Representations of Catalan's Constant

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Abstract

Here, we assemble a fairly exhaustive list of formulae involving Catalan's constant. There are single integral, double integral, infinite series, continued fraction, and asymptotic representations, as well as some additional miscellaneous results of independent interest. A separate section in which the formulae are proved is supplied at the end.

1 Introduction

Catalan's constant may be defined by means of

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = L(2, \chi), \tag{1}$$

where χ is the non-principal character modulo 4.

Formulae for Catalan's constant abound. The purpose of this note is to collect and catalog those formulae and results that are most interesting, and to give simple, elementary proofs for as many as possible. I was inspired by [2] to supply human proofs, as some of the derivations there are simply recitations of formulae hard-coded into *Mathematica*, while others are arrived at through somewhat arcane dilogarithm identities.

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2 Single Integral Formulae

$$G = \int_0^1 \frac{\tan^{-1} x}{x}\, dx. \tag{2}$$

$$G = 2\int_0^1 \left(\tfrac{1}{4}\pi - \tan^{-1} x\right) \frac{dx}{1-x^2}. \tag{3}$$

$$G = \tfrac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}\, dx. \tag{4}$$

$$G = \tfrac{1}{2}\int_0^{\infty} \frac{x}{\cosh x}\, dx. \tag{5}$$

$$G = -\tfrac{1}{4}\pi^2 \int_0^1 (x - \tfrac{1}{2}) \sec(\pi x)\, dx. \tag{6}$$

$$G = -2\int_0^{\pi/4} \log(2\sin x)\, dx. \tag{7}$$

$$G = 2\int_0^{\pi/4} \log(2\cos x)\, dx. \tag{8}$$

$$G = \int_0^{\pi/4} \log(\cot x)\, dx = -\int_0^{\pi/4} \log(\tan x)\, dx. \tag{9}$$

$$G = \tfrac{1}{4}\int_0^{\pi/2} \log\left(\frac{1+\cos x}{1-\cos x}\right)\, dx. \tag{10}$$

$$G = \tfrac{1}{4}\int_0^{\pi/2} \log\left(\frac{1+\sin x}{1-\sin x}\right)\, dx. \tag{11}$$

$$G = -\tfrac{1}{4}\int_0^1 \frac{\log x}{(x+1)\sqrt{x}}\, dx = \tfrac{1}{4}\int_1^{\infty} \frac{\log x}{(1+x)\sqrt{x}}\, dx. \tag{12}$$

$$G = \tfrac{1}{\sqrt{2}}\int_0^{\pi/2} \log\left(\frac{1+\frac{1}{\sqrt{2}}\sin x}{1-\frac{1}{\sqrt{2}}\sin x}\right) \frac{dx}{1+\cos^2 x}. \tag{13}$$

$$G = \tfrac{1}{2}\int_0^{\pi/4} \frac{dx}{\cos x\sqrt{\cos 2x}} \log\left(\frac{1+\sin x}{1-\sin x}\right). \tag{14}$$

$$G = \int_1^{\frac{\sqrt{2}+1}{\sqrt{2}-1}} \frac{(x+1)\log x\, dx}{4x\sqrt{6x-x^2-1}}. \tag{15}$$

$$G = -\int_0^1 \frac{\log x}{1+x^2}\, dx. \tag{16}$$

$$G = \int_1^{\infty} \frac{\log x}{1+x^2}\, dx. \tag{17}$$

$$G = -\int_0^1 \log\left(\tfrac{1}{\sqrt{2}}(1-x)\right) \frac{dx}{1+x^2}. \tag{18}$$

$$G = -\int_0^1 \log\left(\tfrac{1}{2}(1-x^2)\right) \frac{dx}{1+x^2}. \tag{19}$$

$$G = \int_1^\infty \log\left(\tfrac{1}{\sqrt{2}}(x+1)\right) \frac{dx}{1+x^2}. \tag{20}$$

$$G = -\int_1^\infty \log\left(\tfrac{1}{2}(x^2-1)\right) \frac{dx}{1+x^2}. \tag{21}$$

$$G = -\tfrac{1}{4}\pi \log 2 + \int_0^\infty \frac{\log(1+x)}{1+x^2}\, dx. \tag{22}$$

$$G = \tfrac{1}{2}\pi \log 2 - \int_0^1 \frac{\log(1+x^2)}{1+x^2}\, dx. \tag{23}$$

$$G = \tfrac{1}{2}\pi \log 2 - \int_1^{\sqrt{2}} \frac{2\log x}{x\sqrt{x^2-1}}\, dx. \tag{24}$$

$$G = \tfrac{1}{4}\pi \log 2 + \int_0^{\pi/2} \log(\cos x + \sin x)\, dx \tag{25}$$

$$G = \int_0^{\pi/2} \sinh^{-1}(\sin x)\, dx = \int_0^{\pi/2} \sinh^{-1}(\cos x)\, dx. \tag{26}$$

$$G = \int_0^{\pi/2} \operatorname{csch}^{-1}(\csc x)\, dx = \int_0^{\pi/2} \operatorname{csch}^{-1}(\sec x)\, dx. \tag{27}$$

$$G = \tfrac{1}{16}\pi^2 + \tfrac{1}{4}\pi \log 2 - \int_0^1 (\tan^{-1} x)^2\, dx. \tag{28}$$

$$G = \tfrac{1}{16}\pi^2 - \tfrac{1}{4}\pi \log 2 + \int_0^{\pi/4} \frac{x^2}{\sin^2 x}\, dx. \tag{29}$$

$$G = \tfrac{1}{2}\pi \log(1+\sqrt{2}) - \int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x^2}}\, dx. \tag{30}$$

$$G = -\tfrac{3}{2}\int_0^{2-\sqrt{3}} \frac{\log x}{1+x^2}\, dx. \tag{31}$$

$$G = \tfrac{3}{2}\int_{2+\sqrt{3}}^\infty \frac{\log x}{1+x^2}\, dx. \tag{32}$$

$$G = \tfrac{1}{8}\pi \log(2+\sqrt{3}) + \tfrac{3}{2}\int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x}\, dx. \tag{33}$$

$$G = \tfrac{1}{8}\pi \log(2+\sqrt{3}) + \tfrac{3}{4}\int_0^{\pi/6} \frac{x}{\sin x}\, dx. \tag{34}$$

$$2\pi G - \tfrac{7}{2}\zeta(3) = \int_0^{\pi/2} \frac{x^2}{\sin x}\, dx = 4\int_0^1 \left(\tan^{-1} x\right)^2 \frac{dx}{x}. \tag{35}$$

$$G = -\tfrac{1}{4}\pi \log 2 + \int_0^{\pi/2} \frac{x \csc x}{\cos x + \sin x}\, dx \tag{36}$$

$$G = \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi \log 2 - 2\int_0^{\pi/2} \frac{x\cos x}{\cos x + \sin x}\, dx \qquad (37)$$

$$G = -\tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi \log 2 + 2\int_0^{\pi/2} \frac{x\sin x}{\cos x + \sin x}\, dx \qquad (38)$$

$$G = \tfrac{1}{4}\pi \log 2 - P.V. \int_0^{\pi/2} \frac{x\csc x}{\cos x - \sin x}\, dx. \qquad (39)$$

Remarks. Formula (33) can be found on page 266 of [3]. We give an alternative proof, based on an elementary trigonometric identity. The closely related (31) and (34), of which the latter appears as an exercise on page 199 of [6], both follow from the same trigonometric identity. Note that (16) and (17) imply the identity

$$\int_0^\infty \frac{\log x}{1+x^2}\, dx = 0,$$

a result which is typically proved via contour integration. However, this result can also be proved simply and directly by the making the change of variable $y = 1/x$.

3 Double Integral Formulae

$$G = \int_0^1 \int_0^1 \frac{dx\, dy}{1+x^2y^2}. \qquad (40)$$

$$G = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \tan^{-1}(\sin x \sin y)\, \frac{dx\, dy}{\sin x}. \qquad (41)$$

$$G = \tfrac{1}{2} \int_0^1 \int_0^{\pi/2} \frac{d\theta\, dx}{\sqrt{1 - x^2 \sin^2\theta}}. \qquad (42)$$

$$G + \tfrac{1}{2} = \int_0^1 \int_0^{\pi/2} \sqrt{1 - x^2\sin^2\theta}\, d\theta\, dx. \qquad (43)$$

$$G = \tfrac{1}{4} \int_0^1 \int_0^1 \frac{dx\, dy}{(x+y)\sqrt{(1-x)(1-y)}}. \qquad (44)$$

$$2\pi G - \tfrac{7}{2}\zeta(3) = 8\int_0^1 \int_0^1 \frac{\tan^{-1}(xy)}{1+x^2y^2}\, dx\, dy. \qquad (45)$$

$$2\pi G - \tfrac{7}{2}\zeta(3) = 4\int_0^1 \int_0^1 \frac{\tan^{-1} x}{1+x^2y^2}\, dx\, dy. \qquad (46)$$

$$2\pi G - \tfrac{7}{2}\zeta(3) \quad = \quad -\int_0^1 \int_0^1 \frac{\log(1-x^2y^2)}{xy\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy. \qquad (47)$$

Remarks. The innermost integral in (42) is the complete elliptic integral of the first kind. The innermost integral in (43) is the complete elliptic integral of the second kind.

4 Line Integral Formulae

$$G \stackrel{?}{=} \tfrac{1}{15}\pi^2 + \frac{3}{\pi^3 i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(2-s)\zeta(2s)\zeta(4-2s)\,ds, \quad 1 < c < 3/2. \quad (48)$$

5 Asymptotic Formulae

$${}_3F_2\left(\begin{matrix} 1,\ 1,\ \tfrac{1}{2}-\varepsilon \\ \tfrac{3}{2},\ 1-\varepsilon \end{matrix} \,\Bigg|\, -1 \right) = \tfrac{1}{4}\pi + (2G - \tfrac{3}{4}\pi\log 2)\varepsilon + O(\varepsilon^2), \quad \varepsilon \to 0. \quad (49)$$

