k+1}} - \lambda t_\alpha \left( 1 - \frac{1 - (\lambda t_\alpha)^{k+1}}{R (1 - (\lambda t_\alpha)^{k+1})} \right)$$  \hspace{1cm} (43)

$$\tilde{S}_{M_{w \alpha}} (\lambda) = \lambda \bar{t}_\alpha \left( 1 - \frac{k}{R} \right) \frac{1 - (\lambda \bar{t}_\alpha)^k}{1 - (\lambda \bar{t}_\alpha)^{k+1}} - \lambda \bar{t}_\alpha \left( 1 - \frac{1 - (\lambda \bar{t}_\alpha)^{k+1}}{R (1 - (\lambda \bar{t}_\alpha)^{k+1})} \right)$$  \hspace{1cm} (44)

With the $\alpha$-cuts $S_{M_{w \alpha}} (\lambda)$ and $\tilde{S}_{M_{w \alpha}} (\lambda)$ it is carried out the process of optimization described previously with the basic model.

The same as in the basic model, the functions $S_{M_{w \alpha}} (\lambda)$ and $\tilde{S}_{M_{w \alpha}} (\lambda)$ are unimodal functions for $\lambda \geq 0$, so they can be optimized with respect to this variable using methods of non linear optimization of bisection interval, and the optimization problem with non linear restrictions can be solved using Sequential Unconstrained Minimization Techniques (Fiacco and McCormick [37]).

Regarding the discrete function of standardized profit depending on $k$, $S_{M_w} (\lambda, k)$ when $k = 1, 2, \ldots, [R]$, is also unimodal function and their optimum value is obtained in a $k$ value close to the unity.

3.3.2. Method 2: Optimization of functions of fuzzy profit in queuing models

Next we expose the method proposed by us to optimize functions of fuzzy profit.

1. Basic model

In the basic model the fuzzy standardized profit function to be optimized is:

$$\tilde{S}_{M_b} (\bar{\rho}, k) = \bar{\rho} \left( 1 - \frac{k}{R} \right) \frac{1 - \bar{\rho}^k}{1 - \bar{\rho}^{k+1}} - \bar{\rho}.$$  \hspace{1cm} (45)
The fuzzy function $\tilde{S}_{mb}(\tilde{\rho}, k)$ have an uncertain variable $\tilde{\rho}$, but the same characteristics as the non-fuzzy function that correspond to the classic queuing model with publicity and renouncement, $S_{mb}(\rho, k)$: the fuzzy function depend on a variable $k$ discrete, reason why the function come to be function of only one variable $\rho$ according to $k$ take values in $1, 2, \ldots, [R]$. The proposed methodology to find the optimum value of fuzzy standardized profit function consists of:

1. To find the optimum global value of non-fuzzy standardized profit function:

$$S_{mb}(\rho, k) = \rho \left(1 - \frac{k}{R}\right) \frac{1 - \rho^k}{1 - \rho^{k+1}} - \rho \frac{1}{R}.$$  \hspace{1cm} (46)

Therefore, in the first place we consider that the variable $k$ is fixed and we calculate the optimum value of the variable $\rho$, that we denote $\rho_k$ and that satisfies:

$$\max S_{mb} (\rho) \quad \text{s.t.} \quad \rho \geq 0$$  \hspace{1cm} (47)

this way, for each $k$ we find the value $\rho_k$, that optimizes the non-fuzzy standardized profit function $S_{mb}(\rho_k, k)$. The function $S_{mb}(\rho_k, k)$ is unimodal function for $\rho \geq 0$, and we can optimize with respect to this variable using methods of non lineal optimization of bisection interval. When we know $\rho_k$ then we calculate for all $k = 1, \ldots, [R]$ the optimum value of the function of standardized profit $S_{mb}(\rho_k, k)$, so that now we know $S_{mb}(\rho_1, 1), S_{mb}(\rho_2, 2), \ldots, S_{mb}(\rho_{[R]}, [R])$.

