Bifurcation of elastoplastic pressure-sensitive spheres

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Abstract

Instability of a full sphere, under uniform compression or tension, is investigated within the framework of linear bifurcation theory. Material response is modeled by a Hookean-type hypoelastic relation with pressure dependent instantaneous moduli. Exploiting a formal analogy with Navier equations of linear elasticity we obtain exact solution for the bifurcated field. Two families of eigenmodes and eigenvalues are identified and discussed. We examine, in particular, surface instabilities in compression, and twisting modes with absence of radial velocity. The results are further specified for pressure-sensitive plastic solids with an elliptic yield surface. The deformation theory prediction of bifurcation loads for that material are much lower than those obtained from the flow theory version.

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1. Introduction

Spherical particles and spherical cavities are central elements in modeling elastoplastic response of granular solids and porous materials. Yet, while possible bifurcations at a spherical cavity have been examined in detail [1,2], not much appears to be available in the literature on stability loss of spherical bodies. An early paper by Wesolowski [3] considers nonlinear elastic materials with possible modes of bifurcation for a variety of boundary conditions.

In this paper we examine the simple bifurcation problem of a full sphere under uniform radial load, imposed over the surface. Material behavior is governed by a pressure-sensitive elastoplastic theory, based on an elliptic yield function and accounting for arbitrary hardening.

We begin, in the next section, with a derivation of the rate equilibrium equation. With a hydrostatic primary path, rate equilibrium is expressed as the divergent of the Jaumann stress rate. Assuming a hypoelastic Hookean-like constitutive relation, with two pressure dependent instantaneous moduli, we arrive at Navier-type equations for the three components of the perturbed velocity vector. On the surface of the sphere we impose dead-load conditions, that imply vanishing of traction rate components. This formulation generates an eigen-system of three equations with critical pressure eigenvalues.

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Two families of separation of variables solutions are presented in Section 3. We exploit available methods of solution in linear elasticity [4] due to the mathematical similarity with the present equations. The first family of eigenmodes follows from the observation that the rate of dilation must be harmonic. The second family of eigenmodes describes twisting bifurcation with vanishing radial velocity and absence of dilation rate.

Universal equations are obtained for critical loads in both bifurcation patterns. These are examined with emphasis on the smallest eigenvalues and their constitutive sensitivity. Of particular interest are surface bifurcations obtained with the first family of eigensolutions when the sphere is subjected to external compression. The mirror problem, of bifurcation at a spherical cavity embedded in an infinite medium, is briefly discussed. We show that bifurcation loads in the cavity problem can be obtained from the critical loads for the sphere by a simple transformation of the mode number.

An elastoplastic constitutive model is developed in Section 4 with a pressure-sensitive elliptic yield surface. Both flow and deformation theories are derived with the expected observation that deformation theory predicts a much lower instantaneous shear modulus. Finally, in Section 5, we discuss an illustrative example with power-law plastic response. Simple relations are given for critical strains and are shown to depend on plastic pressure sensitivity.

2. Rate equilibrium equations

We consider a spherical body made of pressure-sensitive elastoplastic material and subjected to uniform radial traction over the surface. The class of solids examined in this paper deforms in a spherical pattern with the Cauchy stress field taking the hydrostatic form

\[ \sigma = \sigma_a I, \quad \text{(2.1)} \]

where \( \sigma_a \) is the applied load and \( I \) denotes the second order unit tensor. We assume that the deformation rate is slow, so the problem can be considered quasi-static. Therefore, the equilibrium equation takes the form

\[ \nabla \cdot \sigma = 0, \quad \text{(2.2)} \]

which is satisfied identically by (2.1), with \( \nabla \) standing for the gradient operator. It is conceivable, however, that bifurcation away from the primary path (2.1) may occur at critical load levels.

Onset of bifurcation is governed by the rate equilibrium equation (e.g. [5])

\[ \nabla \cdot (\nabla W + \sigma + \sigma \cdot W + \sigma I \cdot D) = 0, \quad \text{(2.3)} \]

where the Eulerian strain rate \( D \) and the spin \( W \) are, respectively, the symmetric and anti-symmetric parts of the perturbed velocity gradient \( \nabla V \), namely

\[ D = \frac{1}{2}(\nabla V + V \nabla), \quad W = \frac{1}{2}(\nabla V - V \nabla), \quad \text{(2.4)} \]

where \( \nabla V \) is the right gradient of \( V \), and relates to \( \nabla W \) by \( \nabla W = (\nabla V)^T \). The objective Jaumann stress rate is defined by

\[ \nabla \sigma = \dot{\sigma} + W \cdot \sigma - \sigma \cdot W, \quad \text{(2.5)} \]

with the superposed dot representing differentiation with respect to a time-like parameter.