6 Infinite Series Representations

$$G \quad = \quad \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(2k+1)^2}. \qquad (50)$$

$$G \quad = \quad 1 - \sum_{n=1}^{\infty} \frac{n\zeta(2n+1)}{16^n}. \qquad (51)$$

$$G \quad = \quad \tfrac{1}{6} + \tfrac{1}{4}\sum_{n=1}^{\infty} n16^{-n}(3^{2n}-1)\zeta(2n+1). \qquad (52)$$

$$G \quad = \quad \tfrac{1}{2} + \sum_{n=1}^{\infty} n4^{-n}(1-4^{-n})\zeta(2n+1). \qquad (53)$$

$$G \quad = \quad \tfrac{1}{16}\sum_{n=1}^{\infty} 4^{-n}(3^n-1)(n+1)\zeta(n+2). \qquad (54)$$

$$G \quad = \quad \tfrac{1}{2}\sum_{n=1}^{\infty} n2^{-n}(1-2^{-n})\zeta(n+1). \qquad (55)$$

$$G \quad = \quad \tfrac{1}{8}\sum_{n=2}^{\infty} n2^{-n}\zeta(n+1,\tfrac{3}{4}). \qquad (56)$$

$$G = 1 - \tfrac{1}{8}\sum_{n=2}^{\infty} n2^{-n}\zeta(n+1,\tfrac{5}{4}). \tag{57}$$

$$2\pi G - \tfrac{7}{2}\zeta(3) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{n+\frac{1}{2}}{n}^{-2} = {}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ 1\\ 2,\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\ \middle|\ 1\right). \tag{58}$$

$$2\pi G - \tfrac{7}{2}\zeta(3) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\sum_{k=0}^{n-1}\frac{1}{2k+1}. \tag{59}$$

$$2G = \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}}\sum_{k=0}^{n}\frac{1}{2k+1}. \tag{60}$$

$$2G = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}}. \tag{61}$$

$$G = \tfrac{1}{8}\pi\log(2+\sqrt{3}) + \tfrac{3}{8}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\binom{2n}{n}}. \tag{62}$$

$$G = \tfrac{1}{8}\pi^2 - 2\sum_{n=0}^{\infty}\frac{1}{(4n+3)^2}. \tag{63}$$

$$G = -\tfrac{1}{8}\pi^2 + 2\sum_{n=0}^{\infty}\frac{1}{(4n+1)^2}. \tag{64}$$

$$G = \tfrac{1}{2}\pi\log 2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=1}^{n}\frac{1}{k}. \tag{65}$$

$$G = \tfrac{1}{4}\pi\log 2 - 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sum_{k=0}^{n-1}\frac{1}{2k+1}. \tag{66}$$

$$G = \tfrac{1}{2}\pi\log 2 - \tfrac{1}{32}\pi\sum_{n=0}^{\infty}\frac{(2n+1)^2}{(n+1)^3 16^n}\binom{2n}{n}^2. \tag{67}$$

$$G = -\tfrac{1}{4}\pi\log 2 + \sum_{n=0}^{\infty}\frac{\sqrt{2}}{(2n+1)^2\,8^n}\binom{2n}{n}. \tag{68}$$

$$G = \tfrac{1}{8}\pi\log 2 + \sum_{n=1}^{\infty}\frac{\sin(n\pi/4)}{n^2\,2^{n/2}}. \tag{69}$$

$$2\pi G - \tfrac{35}{8}\zeta(3) = -\tfrac{1}{4}\pi^2\log 2 + \sum_{n=0}^{\infty}\frac{2^{n+1}(n!)^2}{(2n+1)!(n+1)^2}. \tag{70}$$

Let n be a non-negative integer. Then

$$G = \tfrac{1}{4}\pi\,{}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ n+\frac{1}{2}\\ 1,\ n+\frac{3}{2}\end{matrix}\ \middle|\ 1\right) - \tfrac{1}{2}\sum_{k=0}^{n-1}\frac{(k!)^2}{(\frac{3}{2})_k^2}. \tag{71}$$

Let

$$z := {}_2F_1\left(\tfrac{1}{2},\ \tfrac{1}{2};\ 1 \,\middle|\, x\right), \quad y := \pi \frac{{}_2F_1\left(\tfrac{1}{2},\ \tfrac{1}{2};\ 1 \,\middle|\, 1-x\right)}{{}_2F_1\left(\tfrac{1}{2},\ \tfrac{1}{2};\ 1 \,\middle|\, x\right)}.$$

Then

$$\begin{aligned} G &= \tfrac{1}{8}\pi y - \tfrac{1}{2} z^{-1} (1-x)^{1/2} {}_3F_2\left(\begin{matrix} 1,\ 1,\ 1 \\ \tfrac{3}{2},\ \tfrac{3}{2} \end{matrix} \,\middle|\, 1-x \right) \\ &\quad + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2 \left(e^{(2n+1)y} + 1\right)}. \end{aligned} \tag{72}$$

Note that (72) generalizes the definition (1) when $y = 0$, $x = 1$ and $z = \infty$.

From [11], we have

$$G = \tfrac{5}{48}\pi^2 - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2 \left(e^{(2n+1)\pi} - 1\right)} - \tfrac{1}{4} \sum_{n=1}^{\infty} \frac{\operatorname{sech}(n\pi)}{n^2}. \tag{73}$$

$$G = \tfrac{1}{4}\pi x + \tfrac{3}{4}\pi \sum_{n=0}^{\infty} (-1)^n (2n+1) \log\left(1 - \frac{4}{(6n+3)^2}\right), \tag{74}$$

$$G = \tfrac{3}{4}\pi \sum_{n=0}^{\infty} (-1)^n (2n+1) \log\left(1 + \frac{x^2}{(2n+1)^2}\right), \tag{75}$$

where $\frac{1}{2}\pi x = \log(2+\sqrt{3})$.

The following representation is due to the author. Let $L(1) = 1$, $L(2) = 3$, and $L(n) = L(n-1) + L(n-2)$, $n > 2$ be the Lucas numbers (M0155 in Sloan and Plouffe's Encyclopaedia). Note the Lucas numbers satisfy the same recurrence as the Fibonacci numbers, but with different initial values. Then

$$G = \tfrac{1}{8}\pi \log\left(\frac{10 + \sqrt{50 - 22\sqrt{5}}}{10 - \sqrt{50 - 22\sqrt{5}}}\right) + \tfrac{5}{8} \sum_{n=0}^{\infty} \frac{L(2n+1)}{(2n+1)^2 \binom{2n}{n}}. \tag{76}$$

Remarks. The formula (51) is apparently due to Glaisher. The formulae (51)–(52) all have similar proofs. Steven Finch [8] attributes (58) to Bill Gosper. It is closely connected with the double integral (47). Several of the most interesting infinite series of this section are due to Ramanujan. In [3], we find (60), (61), (62), (68), (70). See pages 264–269 there. Additionally, the elegant hypergeometric representation (71) can be found on page 45 of [4], and the stunning evaluation (72) appears on page 155 of [5]. The beautiful formulae (73)–(75) were taken directly from Ramanujan's paper [11].

7 Infinite/Matrix Product Representations

$$\begin{bmatrix} 0 & G \\ 0 & 1 \end{bmatrix} = \prod_{n=0}^{\infty} \begin{bmatrix} \frac{32(n+1)^3(2n+1)}{9(6n+7)^2(6n+11)^2} & \frac{580n^2+976n+411}{450} \\ 0 & 1 \end{bmatrix}. \tag{77}$$

Remarks. The matrix product (77), due to Bill Gosper, can be rewritten in terms of a ${}_4F_3$ at 4/729, which Macsyma uses as its internal series representation.

8 Continued Fraction Representations

$$2G = 2 - \frac{1}{3+}\,\frac{2^2}{1+}\,\frac{2^2}{3+}\,\frac{4^2}{1+}\,\frac{4^2}{3+}\,\frac{6^2}{1+}\,\frac{6^2}{3+}\cdots. \tag{78}$$

$$2G = 1 + \frac{1}{\frac{1}{2}+}\,\frac{1^2}{\frac{1}{2}+}\,\frac{1\cdot 2}{\frac{1}{2}+}\,\frac{2^2}{\frac{1}{2}+}\,\frac{2\cdot 3}{\frac{1}{2}+}\,\frac{3^2}{\frac{1}{2}+}\,\frac{3\cdot 4}{\frac{1}{2}+}\cdots. \tag{79}$$

$$G \stackrel{?}{=} \frac{1}{1+}\,\frac{1}{3^2-1+}\,\frac{1}{1^2\cdot 3^{-2}\cdot 5^2-1^2+}\,\frac{1}{3^2\cdot 5^{-2}\cdot 7^2-3^2+}\cdots. \tag{80}$$

Remarks. Both (78) and (79) can be found in [4]. See pages 151 and 153 there.

9 Additional Results

Let $f(n)$ denote the number of perfect matchings which cover the $2n \times 2n$ square planar lattice. Then [9, 12]

$$\lim_{n\to\infty} f(n)^{1/2n^2} = e^{2G/\pi}. \tag{81}$$

10 Proofs

Proof of (2): Use the Maclaurin series expansion for the arctangent, and integrate term by term.

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Proof of (3): Use the addition formula for the arctangent

$$\tan^{-1} u - \tan^{-1} v = \tan^{-1}\left(\frac{u-v}{1+uv}\right)$$

with $u = 1$, $v = x$. Then

$$\begin{aligned} 2\int_0^1 \left(\tfrac{1}{4}\pi - \tan^{-1} x\right) \frac{dx}{1-x^2} &= 2\int_0^1 \tan^{-1}\left(\frac{1-x}{1+x}\right)\frac{dx}{1-x^2} \\ &= 4\int_0^1 \frac{\tan^{-1} y\, dy}{(1+y)^2-(1-y)^2} \\ &= \int_0^1 \frac{\tan^{-1} y}{y}\, dy \\ &= G, \quad \text{by (2)}, \end{aligned}$$

where we have made the substitution $y = (1-x)/(1+x)$ in the antepenultimate line.

Proof of (4): Let $y = \tan(x/2)$ in (2) and apply the double-angle formula for sine. Then

$$\begin{aligned} \int_0^1 \frac{\tan^{-1} y}{y}\, dy &= \tfrac{1}{4}\int_0^{\pi/2} \frac{x\, dx}{\tan(x/2)\cos^2(x/2)} = \tfrac{1}{4}\int_0^{\pi/2} \frac{x\, dx}{\sin(x/2)\cos(x/2)} \\ &= \tfrac{1}{2}\int_0^{\pi/2} \frac{x}{\sin x}\, dx. \end{aligned}$$

Proof of (5): Write the hyperbolic cosine in terms of exponentials, then expand in powers of $\exp(-x)$ using the geometric series formula. Integrate term by term. Then

$$\tfrac{1}{2}\int_0^\infty \frac{x\, dx}{\cosh x} = \int_0^\infty \frac{xe^{-x}\, dx}{1+e^{-2x}} = \sum_{n=0}^\infty (-1)^n \int_0^\infty xe^{-(2n+1)x}\, dx = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}.$$

Proof of (6): Make a change of variable so that the range of integration is from $-\pi/2$ to $\pi/2$. Write the secant in terms of the reciprocal of sine, and then use the fact that the integrand is even.

$$\begin{aligned} -\tfrac{1}{4}\pi^2 \int_0^1 (x - \tfrac{1}{2})\sec(\pi x)\, dx &= -\tfrac{1}{4}\int_0^\pi (u - \pi/2)\sec(u)\, du \\ &= -\tfrac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{v\, dv}{\cos(v+\pi/2)} \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{4}\int_{-\pi/2}^{\pi/2} \frac{v}{\sin v}\,dv \\
&= \tfrac{1}{2}\int_{0}^{\pi/2} \frac{v}{\sin v}\,dv \\
&= G, \quad \text{by (4).}
\end{aligned}
$$

Proof of (7): Use the boundedly convergent Fourier series

$$
-\log|2\sin x| = \sum_{n=1}^{\infty} \frac{\cos 2nx}{n}, \qquad x \in \mathbf{R},
$$

and integrate term by term. Then

$$
\begin{aligned}
-2\int_0^{\pi/4} \log(2\sin x)\,dx &= 2\int_0^{\pi/4} \sum_{n=1}^{\infty} \frac{\cos 2nx}{n}\,dx = \sum_{n=1}^{\infty} \frac{\sin n\pi/2}{n^2} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.
\end{aligned}
$$

Proof of (8): Use (7), write sine in terms of cosine, and apply the well-known evaluation $\int_0^{\pi/2} \log 2\cos\theta\,d\theta = 0$.

$$
\begin{aligned}
-2\int_0^{\pi/4} \log(2\sin x)\,dx &= -2\int_0^{\pi/4} \log(2\cos(\pi/2 - x))\,dx \\
&= -2\int_{\pi/4}^{\pi/2} \log(2\cos\theta)\,d\theta \\
&= -2\int_0^{\pi/2} \log(2\cos\theta)\,d\theta + 2\int_0^{\pi/4} \log(2\cos\theta)\,d\theta.
\end{aligned}
$$

Proof of (9): Add (7) and (8).