Next we calculate the optimum global value of the function of non-fuzzy standardized profit:

$$\max S_{mb} = \max \{S_{mb}(\rho_1, 1), S_{mb}(\rho_2, 2), \ldots, S_{mb}(\rho_{[R]}, [R])\} = S_{mb}(\rho_i, i).$$  \hspace{1cm} (48)

Here, $\rho_i$ denote the value of $\rho$ that optimizes the function of non-fuzzy standardized profit in (48) and $i$ is the value of $k$ that corresponds to the optimum global profit, this way $(\rho_i, i)$ is the pair of values $(\rho_k, k)$ that provide the optimum global value of the function of non-fuzzy standardized profit $S_{mb}$.

The discrete function of non-fuzzy standardized profit depending on $k$, $S_{mb}(\rho_k, k)$ when $k = 1, 2, \ldots, [R]$, is also unimodal function on $k$ and their optimum value is obtained in a value $k$ close to the unity. This way, an efficient procedure of search consists in solving the previous calculation continually with growing values of $k$ (initiating in $k = 1$), until the optimum global value is obtained which will determine the optimum values $i$, $\rho_i$ and $S_{mb}$.

2. Calculated in step 1 the value optimum $\rho_i$, and given that $\rho = \lambda \cdot \tilde{t}$, then the optimum $\lambda$ is:

$$\lambda = \frac{\rho_i}{\tilde{t}}.$$  \hspace{1cm} (49)

Then $\lambda$ is a fuzzy subset and we can calculate with the Zadeh’s extension principle [28], and applying the Buckley and Qu’s method [31], the membership function of the arrival rate $\lambda$, function that optimize the fuzzy queuing models with publicity and renouncement:

$$\mu_{\lambda}(\lambda) = \mu_{\tilde{t}}(t) \cdot \mu_{\rho_i}(t).$$  \hspace{1cm} (50)

In this model $P_{mb} = \lambda$, so we have also calculated $\mu_{\tilde{t}}(P)$.

3. To finish, with the membership functions $\mu_{\lambda}(\lambda)$ and $\mu_{\tilde{t}}(t)$, and using another time the Zadeh’s extension principle [28], and applying the Dong and Shah’s vertex method [38] and Dubois et al. [39], the membership function of the fuzzy standardized profit function is calculated:

$$\mu_{\tilde{S}_{mb}}(S_{mb}) = \sup \min \left\{ \mu_{\lambda}(\lambda), \mu_{\tilde{t}}(t) / S_{mb}(\lambda, t, i) = \lambda \left(1 - \frac{i}{R}\right) \frac{1 - (\lambda \cdot \tilde{t})^i}{1 - (\lambda \cdot \tilde{t})^{i+1}} - \frac{\lambda \cdot \tilde{t}}{R} \right\}.$$  \hspace{1cm} (51)

2. Word of mouth model

The process of optimization is the one proposed for the fuzzy standardized profit function of the basic model keeping in mind that in the word of mouth model the fuzzy standardized profit function is:

$$\tilde{S}_{mw}(\tilde{\rho}, k) = \tilde{\rho} \left(1 - \frac{k}{R}\right) \frac{1 - \tilde{\rho}^k}{1 - \tilde{\rho}^{k+1}} - \tilde{\rho} \frac{1 - (1 - \tilde{\rho}^{k+1})}{R (1 - \tilde{\rho}^k)}.$$  \hspace{1cm} (52)

Here the function $S_{mw}(\rho, k)$ is also unimodal function for $\rho \geq 0$, and we can optimize with respect to $\rho$ using methods of non lineal optimization of bisection interval. And the discrete function of standardized profit depending on $k$, $S_{mw}(\rho_k, k)$ when $k = 1, 2, \ldots, [R]$, is also unimodal function on $k$ and their optimum value is obtained in a value $k$ close to the unity.