Locating the origin of an Eulerian spherical system of coordinates \( (r, \theta, \phi) \) at the center of the sphere, with the associated orthonormal triad \( (e_r, e_\theta, e_\phi) \), we write the perturbed velocity vector as

\[ V = u e_r + v e_\theta + w e_\phi, \quad \text{(2.6)} \]

where the velocity components \( (u, v, w) \) need to be determined. Inserting (2.6) in (2.4) we obtain the strain rate and spin tensors

\[ D = d_{rr} e_r e_r + d_{\theta \theta} e_\theta e_\theta + d_{\phi \phi} e_\phi e_\phi + d_{r \theta} (e_r e_\theta + e_\theta e_r) + d_{r \phi} (e_r e_\phi + e_\phi e_r) + d_{\theta \phi} (e_\theta e_\phi + e_\phi e_\theta) + d_{\phi r} (e_\phi e_r + e_r e_\phi), \quad \text{(2.7)} \]

\[ W = w_{r \theta} (e_r e_\theta - e_\theta e_r) + w_{\theta \phi} (e_\theta e_\phi - e_\phi e_\theta) + w_{\phi r} (e_\phi e_r - e_r e_\phi), \quad \text{(2.8)} \]
with the components
\[
\begin{align*}
    d_{rr} &= u, \\
    d_{r\theta} &= \frac{1}{r} (v, u + v) \\
    d_{\theta\theta} &= \frac{1}{r} (w, \phi + u + v \cot \theta) \\
    d_{\phi r} &= \frac{1}{2r} (w, \phi + u + v \cot \theta) \\
    d_{r\phi} &= \frac{1}{2r} (w, \phi + u + v \cot \theta) \\
    d_{\phi\phi} &= \frac{1}{2r} (w, \phi + u + v \cot \theta)
\end{align*}
\]

(2.9)

Considerable simplification of the rate equilibrium equation (2.3) is gained when the primary stress field is hydrostatic, as in our case. To this end we substitute (2.1) and (2.4) in (2.3) to obtain
\[
\nabla \cdot \sigma - \sigma_a \nabla \cdot (V \nabla - IV \cdot V) = 0.
\]

(2.11)

However, by a standard theory of tensor calculus, the second term in (2.11) vanishes identically, rendering the simple form
\[
\nabla \cdot \sigma = 0.
\]

(2.12)

The superficial resemblance of (2.12) to (2.2) suggests an analogy which can probably be carried further. Thus, in the spirit of linear elasticity we examine the family of hypoelastic solids
\[
\nabla \cdot \sigma = 2\mu_i D + \lambda_i I \cdot D,
\]

(2.13)

where the instantaneous Lamé-type moduli \((\mu_i, \lambda_i)\) depend on the hydrostatic stress \(\sigma_a\). A specific example of the constitutive relation (2.13), for pressure-sensitive materials, is discussed in Sections 4 and 5. Instantaneous volume changes are given by the first invariant of (2.13), namely
\[
I \cdot \sigma = 3\kappa_i I \cdot D
\]

(2.14)

where the instantaneous bulk modulus is given by
\[
\kappa_i = \frac{2}{3} \mu_i + \lambda_i
\]

(2.15)

and may depend on hydrostatic stress.

Now, we substitute (2.13) in (2.12) and use the strain rate components (2.9) to obtain the Navier-type equations
\[
\mu_i \nabla^2 V + (\mu_i + \lambda_i) V I_D = 0
\]

(2.16)

or, in terms of velocity components
\[
\begin{align*}
    \mu_i \left[ \nabla^2 u - \frac{2u}{r^2} - \frac{2(u \sin \theta)}{r^2 \sin \theta} \right] + (\mu_i + \lambda_i) I_{D,r} &= 0 \\
    \mu_i \left[ \nabla^2 v + \frac{v}{r^2 \sin^2 \theta} - \frac{2w, \phi}{r^2 \sin \theta} \right] + (\mu_i + \lambda_i) I_{D,\theta} &= 0 \\
    \mu_i \left[ \nabla^2 w + \frac{w \cot \theta}{r^2 \sin \theta} + \frac{w}{r^2 \sin^2 \theta} \right] + (\mu_i + \lambda_i) - \frac{I_{D,\phi}}{r \sin \theta} &= 0.
\end{align*}
\]