Proof of (10): Start with (9), rescale the argument by one half, and then apply the half-angle formulae for sine and cosine.

$$
\int_0^{\pi/4} \log\cot x\,dx = \tfrac{1}{2}\int_0^{\pi/2} \log\cot(x/2)\,dx = \tfrac{1}{2}\int_0^{\pi/2} \log\sqrt{\frac{1+\cos x}{1-\cos x}}\,dx
$$

Proof of (11): Replace x by $\pi/2 - x$ in (10).

Proof of (12): Make the change of variable $v = (1+\sin x)/(1-\sin x)$ in (11).

Proof of (13): We start with (60), which is proved from first principles later in the sequel. Thus,

$$\begin{aligned} G &= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)\binom{2n}{n}}\sum_{k=0}^{n}\int_0^1 t^{2k}\,dt \\ &= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{2^n}{(2n+1)\binom{2n}{n}}\int_0^1 \frac{1-t^{2n+2}}{1-t^2}\,dt. \end{aligned}$$

Employing the reduction formula[1]

$$\int_0^{\pi/2}(\sin x)^{2n+1}\,dx = \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} = \frac{4^n}{(2n+1)\binom{2n}{n}} \tag{82}$$

for integrating odd powers of sine on $[0,\pi/2]$, we obtain

$$\begin{aligned} G &= \tfrac{1}{2}\int_0^1 \frac{dt}{1-t^2}\sum_{n=0}^{\infty}2^{-n}(1-t^{2n+2})\int_0^{\pi/2}(\sin x)^{2n+1}\,dx \\ &= \tfrac{1}{2}\int_0^1 \frac{dt}{1-t^2}\int_0^{\pi/2}\sin x\left(\frac{1}{1-\frac{1}{2}\sin^2 x}-\frac{t^2}{1-\frac{1}{2}t^2\sin^2 x}\right)dx \\ &= \tfrac{1}{2}\int_0^1\int_0^{\pi/2}\frac{\sin x\,dx\,dt}{(1-\frac{1}{2}\sin^2 x)(1-\frac{1}{2}t^2\sin^2 x)} \\ &= \tfrac{1}{\sqrt{2}}\int_0^{\pi/2}\log\left(\frac{1+\frac{1}{\sqrt{2}}\sin x}{1-\frac{1}{\sqrt{2}}\sin x}\right)\frac{dx}{1+\cos^2 x}. \end{aligned}$$

Proof of (14): Make the change of variable $\sin x = \sqrt{2}\sin v$ in (13).

Proof of (15): Make the change of variable $v = (1+\sin x)/(1-\sin x)$ in (14).

Proof of (16): Integrate (2) by parts.

$$\int_0^1 \frac{\tan^{-1} x}{x}\,dx = (\log x)(\tan^{-1} x)\Big|_0^1 - \int_0^1 \frac{\log x}{1+x^2}\,dx.$$

Alternative Proof of (16): Replace x by x^2 in (12).

Proof of (17): Make the change of variable $y = 1/x$ in (16).

Proof of (18): We begin by proving the following

[1] Integrating by parts once defines a recursion for the integral which can be easily solved to give the closed form product.

Lemma 1

$$\int_0^1 \log\left(\frac{1+x}{\sqrt{2}}\right)\frac{dx}{1+x^2} = 0.$$

Proof. Expand the integrand into a double series, then split into two double sums according to whether the outer summation index is even or odd. Interchange the summation order in one of the double sums so that the two double sums can be re-joined. Thus

$$\begin{aligned}
\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+n+1} \\
&= \sum_{n=1}^{\infty}\frac{1}{2n-1}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+2n} - \sum_{n=1}^{\infty}\frac{1}{2n}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+2n+1} \\
&= \sum_{n=1}^{\infty}\frac{1}{2n-1}\sum_{k=n}^{\infty}\frac{(-1)^{k+n}}{2k} - \sum_{n=1}^{\infty}\frac{1}{2n}\sum_{k=n}^{\infty}\frac{(-1)^{k+n}}{2k+1} \\
&= \sum_{k=1}^{\infty}\frac{(-1)^k}{2k}\sum_{n=1}^{k}\frac{(-1)^n}{2n-1} + \sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\sum_{k=n+1}^{\infty}\frac{(-1)^k}{2k-1}.
\end{aligned}$$

Now switch k and n in the second double series and paste the two double series together. We get

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2k}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = \tfrac{1}{8}\pi\log 2.$$

It follows that

$$\int_0^1 \log\left(\frac{1+x}{\sqrt{2}}\right)\frac{dx}{1+x^2} = \tfrac{1}{8}\pi\log 2 - \tfrac{1}{2}\log 2\int_0^1 \frac{dx}{1+x^2} = 0.$$

To prove (18), add zero in the form of Lemma 1 and combine the two logarithms. Simplify the argument of the logarithm by making a suitable change of variable.

$$\begin{aligned}
& -\int_0^1 \log\left(\frac{1-x}{\sqrt{2}}\right)\frac{dx}{1+x^2} \\
= & -\int_0^1 \log\left(\frac{1-x}{\sqrt{2}}\right)\frac{dx}{1+x^2} + \int_0^1 \log\left(\frac{1+x}{\sqrt{2}}\right)\frac{dx}{1+x^2}
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \log\left(\frac{1+x}{1-x}\right)\frac{dx}{1+x^2} \\
&= \int_1^\infty \frac{\log v}{1+\left(\frac{v-1}{v+1}\right)^2}\cdot\frac{2dv}{(v+1)^2}, \quad v=\frac{1+x}{1-x} \\
&= \int_1^\infty \frac{\log v}{v^2+1}\,dv \\
&= G, \quad \text{by (17).}
\end{aligned}
$$

For (19), factor the argument of the logarithm as a difference of squares, and apply (18).

$$
\begin{aligned}
&-\int_0^1 \log\left(\frac{1-x^2}{2}\right)\frac{dx}{1+x^2} \\
&= -\int_0^1 \log\left(\frac{1-x}{\sqrt{2}}\right)\frac{dx}{1+x^2} - \int_0^1 \log\left(\frac{1+x}{\sqrt{2}}\right)\frac{dx}{1+x^2} \\
&= G, \quad \text{by (18) and Lemma 1.}
\end{aligned}
$$

Proof of (20): Make the change of variable $y = 1/x$ in Lemma 1 and use (17). Then

$$
\begin{aligned}
0 &= \int_1^\infty \log\left(\frac{y+1}{y\sqrt{2}}\right)\frac{dy}{y^2+1} \\
&= \int_1^\infty \log\left(\frac{y+1}{\sqrt{2}}\right)\frac{dy}{y^2+1} - \int_1^\infty \frac{\log y}{y^2+1}\,dy \\
&= \int_1^\infty \log\left(\frac{y+1}{\sqrt{2}}\right)\frac{dy}{y^2+1} - G, \quad \text{by (17).}
\end{aligned}
$$

Proof of (21): Replace x by $1/x$ in (19). Then

$$
\begin{aligned}
G &= -\int_1^\infty \log\left(\frac{x^2-1}{2x^2}\right)\frac{dx}{1+x^2} \\
&= -\int_1^\infty \log\left(\frac{x^2-1}{2}\right)\frac{dx}{1+x^2} + 2\int_1^\infty \frac{\log x}{1+x^2}\,dx \\
&= -\int_1^\infty \log\left(\frac{x^2-1}{2}\right)\frac{dx}{1+x^2} + 2G, \quad \text{by (17).}
\end{aligned}
$$

Proof of (22): From Lemma 1, we have

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \tfrac{1}{8}\pi\log 2. \tag{83}$$

Replacing x by $1/x$ in the latter integral, we obtain

$$\begin{aligned}
\tfrac{1}{8}\pi\log 2 &= \int_1^\infty \log\left(\frac{1+x}{x}\right)\frac{dx}{1+x^2} \\
&= \int_1^\infty \frac{\log(1+x)}{1+x^2}\,dx - \int_1^\infty \frac{\log x}{1+x^2}\,dx \\
&= \int_1^\infty \frac{\log(1+x)}{1+x^2}\,dx - G, \quad \text{by (17).}
\end{aligned} \tag{84}$$

Adding the two equations (83) and (84) yields (22) as required.

Proof of (23): Make the obvious substitution $x = \tan\theta$ in the integral. Then

$$\begin{aligned}
-\int_0^1 \frac{\log(1+t^2)}{1+t^2}\,dt &= -\int_0^{\pi/4} \log(\sec^2\theta)\,d\theta, \quad t=\tan\theta \\
&= 2\int_0^{\pi/4} \log\cos\theta\,d\theta \\
&= 2\int_0^{\pi/4} \log(2\cos\theta)\,d\theta - \tfrac{1}{2}\pi\log 2 \\
&= G - \tfrac{1}{2}\pi\log 2, \quad \text{by (8).}
\end{aligned}$$

Proof of (24): Make the change of variable $x^2 = 1+t^2$.

$$\int_1^{\sqrt{2}} \frac{2\log x}{x\sqrt{x^2-1}}\,dx = \int_0^1 \frac{\log(1+t^2)}{1+t^2}\,dt = -G + \tfrac{1}{2}\pi\log 2, \quad \text{by (23).}$$

Proof of (25): By (22),

$$\begin{aligned}
G &= \int_0^\infty \frac{\log(1+x)}{1+x^2}\,dx - \tfrac{1}{4}\pi\log 2 \\
&= \int_0^{\pi/2} \log(1+\tan\theta)\,d\theta - \tfrac{1}{4}\pi\log 2 \\
&= \int_0^{\pi/2} \log(\cos\theta+\sin\theta)\,d\theta - \tfrac{1}{4}\pi\log 2 - \int_0^{\pi/2} \log\cos\theta\,d\theta \\
&= \int_0^{\pi/2} \log(\cos\theta+\sin\theta)\,d\theta - \tfrac{1}{4}\pi\log 2 - \int_0^{\pi/2} \log(2\cos\theta)\,d\theta + \tfrac{1}{2}\pi\log 2 \\
&= \int_0^{\pi/2} \log(\cos\theta+\sin\theta)\,d\theta + \tfrac{1}{4}\pi\log 2.
\end{aligned}$$

Proof of (26): Write the inverse hyperbolic sine as an integral whose integrand can be expanded into powers of sine. The reduction formula (82) for integrating powers of sine on $[0, \pi/2]$ is then used to arrive at the definition (1).

$$\begin{aligned}
\int_0^{\pi/2} \sinh^{-1}(\sin x)\,dx &= \int_0^{\pi/2} \log\left(\sin x + \sqrt{1+\sin^2 x}\right)\,dx \\
&= \int_0^{\pi/2}\int_0^x \frac{\cos y\,dy}{\sqrt{1+\sin^2 y}}\,dx \\
&= \int_0^{\pi/2}\int_0^x \cos y \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(\sin y)^{2n}\,dy\,dx \\
&= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{2n+1}\int_0^{\pi/2}(\sin x)^{2n+1}\,dx \\
&= \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\frac{1}{2n+1}\cdot\frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.
\end{aligned}$$

Replacing x by $\pi/2 - x$ immediately gives

$$G = \int_0^{\pi/2} \sinh^{-1}(\cos x)\,dx.$$

Proof of (27): Use (26) and the fact that $\operatorname{csch}^{-1}(y) = \sinh^{-1}(1/y)$.

