Global Regulation of Input-Saturated Discrete-Time Linear Systems Subject to Persistent Disturbances

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A novel control strategy for globally stabilizing input-saturated linear plants subject to persistent, arbitrarily large but bounded, disturbances is described. The technique is based on a weaker concept of robust positive invariance, referred to in the paper as positive quasi d-invariance, which is believed to be new and allow the achievement of better control performance if compared with those resulting from using actual control strategies based on the classical notion of positive d-invariance. Simultaneous global internal asymptotic stability and semiglobal external finite-gain BIBO stability is achieved under critical structural conditions on the disturbance-to-state map of the plant. Finally, an example is presented where the achievable regulation performance is shown and contrasted with those achievable by a control solution based on the classical notion of positive d-invariance.

Keywords: Nonlinear Feedback; Persistent Disturbance Rejection; Saturated Control

1. Introduction

In the last years there has been renewed interest in the study of linear systems subject to actuator saturation and relevant results on the topic have been recently found. One of this is the characterization of the class of linear time-invariant (LTI) dynamic systems which can be globally regulated by arbitrarily small bounded controls. They are the so-called asymptotically null-controllable with bounded inputs (ANCBI) systems. A necessary and sufficient condition for a discrete-time LTI system to be ANCBI is that it be stabilizable and have eigenvalues of modulus lower than or equal to one. ANCBI systems are very relevant to process control applications in that they encompass stable systems with chains of integrators of arbitrary complexity.

A second important result is the determination of necessary and sufficient conditions on the structure of the disturbance-to-state map in order to achieve, both in global and semiglobal settings, simultaneous internal asymptotic stability and external finite-gain BIBO stability for ANCBI systems [17,9]. In short, let

\[
\begin{bmatrix}
    x_s(t+1) \\ x_u(t+1)
\end{bmatrix} =
\begin{bmatrix}
    \Phi_s & 0 \\ 0 & \Phi_u
\end{bmatrix}
\begin{bmatrix}
    x_s(t) \\ x_u(t)
\end{bmatrix}
+ \begin{bmatrix}
    G_s \\ G_u
\end{bmatrix} u(t) + \begin{bmatrix}
    \tilde{G}_s \\ \tilde{G}_u
\end{bmatrix} w(t)
\]  

(1)

be a convenient system description (in an appropriate basis) of an ANCBI system, where all eigenvalues of \( \Phi_s \) are asymptotically stable whereas all eigenvalues of \( \Phi_u \) are over the unit circumference. Then, the problem is solvable if and only if \( \tilde{G}_u = 0 \).
It is worth pointing out that such a condition is unavoidable if semiglobal results w.r.t. the disturbance are of interest. Again, although limitative, such a condition is typically satisfied by a large class of stable plants when a cascade of integrators is added to the input terminal for control requirements.

Under the above decoupling condition trivial solutions exist. For example, any control law $u(x)$ which globally asymptotically stabilizes the disturbance-free $x_c$ subsystem under the prescribed input-saturation constraint is a solution. However, such a solution need not be satisfactory because it leaves the $x_s$-subsystem in open-loop under the action of the disturbance $w$. As a consequence, poor control performance in general results under the mentioned strategy.

In this paper, we propose a novel control scheme which achieves simultaneously (1) global internal asymptotic stability and (2) semi-global external finite-gain BIBO stability. Specifically, one wants to find a possibly nonlinear state feedback control law capable of simultaneously ensuring (1) global internal asymptotical stability for the closed-loop system in the absence of disturbances and (2) for any persistent disturbance of arbitrarily large but bounded magnitude $W$, viz. $|w(t)| < W$, $\forall t$, there exists a compact subset of the state space $\mathcal{R} = \mathcal{R}(W)$, possibly depending of $W$ and such that $0_x \in \mathcal{R}$, which contains all possible closed-loop state evolutions emanating from $x(0) = 0_x$ under the action of $w$. The latter requirement is equivalent to the usual semiglobal finite-gain external BIBO stability condition, viz. $\max_{t} |x(t)| < \gamma(W) \max_{t} |w(t)|$, where $\gamma(W)$ is the $L_{\infty}$-induced disturbance-to-state bounded gain, possibly depending on $W$ [18].