In this model:

$$\tilde{P}_{mw} = \tilde{\rho} \left(1 - \frac{(\tilde{\lambda} \tilde{t})^{k+1}}{1 - (\tilde{\lambda} \tilde{t})^k}\right) \cdot \frac{\tilde{\rho} (1 - \tilde{\rho}^{k+1})}{R (1 - \tilde{\rho}^k)}.$$  \hspace{1cm} (53)
so, with the membership functions \( \mu_{\tilde{S}} (\lambda) \) and \( \mu_{i} (t) \), and using the Zadeh’s extension principle [28] the membership function of level of expense in publicity is:

\[
\mu_{\tilde{P}_{MW}} (P) = \sup \min \left\{ \mu_{\tilde{S}} (\lambda), \mu_{i} (t) / P_{MW} = \lambda \frac{1 - (\lambda t)^{i+1}}{1 - (\lambda t)^{i}} \right\}
\]  

and the membership function of the fuzzy standardized profit function is:

\[
\mu_{\tilde{S}_{MW}} (S_{MW}) = \sup \min \left\{ \mu_{\tilde{S}} (\lambda), \mu_{i} (t) / S_{MW} (\lambda, t, i) = \lambda t \left( 1 - \frac{i}{R} \right) \frac{1 - (\lambda t)^{i}}{1 - (\lambda t)^{i+1}} - \lambda t \right\}.
\]  

4. Example

It is considered as an illustration of the process of optimization the case in which the facilities server serves according to exponential distribution with mean value vaguely known, and it is described in form of the triangular fuzzy number \( \tilde{t} = [3, 4, 5] \) per unit of time. The evaluation that the user does of the service is \( R = 15 \) monetary units. In this case, the possible price \( F \) is between 1 and 15. We will calculate the optimum value of the arrival rate of the customers into the system (publicity level) and the price the manager must fix in the facilities to maximize his profit, for both fuzzy queuing models with publicity and renouncement, basic model and word of mouth model, and using the two methods proposed for the optimization.

1. Basic model

A. Method 1

The process of optimization begins by giving values to \( k \) from 1 to 15.

- Case \( M/M/1/1 \) with \( k = 1 \).

In this first case \( F = 14 \). The 4 \( \alpha \)-cuts which have been considered in the study are:

\[
\beta_1 = 0, \quad \beta_2 = 0.25, \quad \beta_3 = 0.5, \quad \beta_4 = 0.75
\]

and with \( \tilde{t} = [3, 4, 5] \) is \( t_{\beta_i} = \left[ \beta_i, \tilde{t}, \tilde{t} \right] = [\beta_i + 3, 5, 3] \), \( j = 1, 2, 3, 4 \). The method leads, as it has been mentioned in the previous section, to the optimization of the 8 functions of profit separately.

Next we only show the results for the \( \alpha \)-cut \( \beta_1 = 0 \), with \( t_{\beta_1} = t_0 = [0, 0] = [3, 5] \). The functions \( S_{M00} (\lambda) \) and \( \tilde{S}_{M00} (\lambda) \) are:

\[
S_{M00} (\lambda) = 3\lambda \left( 1 - \frac{1}{15} \right) \frac{1 - 5\lambda}{1 - (5\lambda)^2} = \frac{5\lambda}{15}, \quad \tilde{S}_{M00} (\lambda) = 5\lambda \left( 1 - \frac{1}{15} \right) \frac{1 - 3\lambda}{1 - (3\lambda)^2} = \frac{3\lambda}{15}.
\]

The optimization problems are:

\[
\begin{align*}
\text{Maximize} & \quad \tilde{S}_{M00} (\lambda) \quad \text{s.t.} \quad \lambda \geq 0 \\
\text{Maximize} & \quad \tilde{S}_{M00} (\lambda) \quad \text{s.t.} \quad \lambda \geq 0.
\end{align*}
\]

Then the optimum solution of each objective, \( S_{M00} (\lambda) \) and \( \tilde{S}_{M00} (\lambda) \), is:

\[
\lambda_{0, \text{min}} = 0.38 \quad \lambda_{0, \text{max}} = 1.27
\]

therefore:

\[
\begin{align*}
S_{0, \text{max}} &= \tilde{S}_{M00} (\lambda_{0, \text{max}}) = \tilde{S}_{M00} (1.27) = 0.978 \\
S_{0, \text{min}} &= \tilde{S}_{M00} (\lambda_{0, \text{min}}) = \tilde{S}_{M00} (0.38) = 0.753 \\
S_{0, \text{min}} &= \tilde{S}_{M00} (\lambda_{0, \text{min}}) = \tilde{S}_{M00} (0.38) = 0.24 \\
S_{0, \text{min}} &= \tilde{S}_{M00} (\lambda_{0, \text{max}}) = \tilde{S}_{M00} (1.27) = 0.06.
\end{align*}
\]