Proof of (28): Integrate by parts twice, then use the appropriate trigonometric substitution to transform the resulting integral into the form of (8).

$$\begin{aligned}
-\int_0^1 \left(\tan^{-1} x\right)^2\,dx &= -x\left(\tan^{-1} x\right)^2\Big|_0^1 + \int_0^1 \frac{2x\left(\tan^{-1} x\right)}{1+x^2}\,dx \\
&= -\tfrac{1}{16}\pi^2 + \log(1+x^2)\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{\log(1+x^2)}{1+x^2}\,dx \\
&= -\tfrac{1}{16}\pi^2 - \tfrac{1}{4}\pi\log 2 + G, \quad \text{by (23).}
\end{aligned}$$

Proof of (29): As in the proof of (28), integrate by parts twice. In this case, however, no trignometric substitution is needed.

$$\int_0^{\pi/4} \frac{x^2}{\sin^2 x}\,dx = -x^2\cot x\Big|_0^{\pi/4} + \int_0^{\pi/4} 2x\cot x\,dx$$

$$\begin{aligned} &= -\tfrac{1}{16}\pi^2 + 2x\log\sin x\Big|_0^{\pi/4} - 2\int_0^{\pi/4}\log\sin x\,dx \\ &= -\tfrac{1}{16}\pi^2 - \tfrac{1}{4}\pi\log 2 - 2\int_0^{\pi/4}\log 2\sin x\,dx + \tfrac{1}{2}\pi\log 2 \\ &= -\tfrac{1}{16}\pi^2 + \tfrac{1}{4}\pi\log 2 + G, \quad \text{by (7)}. \end{aligned}$$

Proof of (30): Start with (26). Integrating by parts, we have

$$\begin{aligned} &\int_0^{\pi/2}\sinh^{-1}(\sin x)\,dx \\ &= \int_0^{\pi/2}\log\left(\sin x + \sqrt{1+\sin^2 x}\right)\,dx \\ &= x\log\left(\sin x + \sqrt{1+\sin^2 x}\right)\Big|_0^{\pi/2} - \int_0^{\pi/2}\frac{x\cos x}{\sqrt{1+\sin^2 x}}\,dx \\ &= \tfrac{1}{2}\pi\log(1+\sqrt{2}) - \int_0^1 \frac{\sin^{-1} u}{\sqrt{1+u^2}}\,du, \quad u = \sin x. \end{aligned}$$

Proof of (31): We shall apply the following elegant, yet elementary trigonometric identity.

Lemma 2

$$2\int_0^{\pi/12}\log\tan(3x)\,dx = \int_0^{\pi/12}\log\tan x\,dx. \tag{85}$$

Deferring the proof of Lemma 2 for the moment, we have, letting $y = 3x$,

$$-\tfrac{3}{2}\int_0^{\pi/12}\log\tan x\,dx = -\int_0^{\pi/4}\log\tan y\,dy,$$

or, since $\tan(\pi/12) = 2-\sqrt{3}$,

$$G = -\int_0^1 \frac{\log x}{1+x^2}\,dx = -\tfrac{3}{2}\int_0^{2-\sqrt{3}}\frac{\log x}{1+x^2}\,dx,$$

by (16). To prove Lemma 2, note (following Chris Hill) that by the addition formula for the tangent,

$$\begin{aligned} \tan(3x) &= \frac{\tan(2x)+\tan x}{1-\tan x\tan 2x} \\ &= \left(\frac{2\tan x}{1-\tan^2 x} + \tan x\right)\Big/\left(1 - \tan x\left(\frac{2\tan x}{1-\tan^2 x}\right)\right) \end{aligned}$$

$$\begin{aligned}
&= \tan x \left(\frac{3 - \tan^2 x}{1 - 3\tan^2 x} \right) \\
&= \tan x \left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x} \right) \left(\frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x} \right) \\
&= \tan(x)\tan(\pi/3 + x)\tan(\pi/3 - x).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^{\pi/12} \log\tan(3x)\,dx \\
&= \int_0^{\pi/12} \log\tan x\,dx + \int_0^{\pi/12} \log\tan(\pi/3 + x)\,dx \\
&\quad + \int_0^{\pi/12} \log\tan(\pi/3 - x)\,dx \\
&= \int_0^{\pi/12} \log\tan x\,dx + \int_{\pi/3}^{5\pi/12} \log\tan x\,dx + \int_{\pi/4}^{\pi/3} \log\tan x\,dx \\
&= \int_0^{\pi/12} \log\tan x\,dx + \int_{\pi/4}^{5\pi/12} \log\tan x\,dx \\
&= \int_0^{\pi/12} \log\tan x\,dx - \int_{\pi/4}^{\pi/12} \log\tan(\pi/2 - y)\,dy \\
&= \int_0^{\pi/12} \log\tan x\,dx + \int_{\pi/4}^{\pi/12} \log\tan x\,dx \\
&= 2\int_0^{\pi/12} \log\tan x\,dx - \int_0^{\pi/4} \log\tan y\,dy \\
&= 2\int_0^{\pi/12} \log\tan x\,dx - 3\int_0^{\pi/12} \log\tan(3x)\,dx.
\end{aligned}$$

Rearranging the latter gives Lemma 2 as required.

Proof of (32): Make the change of variable $y = 1/x$ in (31).

Proof of (33): Simply take (31) and integrate by parts.

Proof of (34): Let $x = \tan(\theta/2)$ in (33) and apply the double-angle formula for sine. Then

$$\begin{aligned}
G &= \tfrac{1}{8}\pi\log(2+\sqrt{3}) + \tfrac{3}{2}\int_0^{2-\sqrt{3}} \frac{\tan^{-1} x}{x}\,dx. \\
&= \tfrac{1}{8}\pi\log(2+\sqrt{3}) + \tfrac{3}{2}\int_0^{\pi/6} \frac{(\theta/2)\sec^2(\theta/2)d\theta/2}{\tan(\theta/2)} \\
&= \tfrac{1}{8}\pi\log(2+\sqrt{3}) + \tfrac{3}{4}\int_0^{\pi/6} \frac{\theta}{\sin\theta}\,d\theta.
\end{aligned}$$

Proof of (35): One the one hand, by the double-angle formula for sine, we have

$$\begin{aligned}\int_0^{\pi/2} \frac{x^2}{\sin x}\,dx &= \int_0^{\pi/2} \frac{x^2}{2\sin(x/2)\cos(x/2)}\,dx \\ &= \int_0^{\pi/2} \frac{x^2 \sec^2(x/2)}{2\tan(x/2)}\,dx \\ &= 4\int_0^1 \left(\tan^{-1}\theta\right)^2 \frac{d\theta}{\theta}, \quad \theta = \tan(x/2).\end{aligned}$$

Thus, it suffices to show that

$$2\pi G - \tfrac{7}{2}\zeta(3) = 4\int_0^1 \left(\tan^{-1} x\right)^2 \frac{dx}{x}.$$

Integrating by parts, we have

$$\begin{aligned}4\int_0^1 \left(\tan^{-1} x\right)^2 \frac{dx}{x} &= 4(\log\theta)(\tan^{-1}\theta)^2\Big|_0^1 - 8\int_0^1 \frac{(\log\theta)(\tan^{-1}\theta)}{1+\theta^2}\,d\theta \\ &= -8\int_0^{\pi/4} x\log\tan x\,dx, \quad \theta = \tan x \\ &= -8\int_0^{\pi/4} x\log(2\sin x)\,dx + 8\int_0^{\pi/4} x\log(2\cos x)\,dx.\end{aligned}$$

Next, we want to express the cosine in terms of sine, and then apply the Fourier series for $-\log|2\sin x|$. Continuing,

$$\begin{aligned}&4\int_0^1 \left(\tan^{-1} x\right)^2 \frac{dx}{x} \\ =\ & -8\int_0^{\pi/4} x\log(2\sin x)\,dx + 8\int_0^{\pi/4} x\log\left(2\sin(\tfrac{1}{2}\pi - x)\right)dx \\ =\ & -8\int_0^{\pi/4} x\log(2\sin x)\,dx + 8\int_{\pi/4}^{\pi/2} (\tfrac{1}{2}\pi - x)\log(2\sin x)\,dx \\ =\ & -8\int_0^{\pi/2} x\log(2\sin x)\,dx + 4\pi\int_{\pi/4}^{\pi/2} \log(2\sin x)\,dx \\ =\ & 8\int_0^{\pi/2} x\sum_{n=1}^{\infty} \frac{\cos 2nx}{n}\,dx + 4\pi\int_0^{\pi/2} \log(2\sin x)\,dx\end{aligned}$$

$$\begin{aligned}
&-4\pi\int_0^{\pi/4}\log(2\sin x)\,dx\\
&= 2\pi G + 2\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^3}, \quad \text{by (7)}\\
&= 2\pi G - 4\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\\
&= 2\pi G - 4\left(\sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{(2n)^3}\right)\\
&= 2\pi G - \tfrac{7}{2}\zeta(3).
\end{aligned}$$

Proof of (36): By (22), it suffices to show that

$$\int_0^\infty \frac{\log(1+x)}{1+x^2}\,dx = \int_0^{\pi/2}\frac{x\csc x}{\cos x+\sin x}\,dx.$$

Integrating by parts, we have

$$\begin{aligned}
\int_0^\infty \frac{\log(1+x)}{1+x^2}\,dx &= \left(\tan^{-1}x - \pi/2\right)\log(1+x)\Big|_0^\infty + \int_0^\infty \frac{\tan^{-1}(1/x)}{1+x}\,dx\\
&= \int_0^\infty \frac{\tan^{-1}y}{y^2(1+1/y)}\,dy, \quad y=1/x\\
&= \int_0^{\pi/2}\frac{\theta\sec^2\theta}{\tan^2\theta(1+\cot\theta)}\,d\theta, \quad y=\tan\theta\\
&= \int_0^{\pi/2}\frac{\theta\csc\theta}{\cos\theta+\sin\theta}\,d\theta.
\end{aligned}$$

Proof of (37): Let

$$I := 2\int_0^{\pi/2}\frac{x\cos x}{\cos x+\sin x}\,dx = 2\int_0^{\pi/2}\frac{(\pi/2-x)\sin x}{\cos x+\sin x}\,dx, \quad x\leftrightarrow \pi/2-x.$$

Averaging the two integrals for I gives

$$I = \tfrac{1}{2}\pi\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx + \int_0^{\pi/2} x\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)\,dx.$$

But

$$\begin{aligned}
\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx &= \int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx, \quad x\leftrightarrow\pi/2-x\\
&= \tfrac{1}{2}\int_0^{\pi/2}\frac{\sin x+\cos x}{\cos x+\sin x}\,dx\\
&= \tfrac{1}{4}\pi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I &= \tfrac{1}{8}\pi^2 + \int_0^{\pi/2} x\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)\,dx \\
&= \tfrac{1}{8}\pi^2 + x\log(\cos x + \sin x)\Big|_0^{\pi/2} - \int_0^{\pi/2}\log(\cos x + \sin x)\,dx \\
&= \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\log 2 - G, \quad \text{by } (25).
\end{aligned}$$

Alternative Proof of (37): Write the denominator of the integrand in terms of a shifted sine wave.