(2.17)
Here the Laplacian is applied in spherical coordinates

$$\nabla^2 = \nabla \cdot \nabla = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

(2.18)

and the rate of dilation follows from (2.9) as

$$I_D = \nabla \cdot \mathbf{V} = u_r + \frac{2u}{r} + \frac{1}{r} (v_\theta + v \cot \theta) + \frac{w_\phi}{r \sin \theta}.$$  

(2.19)

Equations (2.17) are supplemented by homogeneous boundary conditions, imposed over the surface \( r = a \), at the onset of bifurcation. In this study we assume \textit{dead-load} conditions where the traction rate vector \([5]\)

$$\mathbf{t}_r = e_r \cdot (\nabla \sigma - \mathbf{D} \cdot \sigma + \mathbf{W} + \sigma \mathbf{e}_r \mathbf{e}_r \cdot \mathbf{D})$$

(2.20)

vanishes at the surface \( r = a \). For the hydrostatic field (2.1) we have, with the aid of (2.9) and (2.10), the three boundary conditions at \( r = a \):

$$\nabla \sigma_{rr} = 0$$

(2.21a)

$$\nabla \sigma_{r\theta} - \frac{\sigma_\theta}{r} (u_\theta - v) = 0$$

(2.21b)

$$\nabla \sigma_{r\phi} - \frac{\sigma_\phi}{r \sin \theta} (u_\phi - w \sin \theta) = 0.$$  

(2.21c)

Further specification with the hypoelastic materials (2.13) gives the boundary data in terms of velocity components, at \( r = a \):

$$2\mu_i u_r + \lambda_i I_D = 0$$

(2.22a)

$$\mu_i v_r + \frac{1}{r} (\mu_i - \sigma_a) (u_\theta - v) = 0$$

(2.22b)

$$\mu_i w_r + \frac{1}{r \sin \theta} (\mu_i - \sigma_a) (u_\phi - w \sin \theta) = 0$$

(2.22c)

with the dilation rate expressed by (2.19).

To sum up, possible modes of bifurcation, along the spherical symmetric path (2.1), are governed by the homogeneous system (2.17) with boundary conditions (2.22).

Other boundary conditions can be discussed in a similar way, including mixed boundary data. The similarity of (2.17) with the linear elastic Navier equations guarantees elliptic response so that only diffusive-mode bifurcations are expected. A noteworthy observation is that the load (eigenvalue) appears, as an independent-parameter, only in the boundary conditions (2.22). Available separation of variables solutions of (2.17) can be used, yet the load–velocity coupling in the boundary conditions marks the departure from linear elasticity analysis.

3. Eigenfunctions and eigenvalues

An obvious choice of eigensolutions would be a separation of variables representation in the spirit of linear elasticity analysis of spherical fields given in [4]. Here, however, we proceed with a particular version of that analysis which appears to be more instructive. We begin with the observation, again in analogy with linear elasticity, that the rate of dilation \( I_D \) must obey the harmonic

$$\nabla^2 I_D = 0.$$  

(3.1)

This identity follows at once from (2.16) by calculating the divergence of the equation. Thus, (2.19) is a harmonic function and accordingly we assume the eigenvelocity field

$$u(r, \theta, \phi) = f(r) \cos m\phi P^m_r(\theta)$$

(3.2a)
that dilationless branch admits the connection

\[ v(r, \theta, \phi) = g(r) \cos m\phi \frac{dP_n^{(m)}(\theta)}{d\theta}, \quad (3.2b) \]

\[ w(r, \theta, \phi) = h(r) \sin m\phi \frac{P_n^{(m)}(\theta)}{\sin \theta}, \quad (3.2c) \]

where \( f(r), g(r) \) and \( h(r) \) are the radial profiles of the velocity field, \( m \) and \( n \) are positive integers with \( m \leq n \), and \( P_n^{(m)}(\theta) \) are the associated Legendre functions, of degree \( n \) and order \( m \), that satisfy the equation

\[ \frac{d^2 P_n^{(m)}}{d\theta^2} + \cot \theta \frac{dP_n^{(m)}}{d\theta} + \left[ n(n + 1) - \frac{m^2}{\sin^2 \theta} \right] P_n^{(m)} = 0. \quad (3.3) \]