The membership functions of the objectives are:

\[
\begin{align*}
\mu_{\tilde{S}_{M00}} (\lambda) &= \begin{cases} 
1 & \text{if } 0.978 < \tilde{S}_{M00} (\lambda) \\
\frac{S_{M00} (\lambda) - 0.753}{0.978 - 0.753} & \text{if } 0.753 \leq \tilde{S}_{M00} (\lambda) \leq 0.978 \\
0 & \text{if } 0.24 < \tilde{S}_{M00} (\lambda)
\end{cases} \\
\mu_{\tilde{S}_{M00}} (\lambda) &= \begin{cases} 
1 & \text{if } 0.24 \leq \tilde{S}_{M00} (\lambda) < 0.753 \\
\frac{S_{M00} (\lambda) - 0.06}{0.24 - 0.06} & \text{if } 0.06 \leq \tilde{S}_{M00} (\lambda) \leq 0.24 \\
0 & \text{if } \tilde{S}_{M00} (\lambda) < 0.06.
\end{cases}
\end{align*}
\]
optimization of bisection interval we calculate the maximum of the function \( S \), uncertainty of input data, will be very useful for designing fuzzy queuing models with publicity and renouncement.

\[ \lambda \]

\( \alpha \) in which membership function of fuzzystandardized profit is calculated using Dong and Shah’s vertex method [38].

\( S_n \) Table 1

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_{mb} )</td>
<td>[0.599, 0.614]</td>
<td>[0.606, 0.614]</td>
<td>[0.611, 0.614]</td>
<td>[0.613, 0.614]</td>
<td>0.614</td>
</tr>
</tbody>
</table>

And the two constrains of the problem of optimization that correspond to the \( \alpha \)-cut \( \beta_1 = 0 \) are:

\[
\frac{\hat{S}_{mb}(\lambda) - 0.753}{0.978 - 0.753} \leq \alpha \Rightarrow 20.67\alpha \frac{1 - 3\lambda}{1 - 9\lambda^2} - 0.89\lambda - 3.33 \geq \alpha
\]

\[
\frac{\hat{S}_{mb}(\lambda) - 0.06}{0.24 - 0.06} \leq \alpha \Rightarrow 15.41\alpha \frac{1 - 5\lambda}{1 - 25\lambda^2} - 1.83\lambda - 0.22 \geq \alpha.
\]

Repeating this process for the \( \alpha \)-cuts \( \beta_2 = 0.25, \beta_3 = 0.5 \) and \( \beta_4 = 0.75 \), the remaining constrains that are part of the problem of optimization (35) are obtained.

The solution is:

\[ \alpha^* = 0.7095 \quad \lambda_1^* = 0.6959 \]

which has an average standardized profit: \( S_{mb}^*(\lambda_1^*, 1) = S_{mb}^* (0.6959, 1) = 0.5274 \).

We compare these results with which we obtain in the second case with \( k = 2 \).

**Case \( M/M/1/2 \) with \( k = 2 \).**

A similar process to the previous case is carried out, with the difference that \( k = 2 \). The solution of case 2, with \( F = 13 \), is:

\[ \alpha^* = 0.7262 \quad \lambda_2^* = 0.5759 \]

which has an average standardized profit: \( S_{mb}^*(\lambda_2^*, 2) = S_{mb}^* (0.5759, 2) = 0.6443 \).

In case 2 the average standardized profit remains on top of case 1. It is necessary to continue the procedure to observe if it is possible to improve the solution.