$$\begin{aligned}
& -2\int_0^{\pi/2}\frac{x\cos x}{\cos x + \sin x}\,dx \\
&= -2\int_0^{\pi/2}\frac{x\cos x}{\sqrt{2}\sin(\pi/4 + x)}\,dx \\
&= -\sqrt{2}\int_{\pi/4}^{3\pi/4}\frac{(x-\pi/4)\sin(x-\pi/4)}{\sin x}\,dx \\
&= -\sqrt{2}\int_{\pi/4}^{3\pi/4}\frac{x-\pi/4}{\sin x}\left(\frac{\cos x}{\sqrt{2}} + \frac{\sin x}{\sqrt{2}}\right)\,dx \\
&= -\int_{\pi/4}^{3\pi/4}(x-\pi/4)\cot x\,dx - \int_{\pi/4}^{3\pi/4}(x-\pi/4)\,dx \\
&= -(x-\pi/4)\log\sin x\Big|_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4}\log\sin x\,dx - \int_0^{\pi/2} x\,dx \\
&= -\tfrac{1}{2}\pi\log\tfrac{1}{\sqrt{2}} + 2\int_{\pi/4}^{\pi/2}\log\sin x\,dx - \tfrac{1}{8}\pi^2 \\
&= \tfrac{1}{4}\pi\log 2 + 2\int_0^{\pi/4}\log(2\cos x)\,dx - \tfrac{1}{2}\pi\log 2 - \tfrac{1}{8}\pi^2 \\
&= -\tfrac{1}{4}\pi\log 2 + G - \tfrac{1}{8}\pi^2, \quad \text{by } (8).
\end{aligned}$$

Proof of (38): Let

$$J := 2\int_0^{\pi/2}\frac{x\sin x}{\cos x + \sin x}\,dx = 2\int_0^{\pi/2}\frac{(\pi/2 - x)\cos x}{\cos x + \sin x}\,dx, \quad x \leftrightarrow \pi/2 - x.$$

Averaging the two integrals for J gives

$$J = \tfrac{1}{2}\pi\int_0^{\pi/2}\frac{\cos x}{\cos x + \sin x}\,dx + \int_0^{\pi/2} x\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)\,dx.$$

But

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx &= \int_0^{\pi/2} \frac{\sin x}{\cos x + \sin x}\,dx, \quad x \leftrightarrow \pi/2 - x \\
&= \tfrac{1}{2}\int_0^{\pi/2} \frac{\cos x + \sin x}{\cos x + \sin x}\,dx \\
&= \tfrac{1}{4}\pi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J &= \tfrac{1}{8}\pi^2 - \int_0^{\pi/2} x\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)\,dx \\
&= \tfrac{1}{8}\pi^2 - x\log(\cos x + \sin x)\Big|_0^{\pi/2} + \int_0^{\pi/2} \log(\cos x + \sin x)\,dx \\
&= \tfrac{1}{8}\pi^2 - \tfrac{1}{4}\pi\log 2 + G, \quad \text{by (25)}.
\end{aligned}$$

Alternative Proof of (38): Write the denominator of the integrand in terms of a shifted sine wave.

$$\begin{aligned}
&2\int_0^{\pi/2} \frac{x\sin x}{\cos x + \sin x}\,dx \\
&= 2\int_0^{\pi/2} \frac{x\sin x}{\sqrt{2}\sin(x+\pi/4)}\,dx \\
&= \sqrt{2}\int_{\pi/4}^{3\pi/4} \frac{(x-\pi/4)\sin(x-\pi/4)}{\sin x}\,dx \\
&= \sqrt{2}\int_{\pi/4}^{3\pi/4} \frac{x-\pi/4}{\sin x}\left(\frac{\sin x}{\sqrt{2}} - \frac{\cos x}{\sqrt{2}}\right)\,dx \\
&= \int_{\pi/4}^{3\pi/4} (x-\pi/4)\,dx - \int_{\pi/4}^{3\pi/4} (x-\pi/4)\cot x\,dx \\
&= \int_0^{\pi/2} x\,dx - (x-\pi/4)\log\sin x\Big|_{\pi/4}^{3\pi/4} + \int_{\pi/4}^{3\pi/4} \log\sin x\,dx \\
&= \tfrac{1}{8}\pi^2 - \tfrac{1}{2}\pi\log\tfrac{1}{\sqrt{2}} + 2\int_{\pi/4}^{\pi/2} \log\sin x\,dx \\
&= \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi\log 2 + 2\int_0^{\pi/4} \log(2\cos x)\,dx - \tfrac{1}{2}\pi\log 2 \\
&= \tfrac{1}{8}\pi^2 - \tfrac{1}{4}\pi\log 2 + G, \quad \text{by (8)}.
\end{aligned}$$

Proof of (39): We have

$$\begin{aligned}
& P.V.\int_0^{\pi/2} \frac{x \csc x}{\cos x - \sin x}\, dx \\
= \;& P.V.\int_0^{\infty} \frac{\tan^{-1} u}{u(1-u)}\, du \\
= \;& P.V.\left[\int_0^1 \left(\frac{1}{u} + \frac{1}{1-u}\right) \tan^{-1} u\, du + \int_0^1 \left(\frac{1}{u} + \frac{1}{1-u}\right) \tan^{-1} u\, du\right]. \\
= \;& P.V.\Bigg[(\log u)(\tan^{-1} u)\Big|_0^1 - \int_0^1 \frac{\log u}{1+u^2}\, du - \log(1-u)\tan^{-1} u\Big|_0^1 \\
& \quad + \int_0^1 \frac{\log(1-u)}{1+u^2}\, du + (\log u)(\tan^{-1} u)\Big|_1^{\infty} - \int_1^{\infty} \frac{\log u}{1+u^2}\, du \\
& \quad - \log(u-1)\tan^{-1} u\Big|_1^{\infty} + \int_1^{\infty} \frac{\log(u-1)}{1+u^2}\, du\Bigg] \\
= \;& \int_0^{\infty} \frac{\log u}{1+u^2}\, du + \int_0^1 \frac{\log(1-u)}{1+u^2}\, du + \int_1^{\infty} \frac{\log(u-1)}{1+u^2}\, du \\
= \;& \int_0^1 \frac{\log(1-u)}{1+u^2}\, du + \int_0^1 \log\left(\frac{1-u}{u}\right) \frac{du}{1+u^2} \\
= \;& 2\int_0^1 \log\left(\frac{1-u}{\sqrt{2}}\right) \frac{du}{1+u^2} - \int_0^1 \frac{\log u}{1+u^2}\, du + 2\int_0^1 \frac{\log \sqrt{2}}{1+u^2}\, du \\
= \;& -G + \tfrac{1}{4}\pi \log 2, \quad \text{by (16) and (18).}
\end{aligned}$$

Proof of (40): Start with (2) and write the arctangent in terms of the integral which defines it. A double integral results. The result follows on scaling one variable by the other, a device which is often quite useful.

$$\int_0^1 \frac{\tan^{-1} y}{y}\, dy = \int_0^1 \frac{dy}{y} \int_0^y \frac{du}{1+u^2} = \int_0^1 \frac{dy}{y} \int_0^1 \frac{y\, dx}{1+x^2y^2}, \quad u = yx.$$

Proof of (41): Expand the arctangent in powers of sines, and integrate term by term, using the reduction formula (82) for integrating odd powers of sine on $[0, \pi/2]$.

$$\begin{aligned}
& \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \tan^{-1}(\sin x \sin y) \frac{dx\, dy}{\sin x} \\
= \;& \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^{\pi/2} (\sin y)^{2n+1} \int_0^{\pi/2} (\sin x)^{2n}\, dx\, dy
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \int_0^{\pi/2} (\sin y)^{2n+1}\, dy \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)} \frac{2\cdot 4\cdot 6\cdots(2n)}{3\cdot 5\cdot 7\cdots(2n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.
\end{aligned}
$$

Proof of (42):

$$
\begin{aligned}
\tfrac{1}{2}\int_0^1 \int_0^{\pi/2} \frac{d\theta\, dx}{\sqrt{1-x^2\sin^2\theta}} &= \tfrac{1}{2}\int_0^1 \int_0^1 \frac{dv}{\sqrt{1-v^2}} \frac{dx}{\sqrt{1-x^2v^2}}, \quad v=\sin\theta \\
&= \tfrac{1}{2}\int_0^1 \frac{dv}{v\sqrt{1-v^2}} \int_0^v \frac{du}{\sqrt{1-u^2}}, \quad u = xv \\
&= \tfrac{1}{2}\int_0^1 \frac{\sin^{-1} v}{v\sqrt{1-v^2}}\, dv \\
&= \tfrac{1}{2}\int_0^{\pi/2} \frac{\theta\cos\theta\, d\theta}{\sin\theta\cos\theta}, \quad v=\sin\theta \\
&= \tfrac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta \\
&= G, \quad \text{by (4)}.
\end{aligned}
$$

Proof of (43): Let $x\sin\theta = \sin\phi$. Then

$$
\begin{aligned}
\int_0^1 \int_0^{\pi/2} \sqrt{1-x^2\sin^2\theta}\, d\theta\, dx &= \int_0^{\pi/2}\int_0^{\theta} \sqrt{1-\sin^2\phi}\cos\phi\, d\phi \frac{d\theta}{\sin\theta} \\
&= \int_0^{\pi/2}\int_0^{\theta} \cos^2\phi\, d\phi \frac{d\theta}{\sin\theta} \\
&= \int_0^{\pi/2}\int_0^{\theta} \frac{1+\cos 2\phi}{2\sin\theta}\, d\phi\, d\theta \\
&= \tfrac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta + \tfrac{1}{4}\int_0^{\pi/2} \frac{\sin 2\theta}{\sin\theta} d\theta \\
&= G + \tfrac{1}{2}\int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\sin\theta}\, d\theta, \quad \text{by (4)} \\
&= G + \tfrac{1}{2}.
\end{aligned}
$$

Proof of (44): We make the double change of variable $x = u-v$, $y = u+v$. Then (u,v) runs over the square S defined by the corners $(0,0)$, $(1/2,-1/2)$,

$(1,0)$ and $(1/2,1/2)$. The Jacobian of the transformation is 2, and so, by symmetry of the square, we have

$$
\begin{aligned}
&\tfrac14\int_0^1\int_0^1 \frac{dx\,dy}{(x+y)\sqrt{(1-x)(1-y)}}\\
&= \tfrac14\iint_S \frac{2\,du\,dv}{2u\sqrt{(1-u)^2-v^2}}\\
&= \tfrac14\iint_S \frac{du\,dv}{(1-u)\sqrt{u^2-v^2}}\\
&= \tfrac12\int_0^{1/2}\frac{du}{1-u}\int_0^u\frac{dv}{\sqrt{u^2-v^2}}+\tfrac12\int_{1/2}^1\frac{du}{1-u}\int_0^{1-u}\frac{dv}{\sqrt{u^2-v^2}}\\
&= \tfrac14\pi\int_0^{1/2}\frac{du}{1-u}+\tfrac12\int_{1/2}^1\frac{du}{1-u}\sin^{-1}\left(\frac{1-u}{u}\right)\\
&= \tfrac14\pi\log 2+\tfrac12\int_0^1\left(\frac1t-\frac1{1+t}\right)\sin^{-1}t\,dt.
\end{aligned}
$$

Now let $t=\sin\theta$, $\cos\theta$ respectively. Then

$$
\begin{aligned}
&\tfrac14\int_0^1\int_0^1 \frac{dx\,dy}{(x+y)\sqrt{(1-x)(1-y)}}\\
&= \tfrac14\pi\log 2+\tfrac12\int_0^{\pi/2}\frac{\theta\cos\theta}{\sin\theta}\,d\theta-\tfrac12\int_0^{\pi/2}(\pi/2-\theta)\frac{\sin\theta}{1+\cos\theta}\,d\theta\\
&= \tfrac14\pi\log 2+\tfrac12\theta\log\sin\theta\Big|_0^{\pi/2}-\tfrac12\int_0^{\pi/2}\log\sin\theta\,d\theta\\
&\quad -\tfrac12\int_0^{\pi/2}(\pi/2-\theta)\tan(\theta/2)\,d\theta\\
&= \tfrac14\pi\log 2-\tfrac12\int_0^{\pi/2}\log 2\sin\theta\,d\theta+\tfrac14\pi\log 2-2\int_0^{\pi/4}(\pi/4-\phi)\tan\phi\,d\phi\\
&= \tfrac12\pi\log 2+\tfrac12\pi\log\cos\phi\Big|_0^{\pi/4}-2\phi\log\cos\phi\Big|_0^{\pi/4}+2\int_0^{\pi/4}\log\cos\phi\,d\phi\\
&= \tfrac12\pi\log 2+2\int_0^{\pi/4}\log 2\cos\phi\,d\phi-\tfrac12\pi\log 2\\
&= G,\quad \text{by (8).}
\end{aligned}
$$