Now, inserting (3.2) in (2.19) and using (3.3) gives for \( I_D \)

\[ I_D = \left\{ \left[ f' + \frac{2}{r} f - \frac{n(n + 1)}{r} g \right] + \frac{m(mg + h)}{r \sin^2 \theta} \right\} \cos m\phi, \quad (3.4) \]

where the superposed prime denotes differentiation with respect to \( r \). It appears, therefore, reasonable to expect solutions where

\[ h(r) = -mg(r), \quad (3.5) \]

with \( f(r) \) and \( g(r) \) behaving like powers of \( r \), say \( r^p \). We shall see, however, that a second solution exists where \( I_D = 0 \), thus in addition to (3.5) that dilationless branch admits the connection

\[ f' + \frac{2}{r} f = \frac{n(n + 1)}{r} g. \quad (3.6) \]

Indeed, if we substitute (3.2), along with (3.5), in the Navier-type equations (2.17), we find that (2.17b) and (2.17c) become identical. In fact, we are left with the two ordinary differential equations

\[ \begin{align*}
(2\mu_i + \lambda_i) \left( f'' + \frac{2}{r} f' - \frac{2}{r^2} f \right) - \mu_i n(n + 1) \frac{1}{r^2} f - n(n + 1) \left[ (\mu_i + \lambda_i) \frac{1}{r^2} g - (3\mu_i + \lambda_i) \frac{1}{r^2} g' \right] &= 0 \quad (3.7a) \\
(\mu_i + \lambda_i) \frac{1}{r^2} f' + 2(2\mu_i + \lambda_i) \frac{1}{r^2} f + \mu_i g'' + 2\mu_i \frac{1}{r^2} g' - n(n + 1)(2\mu_i + \lambda_i) \frac{1}{r^2} g &= 0. \quad (3.7b)
\end{align*} \]

The characteristic roots of this system are

\[ p_1 = n - 1 \quad p_2 = n + 1 \quad p_3 = -n \quad p_4 = -(n + 2) \quad (3.8) \]

with the appropriate profiles for the full sphere (vanishing velocity components at the origin \( r = 0 \))

\[ \begin{align*}
f_n(r) &= A_n r^{n-1} + B_n r^{n+1} \quad (3.9a) \\
g_n(r) &= \frac{1}{n} A_n r^{n-1} + K_n B_n r^{n+1}. \quad (3.9b)
\end{align*} \]

where \( A_n, B_n \) are integration constants and

\[ K_n = \frac{(n + 9) \frac{\mu_i}{\lambda_i} + 3(n + 3)}{(n + 1) \left[ (n - 6) \frac{\mu_i}{\lambda_i} + 3n \right]}. \quad (3.10) \]

As expected, the \( r^{n-1} \) terms in (3.9) do not contribute to the rate of dilation (3.4), yet they remain an inherent part of the eigensolution. The second pair of characteristic roots in (3.8) is appropriate for the mirror problem of an infinite medium with an embedded spherical cavity under uniform hydrostatic stress. The bifurcated field, in that case, should decay when \( r \to \infty \) as indeed happens with roots \( p_3, p_4 \). We note also that when solving the problem of a spherical cavity embedded in a finite sphere and, again, equally pressurized on both the inner and outer surfaces, all four roots of (3.8) are required.
Fig. 1. Universal bifurcation curves for a full sphere under radial surface load $\sigma_a$. Mode numbers are indicated on the curves with surface instabilities obtained at $n \to \infty$. Compression/tension transition is at vertical asymptotes shown in broken lines.

Turning to the boundary conditions (2.22) we find that, with $h(r) = -mg(r)$, conditions (2.22b) and (2.22c) become identical and we are left with the requirements, at $r = a$,

$$
(2\mu_i + \lambda_i)f' + 2\lambda_i \frac{f}{r} - \lambda_in(n + 1)\frac{g}{r} = 0
$$

(3.11a)

$$
(\mu_i - \sigma_a)\frac{f}{r} + \mu_ig' - (\mu_i - \sigma_a)\frac{g}{r} = 0.
$$

(3.11b)

Substituting solution (3.9) in the boundary condition (3.11) we obtain two linear algebraic equations for the integration constants $A_n, B_n$. A nontrivial solution exists when

$$
\frac{\sigma_a}{2\mu_i} = -\frac{2n(n - 1)\frac{\mu_i}{\kappa_i} + 3(2n^2 + 4n + 3)}{2n(3n + 7)\frac{\mu_i}{\kappa_i} - 3(n + 3)} \quad n \geq 2.
$$