**Case \( M/M/1/3 \) with \( k = 3 \).**

As in previous cases, there is a problem of maximization with 8 restrictions. The solution to case 3, with \( F = 12 \), is:

\[ \alpha^* = 0.7333 \quad \lambda_3^* = 0.4854 \]

which has an average standardized profit: \( S_{mb}^*(\lambda_3^*, 3) = S_{mb}^* (0.4854, 3) = 0.6464 \).

In this case, we observe that \( S_{mb}^*(\lambda_3^*, 3) \) is slightly superior to \( S_{mb}^*(\lambda_2^*, 2) \), for this reason it is preferable the case 3 and the method of successive optimizing must continue.

**Case \( M/M/1/4 \) with \( k = 4 \).**

In the same manner as in the previous cases arises a maximization problem with 8 restrictions. The solution in case 4, with \( F = 11 \), is:

\[ \alpha^* = 0.7360 \quad \lambda_4^* = 0.4278 \]

which has an average standardized profit: \( S_{mb}^*(\lambda_4^*, 4) = S_{mb}^* (0.4278, 4) = 0.6128 \).

In this case, with \( F = 11 \), the fuzzy standardized profit has decreased considerably with respect to the previous case. It is possible to see that the decrease will be greater from now on, for this reason the solution to the problem is found in the previous case with \( \lambda_3^* = 0.4854 \) customers per unit of time, a level of publicity \( P_{mb}^* = 0.4854 \) monetary units and a price \( F^* = 12 \) monetary units.

With these results, the \( \alpha \)-cuts of optimum fuzzy standardized profit \( S_{mb}^* \) of basic model are shown in Table 1. The membership function of fuzzy standardized profit is calculated using Dong and Shah’s vertex method [38] in the function:

\[
S_{mb}^* \left( \lambda_3^*, 3 \right) = \lambda_3^* t_\alpha \left( 1 - \frac{3}{15} \frac{1 - \left( \lambda_3^* t_\alpha \right)^3}{1 - \left( \lambda_3^* t_\alpha \right)^4} - \frac{\lambda_3^* t_\alpha}{15} \right)
\]

in which \( \lambda_3^* = 0.4854 \) and \( \alpha \) take the values 0, 0.25, 0.5, 0.75 and 1. All this information, that conserves completely the uncertainty of input data, will be very useful for designing fuzzy queuing models with publicity and renouncement.

**B. Method 2**

The non-fuzzy standardized profit function to be optimized is:

\[
S_{mb} (\rho, k) = \rho \left( 1 - \frac{k}{15} \right) \frac{1 - \rho^k}{1 - \rho^{k+1}} - \frac{\rho}{15}.
\]

The optimization process starts, as in the previous method, giving values to \( k \) from 1 to 15. Using methods of non lineal optimization of bisection interval we calculate the maximum of the function \( S_{mb} \) and the value of the variable \( \rho \) which
permits reaching the optimum. In Table 2 the values $\rho_k$ are indicated which optimize $S_{MB} (\rho_k, k)$ for the different values of $k$ next to $k = 1$, as well as the value that the optimum standardized profit acquires.

Starting from $k = 3$, the standardized profit decreases considerably with respect to $k = 3$. It is possible to check that the decrease will be greater from $k = 3$, so that:

$$\max S_{MB} = S_{MB} (\rho_3, 3) = 0.6136$$

with $\rho_3 = 1.9327$ and, for this reason, the solution to the problem is found with a price $F^* = 12$ monetary units.

Obtained $k = 3$ and $\rho_3$, the membership functions of $\hat{\lambda}$ and $\tilde{S}_{MB}$ are calculated per $\alpha$-cuts. The $\alpha$-cuts $\lambda_\alpha = \rho_3 / t_\alpha$ are calculated with the Bucley and Qu’s method [31] and the $\alpha$-cuts $S_{Mbu}$ are calculated applying the Dong and Shah’s vertex method [38] and Dubois et al. [39] in the function:

$$\tilde{S}_{Mbu} (\lambda, t) = \lambda t \left( 1 - \frac{3}{15} \frac{1 - (\lambda t \alpha)^3}{1 - (\lambda t \alpha)^4} - \frac{\lambda t \alpha}{15} \right).$$

In this model $P_{MB} = \lambda$, and the optimum fuzzy publicity level is also calculated. The results are shown in Table 3.