Proof of (45): By (35),

$$
2\pi G-\tfrac72\zeta(3) = 4\int_0^1\left(\tan^{-1}x\right)^2\frac{dx}{x}=4\int_0^1\frac{\tan^{-1}x}{x}\int_0^x\frac{du}{1+u^2}\,dx
$$

$$
\begin{aligned}
&= 4\int_0^1 \frac{\tan^{-1} x}{x} \int_0^1 \frac{x\,dy}{1+x^2y^2}\,dx, \quad u = xy, \\
&= 4\int_0^1 \int_0^1 \frac{\tan^{-1} x}{1+x^2y^2}\,dx\,dy.
\end{aligned}
$$

Proof of (46): By (35),

$$
\begin{aligned}
2\pi G - \tfrac{7}{2}\zeta(3) &= 4\int_0^1 \left(\tan^{-1} x\right)^2 \frac{dx}{x} = 8\int_0^1 \frac{dx}{x}\int_0^x \frac{\tan^{-1} u}{1+u^2}\,du \\
&= 8\int_0^1 \frac{dx}{x}\int_0^1 \frac{\tan^{-1}(xy)}{1+x^2y^2}\,x\,dy, \quad u = xy, \\
&= 8\int_0^1 \int_0^1 \frac{\tan^{-1}(xy)}{1+x^2y^2}\,dx\,dy.
\end{aligned}
$$

Proof of (47) due to Joakim Pettersson, Lund:

$$
-\int_0^1 \int_0^1 \frac{\log(1-x^2y^2)}{xy\sqrt{(1-x^2)(1-y^2)}}\,dy\,dy = -\int_0^{\pi/2}\int_0^{\pi/2} \frac{\log(1-\sin^2 u \sin^2 t)}{\sin u \sin t}\,du\,dt.
$$

Let $f(t)$ denote the previous integral times $\sin t$. Then

$$
\begin{aligned}
f'(t) &= \int_0^{\pi/2} \frac{2\sin u \sin t \cos t}{1-\sin^2 u \sin^2 t}\,du \\
&= \int_0^{\pi/2} \frac{\cos t}{1-\sin t \sin u}\,du - \int_0^{\pi/2} \frac{\cos t}{1+\sin t \sin u}\,du.
\end{aligned}
$$

But

$$
\frac{d}{du} 2\tan^{-1}\left(\frac{\tan(u/2) \pm \sin t}{\cos t}\right) = \frac{\cos t}{1 \pm \sin t \sin u}.
$$

Therefore,

$$
\begin{aligned}
f'(t) &= 2\tan^{-1}\left(\frac{1-\sin t}{\cos t}\right) - 2\tan^{-1}(-\sin t/\cos t) \\
&- 2\tan^{-1}\left(\frac{1+\sin t}{\cos t}\right) + 2\tan^{-1}(\sin t/\cos t).
\end{aligned}
$$

But

$$
\frac{1+\sin t}{\cos t} = \cot \tfrac{1}{2}(\tfrac{1}{2}\pi - t), \qquad \frac{1-\sin t}{\cos t} = \tan \tfrac{1}{2}(\tfrac{1}{2}\pi - t).
$$

Therefore,

$$f'(t) = \tfrac{1}{2}\pi - t + 2t - 2\left(\tfrac{1}{2}\pi - \tfrac{1}{2}(\tfrac{1}{2}\pi - t)\right) + 2t = 2t.$$

Since $f(0) = 0$, it follows that $f(t) = t^2$. Thus,

$$-\int_0^1\int_0^1 \frac{\log(1-x^2y^2)}{xy\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy = \int_0^{\pi/2} \frac{t^2}{\sin t}\,dt = 2\pi G - \tfrac{7}{2}\zeta(3),$$

by (35).

Proof of (49): Let

$$F(\varepsilon) := {}_3F_2\left(\begin{array}{c} 1,\ 1,\ \frac{1}{2}-\varepsilon \\ \frac{3}{2},\ 1-\varepsilon \end{array}\ \Big|\ -1\right).$$

Then

$$F(0) = \sum_{n=0}^{\infty}(-1)^n(\tfrac{1}{2})_n/(\tfrac{3}{2})_n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \tfrac{1}{4}\pi.$$

Next, we need to show that $F'(0) = 2G - \frac{3}{4}\pi\log 2$.

Let $f(\varepsilon) := (\frac{1}{2}-\varepsilon)_n/(1-\varepsilon)_n$, so that

$$F(\varepsilon) = \sum_{n=0}^{\infty}(-1)^n f(\varepsilon)(1)_n/(\tfrac{3}{2})_n,$$

$$\begin{aligned} F'(0) &= \sum_{n=0}^{\infty}(-1)^n f(0)\frac{(1)_n}{(\frac{3}{2})_n}\cdot\frac{f'}{f}(0) \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(\frac{1}{2})_n}{(\frac{3}{2})_n}\cdot\frac{f'}{f}(0) \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\cdot\frac{f'}{f}(0). \end{aligned}$$

But,

$$f(\varepsilon) = \frac{\Gamma(1/2+n-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1+n-\varepsilon)\Gamma(1/2-\varepsilon)}.$$

Thus,

$$\frac{f'}{f}(0) = \psi(1+n) - \psi(1) + \psi(1/2) - \psi(1/2+n)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{2k-1} \\
&= \int_0^1 \frac{1-t^n}{1-t}\,dt - 2\int_0^1 \frac{1-u^{2n}}{1-u^2}\,du \\
&= 2\int_0^1 \frac{u-u^{2n+1}}{1-u^2}\,du - 2\int_0^1 \frac{1-u^{2n}}{1-u^2}\,du,
\end{aligned}
$$

and so

$$
\begin{aligned}
F'(0) &= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 (u-u^{2n+1}-1+u^{2n}) \frac{du}{1-u^2} \\
&= -2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{1-u}{1-u^2}\,du + 2\int_0^1 \frac{\tan^{-1} u}{1-u^2}(1/u-1)\,du \\
&= -2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \log 2 + 2\int_0^1 \frac{\tan^{-1} u}{u(1+u)}\,du \\
&= -\tfrac{1}{2}\pi \log 2 + 2\int_0^1 \left(\frac{1}{u} - \frac{1}{1+u}\right) \tan^{-1} u\,du \\
&= -\tfrac{1}{2}\pi \log 2 + 2G - 2\int_0^1 \frac{\tan^{-1} u}{1+u}\,du \\
&= -\tfrac{1}{2}\pi \log 2 + 2G - 2\log(1+u)\tan^{-1} u\,\Big|_0^1 + 2\int_0^1 \frac{\log(1+u)}{1+u^2}\,du \\
&= \tfrac{1}{2}\pi \log 2 + 2G - \tfrac{1}{2}\pi \log 2 + \tfrac{1}{4}\log 2 \\
&= 2G - \tfrac{3}{4}\pi \log 2.
\end{aligned}
$$

Proof of (50): Apply Euler's acceleration method for alternating series to the definition (1). Alternatively, from (2), we have

$$
\begin{aligned}
G &= \int_0^1 \frac{\tan^{-1} x}{x}\,dx = \int_0^1 \frac{dx}{x} \int_0^x \frac{du}{2-(1-u^2)} \\
&= \tfrac{1}{2}\int_0^1 \frac{dx}{x} \int_0^x \frac{du}{1-\frac{1}{2}(1-u^2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 \frac{dx}{x} \int_0^x (1-u^2)^n\,du \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 \frac{dx}{x} \int_0^x \sum_{k=0}^{n} \binom{n}{k} (-1)^k u^{2k}\,du \\
&= \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(2k+1)^2}.
\end{aligned}
$$

Proof of (51): As customary, we denote the logarithmic derivative of the Euler gamma function by ψ. The idea is to bisect the Maclaurin series for the deriviative of ψ. We have [1]

$$-\gamma - \psi(1-z) := \sum_{n=1}^{\infty}\left(\frac{1}{z-n} - \frac{1}{n}\right) = \sum_{n=2}^{\infty}\zeta(n)z^{n-1}, \quad |z| < 1. \tag{86}$$

Thus,

$$4\sum_{n=1}^{\infty} n\zeta(2n+1)z^{2n} = z\psi'(1-z) - z\psi'(1+z). \tag{87}$$

Also,

$$\psi'(1-z) = \sum_{n=1}^{\infty}\frac{1}{(n-z)^2}. \tag{88}$$

Putting $z = 1/4$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{n\zeta(2n+1)}{16^n} &= \tfrac{1}{16}\left\{\psi'(\tfrac{3}{4}) - \psi'(\tfrac{5}{4})\right\} = \tfrac{1}{16}\sum_{n=1}^{\infty}\left\{\frac{1}{(n-\frac{1}{4})^2} - \frac{1}{(n+\frac{1}{4})^2}\right\} \\
&= \sum_{n=1}^{\infty}\left\{\frac{1}{(4n-1)^2} - \frac{1}{(4n+1)^2}\right\} \\
&= 1 - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \\
&= 1 - G.
\end{aligned}$$

Proof of (52): As in the proof of (51), we have [1] (86), and hence (87). Thus, in view of (88) and (1), we have

$$\begin{aligned}
&\tfrac{1}{4}\sum_{n=1}^{\infty} n4^{-2n}(3^{2n}-1)\zeta(2n+1) \\
= \;& \tfrac{3}{64}\psi'(1-\tfrac{3}{4}) - \tfrac{3}{64}\psi'(1+\tfrac{3}{4}) - \tfrac{1}{64}\psi'(1-\tfrac{1}{4}) + \tfrac{1}{64}\psi'(1+\tfrac{1}{4}) \\
= \;& \tfrac{3}{64}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{3}{4})^2} - \tfrac{3}{64}\sum_{n=1}^{\infty}\frac{1}{(n+\frac{3}{4})^2} - \tfrac{1}{64}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{4})^2} + \tfrac{1}{64}\sum_{n=1}^{\infty}\frac{1}{(n+\frac{1}{4})^2} \\
= \;& \tfrac{3}{4}\sum_{n=1}^{\infty}\frac{1}{(4n-3)^2} - \tfrac{3}{4}\sum_{n=1}^{\infty}\frac{1}{(4n+3)^2} + \tfrac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(4n+1)^2} - \tfrac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} \\
= \;& \tfrac{3}{4}(G + \tfrac{1}{9}) + \tfrac{1}{4}(G-1) \\
= \;& G - \tfrac{1}{6}.
\end{aligned}$$