(3.12)

The eigenvalues (3.12) are valid universally for the entire family (2.13) and can be traced (Fig. 1) with any particular constitutive response. In compression ($\sigma_a < 0$) bifurcation will occur in surface modes, with $n \to \infty$, at pressure levels of

$$
-\frac{\sigma_a}{2\mu_i} = \frac{\kappa_i}{\mu_i} + \frac{1}{3}.
$$

(3.13)

In tension ($\sigma_a > 0$) the smallest bifurcation load is obtained, with $n = 2$, at critical values given by

$$
\frac{\sigma_a}{2\mu_i} = \frac{57 + 4\frac{\mu_i}{\kappa_i}}{15 - 52\frac{\mu_i}{\kappa_i}} \quad \frac{\mu_i}{\kappa_i} < \frac{15}{52}.
$$

(3.14)

No bifurcation is possible in tension for moduli ratio above the upper bound in (3.14).

It is apparent from Fig. 1 that the eigenvalues display considerable constitutive sensitivity. With the same mode number $n$ there is an abrupt compression/tension transition in possible bifurcation modes near the moduli ratio

$$
\frac{\mu_i}{\kappa_i} = \frac{3(n + 3)}{2n(3n + 7)}
$$

(3.15)

displayed in Fig. 1 by asymptotic lines.
A second family of possible eigenmodes, inspired by the structure of $I_D$ in (2.19) is the twisting pattern

\begin{align}
\text{(3.16a)} & \quad u(r, \theta, \phi) = 0 \\
\text{(3.16b)} & \quad v(r, \theta, \phi) = g(r) \cos m\phi \frac{P_n^m(\theta)}{\sin \theta} \\
\text{(3.16c)} & \quad w(r, \theta, \phi) = h(r) \sin m\phi \frac{dP_n^m(\theta)}{d\theta}.
\end{align}

The rate of dilation is now

\[ I_D = \frac{g + mh}{r \sin \theta} \cos m\phi \frac{dP_n^m(\theta)}{d\theta} \tag{3.17} \]

which suggests the choice of the dilationless solution relation

\[ g(r) = -mh(r). \tag{3.18} \]

The radial equilibrium rate (2.17a) is satisfied identically while the pair (2.17b) and (2.17c) coincides, leaving us with the simple ordinary differential equation

\[ (r^2 g')' - n(n+1)g = 0. \tag{3.19} \]

Of the two independent solutions of (3.19), namely

\[ r^n, \quad r^{-n-1} \tag{3.20} \]

the appropriate one for the full sphere is

\[ g_n(r) = A_n r^n, \tag{3.21} \]

where $A_n$ is an integration constant. Boundary condition (2.22a) is now immaterial, and (2.22b) and (2.22c) coincide to produce the boundary data at $r = a$:

\[ \mu_i g' - \frac{1}{r} (\mu_i - \sigma_a) g = 0. \tag{3.22} \]

Clearly, with (3.21) a bifurcation solution may develop, only with vanishing shear traction rates over the surface, when

\[ \sigma_a = -(n - 1) \mu_i \quad n \geq 2 \tag{3.23} \]

independent of the bulk modulus $\kappa_i$.

By comparison with (3.12), only compression instabilities are predicted by (3.23) with constant discrete values that are multiples of the instantaneous shear modulus.

The circumferential wave number $m$, of eigenmodes (3.2) and (3.16), does not appear directly in the eigenvalues (3.12) and (3.23). A multiplicity of eigenmodes, over the range of integers $m \leq n$, is therefore associated with each eigenvalue. There is also a definite possibility of identical eigenvalue predicted by (3.12) and (3.23), though not necessarily for the same mode number $n$.

The main conclusion that emerges from the values of critical loads is that at bifurcation the applied load should have the same order of magnitude as the instantaneous shear modulus. This may happen in, for example, isotropic hyperelastic solids [6], or in pressure-sensitive plastic materials discussed next.