Comparing the solutions obtained with both methods, we will find out that the solutions supply in both cases very similar results, the validity of both procedures is demonstrated, with the advantage that method 2 afford a computation cost lower than method 1.

2. Word of mouth model

Under the same considerations as in the basic model, we study the case of word of mouth model with publicity and renouncement.

A. Method 1

The optimization process starts giving values to $k$ of 1 to 15. The procedure is similar to the one shown in the preceding model with the difference that now the $\alpha$-cuts the functions of fuzzy standardized profit are:

$$S_{Mwu} (\lambda) = \lambda t_\alpha \left( 1 - \frac{k}{15} \frac{1 - (\lambda t \alpha)^3}{1 - (\lambda t \alpha)^4} - \frac{\lambda t \alpha}{15} \right)$$

$$\tilde{S}_{Mwu} (\lambda) = \lambda t_\alpha \left( 1 - \frac{k}{15} \frac{1 - (\lambda t \alpha)^3}{1 - (\lambda t \alpha)^4} - \frac{\lambda t \alpha}{15} \right)$$

In Table 4 the results obtained for $\alpha^*, \lambda_k^*, F$ and $S_{Mwu} (\lambda_k^*, k)$ are shown.

In this model, we can observe that arrival rates and profits are lower than the basic model, which is due to the construction of the profit function.

For the cases $k = 2$ and $k = 3$ the solution has a mean standardized profit $S_{Mwu} (\lambda_k^*, k)$ superior to the profit of the previous case and the process of optimization must continue.

With $k = 4$ the standardized profit decreases sensibly with respect to the previous case and the decrease will be greater from $k = 4$ so that the solution to the problem is found in the $M/\tilde{M}/1/3$ model, with $k = 3, \lambda_k^* = 0.3367$ customers per unit of time and a price $F^* = 12$ monetary units.

---

**Table 2**

$\rho_k$ and $S_{MB} (\rho_k, k)$, Basic model.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_k$</td>
<td>2.742</td>
<td>2.286</td>
<td>1.933</td>
<td>1.706</td>
<td>1.552</td>
<td>...</td>
</tr>
<tr>
<td>$S_{MB} (\rho_k, k)$</td>
<td>0.501</td>
<td>0.613</td>
<td>0.614</td>
<td>0.581</td>
<td>0.535</td>
<td>...</td>
</tr>
</tbody>
</table>

**Table 3**

$\lambda_\alpha$ and $S_{Mbu}$, Basic model.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_\alpha = P_{Mbu}$</td>
<td>[0.387, 0.644]</td>
<td>[0.407, 0.595]</td>
<td>[0.429, 0.552]</td>
<td>[0.455, 0.515]</td>
<td>0.483</td>
</tr>
<tr>
<td>$S_{Mbu}$</td>
<td>[0.569, 0.614]</td>
<td>[0.588, 0.614]</td>
<td>[0.602, 0.614]</td>
<td>[0.611, 0.614]</td>
<td>0.614</td>
</tr>
</tbody>
</table>

**Table 4**

$\alpha^*, \lambda_k^*, F$ and $S_{Mwu} (\lambda_k^*, k)$, Word of mouth model.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha^*$</td>
<td>0.666</td>
<td>0.681</td>
<td>0.686</td>
<td>0.687</td>
<td>...</td>
</tr>
<tr>
<td>$\lambda_k^*$</td>
<td>0.281</td>
<td>0.339</td>
<td>0.337</td>
<td>0.326</td>
<td>...</td>
</tr>
<tr>
<td>$F$</td>
<td>14</td>
<td>13</td>
<td>12</td>
<td>11</td>
<td>...</td>
</tr>
<tr>
<td>$S_{Mwu} (\lambda_k^*, k)$</td>
<td>0.344</td>
<td>0.514</td>
<td>0.552</td>
<td>0.540</td>
<td>...</td>
</tr>
</tbody>
</table>
In the word of mouth model with publicity and renouncement, the optimum publicity $P_{MW}$:

$$P_{MW} = \lambda \frac{1 - \rho^{k+1}}{1 - \rho^k} = \frac{1 - (\lambda t)^{k+1}}{1 - (\lambda t)^k}$$

acquires the form of a fuzzy number whose $\alpha$-cuts have the expression (it is calculated following the Bucley and Qu's method [31]):

$$P_{MW, \alpha} = \left[ P_{MW, \alpha}, \tilde{P}_{MW, \alpha} \right] = \left[ \lambda_3^{\alpha} \frac{1 - (\lambda^3 t_\alpha)^4}{1 - (\lambda^3 t_\alpha)^3}, \lambda_4^{\alpha} \frac{1 - (\lambda^4 t_\alpha)^4}{1 - (\lambda^4 t_\alpha)^3} \right]$$

with $\lambda_3^\ast = 0.3367$ and $\tilde{t} = [3, 4, 5]$, then $\lambda_4^\ast \tilde{t} = [1.0101, 1.3468, 1.6835]$ and we obtain:

$$P_{MW, \alpha} = \left[ 0.3367 \frac{1 - (1.0101 + 0.3367 \alpha)^4}{1 - (1.0101 + 0.3367 \alpha)^3}, 0.3367 \frac{1 - (1.6835 - 0.3367 \alpha)^4}{1 - (1.6835 - 0.3367 \alpha)^3} \right].$$

The $\alpha$-cuts $S_{MW, \alpha}$ are calculated applying the Dong and Shah's vertex method [38] and Dubois et al. [39] in the function:

$$S_{MW, \alpha} (\lambda_3^\ast, t) = \lambda_3^\ast t_\alpha \left( 1 - \frac{3}{15} \right) \frac{1 - (\lambda_3^\ast t_\alpha)^3}{1 - (\lambda_3^\ast t_\alpha)^4} - \lambda_4^\ast t_\alpha \left( 1 - (\lambda_4^\ast t_\alpha)^4 \right) \frac{1 - (\lambda_3^\ast t_\alpha)^4}{15 \left( 1 - (\lambda_3^\ast t_\alpha)^3 \right)}.$$

In Table 5 the optimum fuzzy values of the publicity $P_{MW, \alpha}$, and the fuzzy standardized profit $S_{MW, \alpha}$ are shown, for the 4 $\alpha$-cuts analyzed.

**B. Method 2**

In the word of mouth model, the non-fuzzy standardized profit function to be optimized is:

$$S_{MW} (\rho, k) = \rho \left( 1 - \frac{k}{15} \right) \frac{1 - \rho^k}{1 - \rho^{k+1}} - \rho \frac{1 - \rho^{k+1}}{15 \left( 1 - \rho^k \right)}.$$  

As in the previous methods, the process of optimization starts by giving values to $k$ from 1 to 15. Using methods of non linear optimization of bisection interval, we calculate the maximum of the function $S_{MW}$ and the corresponding value $\rho$. In Table 6 the values $\rho_k$ that optimize $S_{MW} (\rho_k, k)$ are shown, for the different values of $k$ next to $k = 1$, as well as the values that the optimum standardized benefit acquires.

Comparing the results we observe that max $S_{MW} = S_{MW} (\rho_3, 3) = 0.5364$ with $\rho_3 = 1.3254$. Similar to what happens in the basic model, the optimum solution is obtained by $F^* = 12$ monetary units.

Once determined the values of $k = 3$ and $\rho_3$, we calculate by $\alpha$-cuts the membership functions $\hat{\lambda}$ and $\tilde{S}_{MW}$. The $\alpha$-cuts $\lambda_\alpha = \rho_3 / t_\alpha$ are calculated following the Bucley and Qu's method [31] and the $\alpha$-cuts $S_{MW, \alpha}$ are calculated applying the Dong and Shah's vertex method [38] and Dubois et al. [39] in the function:

$$S_{MW, \alpha} (\lambda_\alpha, t) = \lambda_\alpha t_\alpha \left( 1 - \frac{3}{15} \right) \frac{1 - (\lambda_\alpha t_\alpha)^3}{1 - (\lambda_\alpha t_\alpha)^4} - \lambda_\alpha t_\alpha \left( 1 - (\lambda_\alpha t_\alpha)^4 \right) \frac{1 - (\lambda_\alpha t_\alpha)^4}{15 \left( 1 - (\lambda_\alpha t_\alpha)^3 \right)}.$$

The results are shown in Table 7.