Proof of (53): As in the proof of (51), we have [1] (86), and hence (87). Thus, in view of (88) and (1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n4^{-n}(1-4^{-n})\zeta(2n+1) \\
= \;& \tfrac{1}{4}\left\{\tfrac{1}{2}\psi'(1-\tfrac{1}{2}) - \tfrac{1}{2}\psi'(1+\tfrac{1}{2})\right\} - \tfrac{1}{4}\left\{\tfrac{1}{4}\psi'(1-\tfrac{1}{4}) - \tfrac{1}{4}\psi'(1+\tfrac{1}{4})\right\} \\
= \;& \tfrac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{2})^2} - \tfrac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(n+\frac{1}{2})^2} - \tfrac{1}{16}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{4})^2} + \tfrac{1}{16}\sum_{n=1}^{\infty}\frac{1}{(n+\frac{1}{4})^2} \\
= \;& \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} - \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2} - \sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} + \sum_{n=1}^{\infty}\frac{1}{(4n+1)^2} \\
= \;& -\tfrac{1}{2} + G, \quad \text{by (1).}
\end{aligned}$$

Proof of (54): As in the proof of (51), we have [1] (86), so that

$$-\psi'(1-z) = \sum_{n=0}^{\infty}(n+1)\zeta(n+2)z^n, \quad |z| < 1. \tag{89}$$

In view of (88), it follows that

$$\begin{aligned}
\tfrac{1}{16}\sum_{n=1}^{\infty}\frac{3^n-1}{4^n}(n+1)\zeta(n+2) &= \tfrac{1}{16}\left\{\psi'(1-\tfrac{3}{4}) - \psi'(1-\tfrac{1}{4})\right\} \\
&= \tfrac{1}{16}\sum_{n=1}^{\infty}\left\{\frac{1}{(n-\frac{3}{4})^2} - \frac{1}{(n-\frac{1}{4})^2}\right\} \\
&= \sum_{n=1}^{\infty}\left\{\frac{1}{(4n-3)^2} - \frac{1}{(4n-1)^2}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}.
\end{aligned}$$

Proof of (55): As in the proof of (51), we have [1] (86), so that (89) holds. In view of (88), it follows that

$$\begin{aligned}
&\tfrac{1}{2}\sum_{n=1}^{\infty} n2^{-n}(1-2^{-n})\zeta(n+1) \\
= \;& \tfrac{1}{8}\psi'(1-\tfrac{1}{4}) - \tfrac{1}{4}\psi'(1-\tfrac{1}{2}) \\
= \;& \tfrac{1}{8}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{4})^2} - \tfrac{1}{4}\sum_{n=1}^{\infty}\frac{1}{(n-\frac{1}{2})^2}
\end{aligned}$$

$$
\begin{aligned}
&= 2\sum_{n=1}^{\infty}\frac{1}{(4n-1)^2} - \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \\
&= 2\sum_{n=0}^{\infty}\frac{1}{(4n+3)^2} - \sum_{n=0}^{\infty}\left\{\frac{1}{(4n+3)^2}+\frac{1}{(4n+1)^2}\right\} \\
&= \sum_{n=0}^{\infty}\left\{\frac{1}{(4n+3)^2}-\frac{1}{(4n+1)^2}\right\} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \\
&= G, \quad \text{by (1).}
\end{aligned}
$$

Proof of (56): Since

$$\sum_{n=2}^{\infty} nx^n = \frac{x}{(1-x)^2} - x, \quad |x| < 1,$$

it follows that

$$
\begin{aligned}
\sum_{n=2}^{\infty} n2^{-n}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{3}{4})^{n+1}} &= \sum_{k=0}^{\infty}\frac{1}{k+\frac{3}{4}}\sum_{n=2}^{\infty}\frac{n}{(2k+\frac{3}{2})^n} \\
&= \sum_{k=0}^{\infty}\frac{1}{(k+\frac{3}{4})(2k+\frac{3}{2})\left(1-\frac{1}{2k+\frac{3}{2}}\right)^2} \\
&\qquad -\sum_{k=0}^{\infty}\frac{1}{(k+\frac{3}{4})(2k+\frac{3}{2})} \\
&= \sum_{k=0}^{\infty}\frac{2}{(2k+\frac{1}{2})^2} - \sum_{k=0}^{\infty}\frac{2}{(2k+\frac{3}{2})^2} \\
&= 8\sum_{k=0}^{\infty}\frac{1}{(4k+1)^2} - 8\sum_{k=0}^{\infty}\frac{1}{(4k+3)^2} \\
&= 8G, \quad \text{by (1).}
\end{aligned}
$$

Proof of (57): Since

$$\sum_{n=2}^{\infty} nx^n = \frac{x}{(1-x)^2} - x, \quad |x| < 1,$$

it follows that

$$\sum_{n=2}^{\infty} n2^{-n}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{5}{4})^{n+1}} = \sum_{k=0}^{\infty}\frac{1}{k+\frac{5}{4}}\sum_{n=2}^{\infty}\frac{n}{(2k+\frac{5}{2})^n}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{(k+\frac{5}{4})(2k+\frac{5}{2})\left(1-\frac{1}{2k+\frac{5}{2}}\right)^2} \\
&\qquad - \sum_{k=0}^{\infty} \frac{1}{(k+\frac{5}{4})(2k+\frac{5}{2})} \\
&= \sum_{k=0}^{\infty} \frac{2}{(2k+\frac{3}{2})^2} - \sum_{k=0}^{\infty} \frac{2}{(2k+\frac{5}{2})^2} \\
&= 8\sum_{k=0}^{\infty} \frac{1}{(4k+3)^2} - 8\sum_{k=0}^{\infty} \frac{1}{(4k+5)^2} \\
&= 8(1-G), \quad \text{by (1).}
\end{aligned}
$$

Proof of (58): Rewrite the binomial coefficient in terms of the beta integral. Thus

$$
\binom{n+\frac{1}{2}}{n}^{-1} = \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+1+\frac{1}{2})} = \tfrac{1}{2}\int_0^1 x^n(1-x)^{-1/2}\,dx,
$$

and (58) becomes

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{n+\frac{1}{2}}{n}^{-2} &= \tfrac{1}{4}\sum_{n=0}^{\infty} \frac{1}{n+1}\int_0^1\int_0^1 x^n y^n (1-x)^{-1/2}(1-y)^{-1/2}\,dx\,dy \\
&= -\tfrac{1}{4}\int_0^1\int_0^1 \frac{\log(1-xy)}{xy\sqrt{(1-x)(1-y)}}\,dx\,dy \\
&= -\int_0^1\int_0^1 \frac{\log(1-x^2y^2)}{xy\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy \\
&= 2\pi G - \tfrac{7}{2}\zeta(3),
\end{aligned}
$$

by (47).

Proof of (59): Integrate the series expansion for $(\tan^{-1} x)/(1+x^2)$ to obtain

$$
\left(\tan^{-1} x\right)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{2n} \sum_{k=0}^{n-1} \frac{1}{2k+1}.
$$

Substitute the latter into (35) and integrate term by term.

Proof of (60): From (2), we have

$$
2G = 2\int_0^1 \frac{\tan^{-1} x}{x}\,dx = \int_0^1 \frac{dx}{x}\int_0^x \frac{2\,du}{2-(1-u^2)}
$$

$$
\begin{aligned}
&= \int_0^1 \frac{dx}{x} \int_0^x \frac{du}{1-\frac{1}{2}(1-u^2)} \\
&= \sum_{n=0}^{\infty} \frac{1}{2^n} \int_0^1 \frac{dx}{x} \int_0^x (1-u^2)^n \, du \\
&= \sum_{n=0}^{\infty} 2^{-n} \int_0^{\pi/2} \cot\phi \int_0^{\phi} (\cos\theta)^{2n+1} \, d\theta \, d\phi \\
&= \sum_{n=0}^{\infty} 2^{-n} \int_0^{\pi/2} I_{2n+1}(\phi) \cot\phi \, d\phi, \qquad (90)
\end{aligned}
$$

where

$$I_{2n+1}(\phi) := \int_0^{\phi} (\cos\theta)^{2n+1} \, d\theta.$$

Integrating by parts, we have, for $n > 0$,

$$
\begin{aligned}
I_{2n+1}(\phi) &= \int_0^{\phi} (\cos\theta)^{2n} \cos\theta \, d\theta \\
&= (\cos\theta)^{2n} \sin\theta \Big|_0^{\phi} + 2n \int_0^{\phi} (\cos\theta)^{2n-1} \sin^2\theta \, d\theta \\
&= (\cos\phi)^{2n} \sin\phi + 2n \int_0^{\phi} (\cos\theta)^{2n-1} (1-\cos^2\theta) \, d\theta.
\end{aligned}
$$

Thus for $n > 0$,

$$I_{2n+1}(\phi) = \frac{(\cos\phi)^{2n}}{2n+1} \sin\phi + \frac{2n}{2n+1} I_{2n-1}(\phi),$$

a recurrence which gives, when iterated,

$$
\begin{aligned}
I_{2n+1}(\phi) &= \frac{\sin\phi}{2n+1} \sum_{k=0}^{n} \frac{(2k+2)(2k+4)\cdots(2n)}{(2k+1)(2k+3)\cdots(2n-1)} (\cos\phi)^{2k} \\
&= \frac{4^n \sin\phi}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n} \binom{2k}{k} (\tfrac{1}{2}\cos\phi)^{2k}.
\end{aligned}
$$

Substituting for $I_{2n+1}(\phi)$ in (90), we obtain

$$
\begin{aligned}
2G &= \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n} \binom{2k}{k} 4^{-k} I_{2k+1}(\pi/2) \\
&= \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n} \frac{1}{2k+1},
\end{aligned}
$$

as required.

Alternative Proof of (60): Set $x = -1/2$ in [3], Chapter 9, Entry 34, p. 293, Formula (34.1):

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \left(\frac{x}{1+x}\right)^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{n+1}}{(2n+1)\binom{2n}{n}} \sum_{k=0}^{n} \frac{1}{2k+1}, \quad -\tfrac{1}{2} < x < 1.$$

Remark. Fee [7] proved (60) by first deriving (50) and then simplifying the inner sum.

Proof of (61): Use (4) and the fact that

$$\begin{aligned} \int_0^{\pi/2} \frac{\theta}{\sin\theta}\, d\theta &= \int_0^1 \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} \cdot \frac{dx}{2x^2} = \int_0^1 \frac{dx}{2x^2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}} \\ &= \sum_{n=1}^{\infty} \frac{4^n}{2n(2n-1)\binom{2n}{n}} \\ &= \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}}. \end{aligned}$$

Alternative proof of (61): Start with (11). Expand the integrand into odd powers of sine, interchange the order of summation and integration, and apply the reduction formula (82) for integrating odd powers of sine on $[0, \pi/2]$. Thus,

$$\begin{aligned} 2G &= \tfrac{1}{2} \int_0^{\pi/2} \log\left(\frac{1+\sin x}{1-\sin x}\right) dx \\ &= \int_0^{\pi/2} \sum_{n=0}^{\infty} \frac{1}{2n+1} (\sin x)^{2n+1}\, dx \\ &= \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \\ &= \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2\binom{2n}{n}}. \end{aligned}$$

Proof of (62): Use (34) and the fact that

$$\int_0^{\pi/6} \frac{\theta}{\sin\theta}\, d\theta = \int_0^{1/2} \frac{2x \sin^{-1} x}{\sqrt{1-x^2}} \cdot \frac{dx}{2x^2} = \int_0^{1/2} \frac{dx}{2x^2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n\binom{2n}{n}}$$

$$= 2\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)\binom{2n}{n}}$$

$$= \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\binom{2n}{n}}.$$

Proof of (63): Start from the definition (1), splitting the sum according to whether n is even or odd. Then

$$\begin{aligned}
G &= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= \sum_{n=0}^{\infty} \left\{ \frac{1}{(4n+3)^2} + \frac{1}{(4n+1)^2} \right\} - 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} - 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n-1)^2} + \frac{1}{(2n)^2} \right\} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= \tfrac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= \tfrac{1}{8}\pi^2 - 2\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2}.
\end{aligned}$$

Proof of (64): As in the proof of (63), start from the definition (1), splitting the sum according to whether n is even or odd. Then

$$\begin{aligned}
G &= \sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(4n+3)^2} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \left\{ \frac{1}{(4n+3)^2} + \frac{1}{(4n+1)^2} \right\} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \sum_{n=1}^{\infty} \left\{ \frac{1}{(2n-1)^2} + \frac{1}{(2n)^2} \right\} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \tfrac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(4n+1)^2} - \tfrac{1}{8}\pi^2.
\end{aligned}$$

Proof of (65): It is convenient to start with the double sum. First, interchange summation order, then write the inner sum as the integral of a geometric series. The outer sum gives a logarithm, which can be massaged into (23).