Eigenvalues for the mirror problem of a spherical cavity embedded in an infinite medium, under a hydrostatic primary state, are obtained from the present analysis by a simple transformation. Expecting the eigenmodes to decay with radial distance, we retain in (3.8) only the roots $p_3$ and $p_4$. These roots can be obtained from $p_1$ and $p_2$ by replacing $n$ with $(-n - 1)$. Since the same transformation leaves Eq. (3.7) and boundary data (3.11) unchanged, we find the cavity eigenvalues from (3.12) simply by replacing $n$ with $(-n - 1)$

\[ \frac{\sigma_a}{2 \mu_i} = \frac{-2(n + 1)(n + 2) \frac{\mu_i}{\kappa_i} + 3(2n^2 + 1)}{2(n + 1)(3n - 4) \frac{\mu_i}{\kappa_i} + 3(n - 2)}. \tag{3.24} \]
Similarly, the twisting-mode eigenvalues for the cavity problem, follow from (3.23) as

\[ \sigma_n = (n + 2)\mu_i. \tag{3.25} \]

Here, twisting modes (3.25) are possible only in tension. Eigenvalues of the surface mode of (3.24) are identical with those of the full sphere (3.13).

4. Pressure-sensitive elastoplastic solids

An illustrative example of constitutive relation (2.13) is provided by pressure-sensitive elastoplastic solids with an elliptical yield function. Within the standard framework of elastoplastic theory we assume that the total strain rate \( \mathbf{D} \) can be linearly decomposed into an elastic part \( \mathbf{D}^E \) and a plastic part \( \mathbf{D}^P \)

\[ \mathbf{D} = \mathbf{D}^E + \mathbf{D}^P. \tag{4.1} \]

The elastic branch is given by a Hookean-type hypoelastic relation

\[ \mathbf{D}^E = \frac{1}{2\mu} \mathbf{\sigma} - \frac{1}{3} \left( \frac{1}{2\mu} - \frac{1}{3\kappa} \right) \mathbf{I} \cdot \mathbf{\sigma}, \tag{4.2} \]

where \( \mu, \kappa \) are the constant shear and bulk moduli.

The plastic branch in (4.1) is associated with the effective stress \( \sigma_e^2 \)

\[ \sigma_e^2 = Q^2 + \eta^2 \sigma_h^2, \tag{4.3} \]

where \( Q \) denotes the von Mises effective stress.

\[ Q = \left( \frac{3}{2} \mathbf{S} \cdot \mathbf{S} \right)^{1/2} \quad \mathbf{S} = \mathbf{\sigma} - \sigma_h \mathbf{I}, \tag{4.4} \]

\( \mathbf{S} \) being the stress deviator and \( \sigma_h \) the hydrostatic stress,

\[ \sigma_h = \frac{1}{3} \mathbf{I} \cdot \mathbf{\sigma}. \tag{4.5} \]

Parameter \( \eta \) in (4.3) reflects the pressure sensitivity, with the von Mises condition recovered when \( \eta = 0 \). In associated plasticity the plastic potential is identical with the effective stress and the normality rule gives, for the plastic strain rate tensor

\[ \mathbf{D}^P = \Lambda \frac{\partial \sigma_e}{\partial \mathbf{\sigma}}, \tag{4.6} \]

where \( \Lambda \) is a scaling parameter. Hence, following (4.3)

\[ \mathbf{D}^P = \frac{\Lambda}{\sigma_e} \left( \frac{3}{2} \mathbf{S} + \frac{1}{3} \eta^2 \sigma_h \mathbf{I} \right). \tag{4.7} \]

Invoking now the principle of plastic power equivalence

\[ \mathbf{\sigma} \cdot \mathbf{D}^P = \sigma_e \dot{\epsilon}_p, \tag{4.8} \]

where the total plastic strain \( \epsilon_p \) is a known, experimentally determined, function of the effective stress, we find that \( \Lambda = \dot{\epsilon}_p \). Thus, the plastic strain rate (4.7) becomes

\[ \mathbf{D}^P = \frac{3\dot{\epsilon}_p}{2\sigma_e} \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right). \tag{4.9} \]

Recalling, however, that

\[ \dot{\epsilon}_p = \frac{d\epsilon_p}{d\sigma_e} \sigma_e = \frac{d\epsilon_p}{d\sigma_e} \frac{d\sigma_e}{\sigma} \cdot \mathbf{\sigma} = \frac{3}{2\sigma_e} \frac{d\epsilon_p}{d\sigma_e} \left( \mathbf{S} \cdot \mathbf{\sigma} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \cdot \mathbf{\sigma} \right) \tag{4.10} \]
we can put the plastic strain rate (4.9) in the form

\[
\mathbf{D}^p = \left( \frac{3}{2\sigma_e} \right)^2 \frac{d\mathbf{e}_p}{d\sigma_e} \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right) \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right) \cdot \mathbf{\vartheta}. \tag{4.11}
\]