In this model the fuzzy level of publicity acquires the expression:

$$\tilde{P}_{MW} = \frac{1 - (\hat{\lambda} t_\alpha)^4}{1 - (\hat{\lambda} t_\alpha)^3}.$$
Given that the non-fuzzy function of publicity level $P_{Mw}$, which depends on $\lambda$ and $t$, is monotonous growing with respect to both variables, we can obtain the membership function of the fuzzy variable $\tilde{P}_{Mw}$ through the use of $\alpha$-cuts, for the 4 $\alpha$-cuts established previously, using the Bucley and Qu’s method \cite{31}. Every $\alpha$-cut has the expression:

$$P_{Mw,\alpha} = \frac{1 - (\tilde{\lambda}_0 \tilde{t}_0)^{\frac{4}{3}}}{1 - (\tilde{\lambda}_0 \tilde{t}_0)^{\frac{1}{3}}}$$

In Table 8 the results $\tilde{P}_{Mw}$ are shown.

When in the word of mouth model we compare the solutions obtained with both methods, we discover that they attain very similar results, logically we observe differences in the expense level in publicity $\tilde{P}_{Mw}$ and in the fuzzy standardized profit $\tilde{S}_{Mw}$ as a consequence that in method 1 the result of the arrival rate which we obtain with the optimization is a non-fuzzy mean value $\lambda^*_\alpha$, different to the obtained result when applying method 2, where the arrival rate which optimizes the queuing model with publicity and renouncement is fuzzy $\tilde{\lambda}$, uncertainty that is reflected in the results, and that provokes in turn a larger uncertainty in $\tilde{P}_{Mw}$ and $\tilde{S}_{Mw}$, but this difference doesn’t interfere in the demonstration of the validity of both procedures, with the advantage that method 2 supposes, in the same way as in the basic model, a computation cost lower to the method 1.

5. Conclusions

The Fuzzy Sets Theory has been applied to some systems of simple queues in order to supply larger applications in many fields, but relatively few papers have been published for fuzzy queuing models complicated, such as queuing models with publicity and renouncement or queue decision models.

In real systems, the parameters of the queue decision models can be known in an uncertain way due to uncontrollable factors, in such a way that the performance measures of the systems, such as the number of clients in a queue or the probability of access to the system, as well as the standardized profit, also becomes fuzzy. We clearly lose useful information if the obtained results are crisp values. In this paper, the standardized profit functions are expressed as membership functions which completely keep the uncertainty of the initial information when the service time is fuzzy. The proposed method allows us to obtain reasonable solutions, appropriate for all cases, with different levels of possibility ranging from the pessimist case to the optimistic, thus more information for the design of the queuing system with publicity and renouncement is supplied.

The incorporation of service times as fuzzy numbers implies that the objective functions of standardized profit to optimize have fuzzy coefficients, for this reason it is necessary to use fuzzy optimizing techniques for the solution of the problem. Besides, a new technique proposed for to optimize the functions of fuzzy profit of queuing models, together with satisfactory resolution of a fuzzy queuing system, often found in real situations, permits to demonstrate the validity of the procedure established.

Clearly, the proposed approach is not confined to the fuzzy queuing model with uncertainty service rate, or with publicity and renouncement. The concept embedded in the proposed approach can be expanded to the queuing problems with both, arrivals rate and service rate, being fuzzy, or can be also extended to evaluate and solve more complicated fuzzy queuing models.

The ability to analyze fuzzy queuing models with publicity and renouncement developed in this paper, and the satisfactory extension of decision models to fuzzy environments allows that the queuing decision models with publicity and renouncement can have a broader range of applications.

References