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^{n} \frac{1}{k} &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=k}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=k}^{\infty} (-1)^n \int_0^1 t^{2n}\, dt \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 \frac{(-t^2)^k}{1+t^2}\, dt \\
&= -\int_0^1 \frac{dt}{1+t^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} t^{2k} \\
&= -\int_0^1 \frac{\log(1+t^2)}{1+t^2}\, dt \\
&= G - \tfrac{1}{2}\pi \log 2, \quad \text{by (23)}.
\end{aligned}$$

Alternative Proof of (65): Again, start with the double sum, but instead of interchanging summation order, write the inner harmonic sum as an integral of a finite geometric series. The outer sum gives an arctangent, which can be massaged into (3).

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^{n} \frac{1}{k} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{1-t^n}{1-t}\, dt \\
&= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int_0^1 \frac{u-u^{2n+1}}{1-u^2}\, du \\
&= \tfrac{1}{2}\pi \int_0^1 \frac{u}{1-u^2}\, du - 2\int_0^1 \frac{\tan^{-1} u}{1-u^2}\, du \\
&= -\tfrac{1}{2}\pi \int_0^1 \frac{1-u}{1-u^2}\, du + 2\int_0^1 \left(\tfrac{1}{4}\pi - \tan^{-1} u\right) \frac{du}{1-u^2} \\
&= G - \tfrac{1}{2}\pi \log 2, \quad \text{by (3)}.
\end{aligned}$$

Proof of (66): Start with the double sum, and write the inner sum as an integral of a finite geometric series. The outer sum gives an arctangent, which can be massaged into Lemma 1. Thus,

$$-2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sum_{k=0}^{n-1} \frac{1}{2k+1}$$

$$
\begin{aligned}
&= -2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1 \frac{1-t^{2n}}{1-t^2}\,dt \\
&= -2\int_0^1\left(\frac{\pi}{4}-\frac{\tan^{-1}t}{t}\right)\frac{dt}{1-t^2} \\
&= -G+2\int_0^1\left(\tfrac{1}{4}\pi-\tan^{-1}t\right)\frac{dt}{1-t^2}-2\int_0^1\left(\frac{\pi}{4}-\frac{\tan^{-1}t}{t}\right)\frac{dt}{1-t^2} \quad \text{by (3)} \\
&= -G+2\int_0^1\frac{\tan^{-1}t}{1-t^2}(1/t-1)\,dt \\
&= -G+2\int_0^1\frac{\tan^{-1}t}{t(1+t)}\,dt \\
&= -G+2\int_0^1\left(\frac{1}{t}-\frac{1}{1+t}\right)\tan^{-1}t\,dt \\
&= G-2\int_0^1\frac{\tan^{-1}t}{1+t}\,dt \\
&= G-2\log(1+t)\tan^{-1}t\,\Big|_0^1+2\int_0^1\frac{\log(1+t)}{1+t^2}\,dt \\
&= G-\tfrac{1}{2}\pi\log 2+\tfrac{1}{4}\pi\log 2, \quad \text{by Lemma 1} \\
&= G-\tfrac{1}{4}\log 2.
\end{aligned}
$$

Proof of (69): Put $z=\frac{1+i}{2}=\frac{1}{\sqrt{2}}e^{i\pi/4}$ in Landen's transformation ([10])

$$
\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{-z}{1-z}\right)=-\tfrac{1}{2}\log^2(1-z)
$$

for the dilogarithm, and take imaginary parts.

Proof of (71) when $n=0$: Start with (42) and expand the integrand by the binomial theorem. Integrate term by term, applying reduction formulae as needed to integrate even powers of sines.

$$
\begin{aligned}
G &= \tfrac{1}{2}\int_0^1\int_0^{\pi/2}\sum_{n=0}^{\infty}(-1)^n\binom{-\frac{1}{2}}{n}x^{2n}\sin^{2n}\theta\,d\theta\,dx \\
&= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{-\frac{1}{2}}{n}\int_0^{\pi/2}\sin^{2n}\theta\,d\theta \\
&= \tfrac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{-\frac{1}{2}}{n}\frac{\pi}{4}\cdot\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{4}\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \cdot \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 \\
&= \tfrac{1}{4}\pi \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{(\frac{3}{2})_n} \left(\frac{(\frac{1}{2})_n}{n!} \right)^2 \\
&= \tfrac{1}{4}\pi\, {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \frac{3}{2} \end{matrix} \;\middle|\; 1 \right).
\end{aligned}
$$

Proof of (76): As in the proof of (31), we begin with an elementary trigonometric identity, which should be compared with Lemma 1.

Lemma 3

$$
\int_0^{\pi/20} \log \tan x \, dx + 2 \int_0^{\pi/20} \log \tan(5x) \, dx = 3 \int_0^{\pi/20} \log \tan(3x) \, dx.
$$

Proof. By the addition formula for the tangent,

$$
\tan(5x) = \tan(x) \tan(\pi/5 + x) \tan(\pi/5 - x) \tan(2\pi/5 + x) \tan(2\pi/5 - x).
$$

Thus,

$$
\begin{aligned}
&\int_0^{\pi/20} \log \tan(5x) \, dx \\
&= \int_0^{\pi/20} \log \tan x \, dx + \int_0^{\pi/20} \{\log \tan(\pi/5 + x) + \log \tan(\pi/5 - x)\} \, dx \\
&\qquad + \int_0^{\pi/20} \{\log \tan(2\pi/5 + x) + \log \tan(2\pi/5 - x)\} \, dx \\
&= \int_0^{\pi/20} \log \tan x \, dx + \int_{3\pi/20}^{\pi/4} \log \tan x \, dx + \int_{7\pi/20}^{9\pi/20} \log \tan x \, dx \\
&= \int_0^{\pi/20} \log \tan x \, dx + \int_{3\pi/20}^{\pi/4} \log \tan x \, dx - \int_{3\pi/20}^{\pi/4} \log \tan(\pi/2 - x) \, dx \\
&= \int_0^{\pi/20} \log \tan x \, dx + \int_{3\pi/20}^{\pi/4} \log \tan x \, dx - \int_{\pi/20}^{3\pi/20} \log \tan x \, dx \\
&= 2 \int_0^{\pi/20} \log \tan x \, dx + \int_{3\pi/20}^{\pi/4} \log \tan x \, dx - \int_0^{3\pi/20} \log \tan x \, dx \\
&= 2 \int_0^{\pi/20} \log \tan x \, dx + \int_0^{\pi/4} \log \tan x \, dx - 2 \int_0^{3\pi/20} \log \tan x \, dx \\
&= 2 \int_0^{\pi/20} \log \tan x \, dx + 5 \int_0^{\pi/20} \log \tan(5x) \, dx - 6 \int_0^{\pi/20} \log \tan(3x) \, dx.
\end{aligned}
$$

We now proceed to prove (76). Rescaling the variables in Lemma 3, it follows (9) that

$$
\begin{aligned}
G &= -\int_0^{\pi/4} \log\tan x\,dx = -\tfrac{5}{2}\int_{\pi/20}^{3\pi/20} \log\tan x\,dx \\
&= -\tfrac{5}{2}x\log\tan x\Big|_{\pi/20}^{3\pi/20} + \tfrac{5}{2}\int_{\pi/20}^{3\pi/20} \frac{x\sec^2 x}{\tan x}\,dx \\
&= -\tfrac{5}{2}\left(\tfrac{3}{20}\pi\log\tan\tfrac{3}{20}\pi - \tfrac{1}{20}\pi\log\tan\tfrac{1}{20}\pi\right) + \tfrac{5}{2}\int_{\pi/10}^{3\pi/10} \frac{\theta/2\,d\theta/2}{\sin\theta/2\cos\theta/2} \\
&= \tfrac{1}{8}\pi\left(\log\tan\tfrac{1}{20}\pi - \log\tan^3\tfrac{3}{20}\pi\right) + \tfrac{5}{4}\int_{\pi/10}^{3\pi/10} \frac{\theta}{\sin\theta}\,d\theta. \qquad (91)
\end{aligned}
$$

But,

$$
\begin{aligned}
\int_{\pi/10}^{3\pi/10} \frac{\theta}{\sin\theta}\,d\theta &= 2\int_{\sin(\pi/10)}^{\sin(3\pi/10)} \frac{2x\sin^{-1}x}{\sqrt{1-x^2}}\frac{dx}{4x^2} \\
&= 2\int_{\sin(\pi/10)}^{\sin(3\pi/10)} \sum_{n=1}^{\infty} \frac{(2x)^{2n-2}}{n\binom{2n}{n}}\,dx \\
&= \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{\tau^{2n+1}-\phi^{2n+1}}{(2n+1)^2\binom{2n}{n}}, \qquad (92)
\end{aligned}
$$

where $\tau := (\sqrt{5}+1)/2$ and $\phi := (\sqrt{5}-1)/2$. Since the Lucas numbers have the explicit formula (cf. Binet's formula for the Fibonacci numbers)

$$
L(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad 0 \le n \in \mathbf{Z},
$$

it follows (91), (92) that

$$
G = \tfrac{1}{8}\pi\log\left(\frac{\tan(\pi/20)}{\tan^3(3\pi/20)}\right) + \tfrac{5}{8}\sum_{n=0}^{\infty} \frac{L(2n+1)}{(2n+1)^2\binom{2n}{n}}. \qquad (93)
$$

But

$$
\tan\tfrac{1}{20}\pi = \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}}, \quad \tan\tfrac{3}{20}\pi = \frac{\sqrt{5+2\sqrt{5}}-1}{\sqrt{5+2\sqrt{5}}+1}.
$$

Therefore, we may write

$$
\frac{\tan(\pi/20)}{\tan^3(3\pi/20)} = \left(\frac{\sqrt{5+2\sqrt{5}}+1}{\sqrt{5+2\sqrt{5}}-1}\right)^3 \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}} = \frac{a+b}{a-b},
$$

where a and b are to be determined. Cross multiplying, we have

$$a(\sqrt{5}-3)(5+2\sqrt{5})+5b(\sqrt{5}+3)\sqrt{5+2\sqrt{5}}=0,$$

or

$$\frac{a(5+\sqrt{5})}{3+\sqrt{5}}=\left(\frac{5-\sqrt{5}}{2}\right)a=5b\sqrt{5+2\sqrt{5}}.$$

Cross multiplying again, we have

$$10b=\left(\frac{5-\sqrt{5}}{\sqrt{5+2\sqrt{5}}}\right)a=a\sqrt{50-22\sqrt{5}}.$$

Therefore, if we take $b=\sqrt{50-22\sqrt{5}}$ and $a=10$, then (93) becomes

$$G=\tfrac{1}{8}\pi\log\left(\frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}\right)+\tfrac{5}{8}\sum_{n=0}^{\infty}\frac{L(2n+1)}{(2n+1)^2\binom{2n}{n}},$$

as required.

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