When the state of stress is hydrostatic (2.1) there is no active deviator \((\mathbf{S} = 0)\) and the effective stress (4.3) reduces to

\[
\sigma_e = \eta|\sigma_h| = \eta \sigma_h \text{sgn} \sigma_h. \tag{4.12}
\]

The plastic strain rate (4.11) is accordingly

\[
\mathbf{D}^p = \frac{1}{9} \eta^2 \frac{d\mathbf{e}_p}{d\sigma_e} \mathbf{II} \cdot \mathbf{\vartheta}. \tag{4.13}
\]

Inserting (4.2) and (4.13) in (4.1) we obtain the hypoelastic relation (2.13) with the instantaneous moduli

\[
\mu_i = \mu = \frac{1}{\kappa_i} = \frac{1}{\kappa} + \eta^2 \frac{d\mathbf{e}_p}{d\sigma_e}. \tag{4.14}
\]

Thus, in the flow theory version of this material, the instantaneous shear modulus remains constant, while the instantaneous bulk modulus decreases with hydrostatic pressure for strain hardening response.

An alternative formulation of plastic response is provided by the deformation theory version associated with (4.3). The small strain deviation has been given in [7] and is here further developed for the large strain range. First, we rewrite (4.1), which is the basis of flow theory formulation, as the sum of (4.2) and (4.9), in the form

\[
\mathbf{D} = \frac{1}{2\mu} \mathbf{\vartheta} - \frac{1}{3} \left( \frac{1}{2\mu} - \frac{1}{3\kappa} \right) \mathbf{II} \cdot \mathbf{\vartheta} + \frac{3\mathbf{e}_p}{2\sigma_e} \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right). \tag{4.15}
\]

Next, we construct, in view of (4.15), the nonlinear strain–stress relation

\[
\mathbf{E}_L = \frac{1}{2\mu} \mathbf{\vartheta} - \frac{1}{3} \left( \frac{1}{2\mu} - \frac{1}{3\kappa} \right) \mathbf{II} \cdot \mathbf{\vartheta} + \frac{3\mathbf{e}_p}{2\sigma_e} \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right), \tag{4.16}
\]

where \(\mathbf{E}_L\) is the logarithmic strain tensor.

Now, we take the Jaumann rate of (4.16) to obtain, with the aid of (4.10), the rate form

\[
\mathbf{\overline{\nabla} E}_L = \frac{1}{2\mu} \mathbf{\overline{\nabla} \vartheta} - \frac{1}{3} \left( \frac{1}{2\mu} - \frac{1}{3\kappa} \right) \mathbf{II} \cdot \mathbf{\overline{\nabla} \vartheta} + \frac{3\mathbf{e}_p}{2\sigma_e} \left( \mathbf{\overline{\nabla} S} + \frac{2}{27} \eta^2 \mathbf{II} \cdot \mathbf{\overline{\nabla} \vartheta} \right)
\]

\[
+ \left( \frac{3}{2\sigma_e} \right)^2 \left( \frac{d\mathbf{e}_p}{d\sigma_e} - \frac{\mathbf{e}_p}{\sigma_e} \right) \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right) \left( \mathbf{S} + \frac{2}{9} \eta^2 \sigma_h \mathbf{I} \right) \cdot \mathbf{\vartheta}. \tag{4.17}
\]

Observing, however, that when the principal stretches are equal, tensor \(\mathbf{\overline{\nabla} E}_L\) coincides [8] with \(\mathbf{D}\), and that in our primary state \(\mathbf{S} \equiv 0\), we have the reduced form of (4.17)

\[
\mathbf{D} = \frac{1}{2\mu} \mathbf{\vartheta} - \frac{1}{3} \left( \frac{1}{2\mu} - \frac{1}{3\kappa} \right) \mathbf{II} \cdot \mathbf{\vartheta} + \frac{3\mathbf{e}_p}{2\sigma_e} \mathbf{\overline{\nabla} S} + \frac{1}{9} \eta^2 \frac{d\mathbf{e}_p}{d\sigma_e} \mathbf{II} \cdot \mathbf{\vartheta}. \tag{4.18}
\]

Relation (4.18) is exactly rewritten in the Hookean form (2.13) with the instantaneous moduli

\[
\frac{1}{\mu_i} = \frac{1}{\mu} + 3\frac{\mathbf{e}_p}{\sigma_e}, \tag{4.19a}
\]

\[
\frac{1}{\kappa_i} = \frac{1}{\kappa} + \eta^2 \frac{d\mathbf{e}_p}{d\sigma_e}. \tag{4.19b}
\]

Thus, while the bulk modulus, predicted by deformation theory (4.19b), is identical with that of flow theory (4.14), the instantaneous shear modulus (4.19a) is much smaller than its initial elastic value (4.14). This softer modeling of instantaneous response has made deformation theory predictions more reliable in both bifurcation analysis [1,5] and buckling estimations [9] of plastic solids.
5. Illustrative example and discussion

A characteristic plastic response function of strain hardening solids is given by the power-law relation

\[ \epsilon_p = \epsilon_0 \left( \frac{\sigma_e}{\sigma_y} \right)^\alpha, \]  

(5.1)

where \( \sigma_y \) is a nominal yield stress, \( \epsilon_0 \) is a reference strain and \( \alpha \) is the hardening exponent.

The instantaneous moduli of the flow theory (4.14) become

\[ \mu_i = \mu - \frac{1}{\kappa_i} = \frac{1}{\kappa} + \eta^2 \frac{\alpha \epsilon_0}{\sigma_y} \left( \frac{\sigma_e}{\sigma_y} \right)^{\alpha - 1} \]  

(5.2)

while for deformation theory (4.19) the bulk modulus is identical with that of (5.2), and the shear modulus is

\[ \frac{1}{\mu_i} = \frac{1}{\mu} + 3 \frac{\epsilon_0}{\sigma_y} \left( \frac{\sigma_e}{\sigma_y} \right)^{\alpha - 1}. \]  

(5.3)

It is conceivable that no practical bifurcation loads will be predicted with flow theory moduli. Both (3.13), (3.14) and (3.23) require critical loads of the same order of magnitude as the elastic shear modulus \( \mu \), which are not likely to be attained in practice. By contrast, the instantaneous shear modulus of the deformation theory (5.3) decreases considerably as plasticity progresses. For common solids we may neglect the elastic part of the instantaneous moduli and write in the deep plastic zone, where bifurcation is expected

\[ \mu_i = \frac{\sigma_e}{3 \epsilon_p}, \quad \kappa_i = \frac{\sigma_e}{\eta^2 \alpha \epsilon_p}. \]  

(5.4)

Thus, with the deformation theory we have a constant moduli ratio

\[ \frac{\mu_i}{\kappa_i} = \frac{\eta^2 \alpha}{3} \]  

(5.5)

and surface bifurcations (3.13) are possible in compression when the total strain reaches the value

\[ \epsilon_p = 2 \left( \frac{1}{\eta \alpha} + \frac{\eta}{9} \right). \]  

(5.6)

For a typical value of \( \eta = 1 \) and nearly perfect-plasticity (\( \alpha = 10 \)) relation (5.6) predicts a bifurcation strain of \( \epsilon_p \approx 0.42 \) which is comparable to existing results in classical plasticity [1]. Strain hardening lowers the eigenstrain in (5.6) with the perfectly-plastic limit (\( \alpha \to \infty \))

\[ \epsilon_p = \frac{2\eta}{9}. \]  

(5.7)

Twisting bifurcation modes (3.23) are encountered in compression with the lowest load obtained at \( n = 2 \). For the deformation theory (5.4) we find the critical plastic strain

\[ \epsilon_p = \frac{1}{3 \eta} \]  

(5.8)

which is comparable to (5.6) and (5.7).

The level of applied pressure at bifurcation is calculated from (5.1) by

\[ \sigma_{a} = \pm \frac{\sigma_y}{\eta} \left( \frac{\epsilon_p}{\epsilon_0} \right)^\frac{1}{\alpha}, \]  

(5.9)

where the +/- signs are for tension/compression, respectively. With the surface bifurcation strain (5.6) and \( \eta = 1, \alpha = 10 \) we have the critical pressure of \( \sigma_{a} \approx -2.3\sigma_y \) for a typical value of \( \epsilon_0 = 10^{-4} \). By comparison, with the same parameters, uniaxial necking happens when \( \sigma_e \approx 2\sigma_y \) at a plastic strain level of \( \epsilon_p \approx 0.1 \).
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