The scheme is based on a weaker and novel notion of positive invariance under persistent disturbances, referred to in the paper as positive quasi-d-invariance, which is believed to be new. Its main advantage is that it allows one to face more naturally the global nature of the problem and achieve less conservative regulation than methods which hinge upon the classic concept of positive d-invariance [7,8] and relative control design methodologies [9,18,2], which typically give rise to worst-case solutions. On the contrary, it will be shown that the performance of the proposed method depends for a certain extent on the actual disturbance realization.

The control action is of the form $u(t) = F_\rho(x(t)$, where $\rho$ is the control penalizing weight in the LQ performance index (9) and $F_\rho$ the corresponding LQ-optimal state-feedback gain. The smallest $\rho(t)$ at each time instant $t$ is selected on the basis of the current state $x(t)$ and subject to the following monotonicity property $\rho(t) \leq \rho(t-1)$, in such a way that all closed-loop evolutions emanating from the current state satisfy the constraints from $t$ onward regardless of any possible disturbance realization. A suitable lower-bound on $\rho(t)$ is usually prescribed, say $\rho_0$, which defines a convenient LQ controller $F_0$ and a corresponding desirable level of regulation performance in steady-state. Then, the control strategy consists of a linear piecewise-constant state-feedback action $F_\rho(t)$, with $\rho(t)$ converging from the above to a constant value as $t \to \infty$.

It is fair pointing out that the proposed solution needs further investigations in that no optimality results, e.g. regarding the minimization of the closed-loop $L_{\infty}$-induced disturbance-to-state gain, are claimed. However, the rationale for using a $\rho$-dependent LQ feedback-gain for improving the control performance hinges upon the monotonic properties of the LQ state-feedback solution, well-known in a disturbance-free deterministic setting [19]. Monotonicity results for the $L_{\infty}$-induced gain of the closed-loop disturbance-to-state map are not available so far. However, similar monotonicity results have been proved in the optimal LQ stochastic framework, where the disturbance is modelled as a stochastic process and the cost weights the variances of the closed-loop state and input. To be specific, it is shown in [15] that $\langle x_{r.m.s.}\rangle_{\rho_1} \leq \langle x_{r.m.s.}\rangle_{\rho_2}$ if $\rho_1 \leq \rho_2$, where $x_{r.m.s.} := \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} |x_0(k)|^2_{\rho_s}$ is the r.m.s. (semi) norm and $\langle x_{r.m.s.}\rangle_{\rho}$ denotes the variance of the state corresponding to the choice $\rho$ in the cost index. The above condition allows one to postulate that also in a deterministic setting the $L_{\infty}$-induced gain diminishes (not necessarily monotonically, although monotonicity never has been contradicted by numerous simulative experiments carried out on ANCBI systems) from that in open-loop by decreasing $\rho$, and the better would result in selecting the smallest possible $\rho$ compatible with other prescriptions.

Related contributions on global stabilization of disturbance-free ANCBI linear systems can be found e.g. in [16,1,5] in the presence of magnitude input saturation and in [13,3] for the case of both magnitude and rate input saturations. From the applications side see [12], where an adhoc gain-scheduling state-feedback strategy, not based on the LQ framework, has been used to locally stabilize in a disturbance-free situation the pitch angle of a flight control system in the presence of both magnitude and rate saturations. Significantly, the experiment shows that good control performance can be obtained in constrained control problem by gain-scheduling strategies at modest design efforts. For global stabilization in the presence of $L_2/l_2$ disturbances see [14,19] for
continuous-time linear plants and [4,9] for discrete-
time ones. The more general case of disturbances in
$L_p/L_p$, $p \in [1,\infty]$, has been considered in [9,17] under
the quite limiting condition of input-additive dis-
turbances, viz. $\left[ G_i \right] - \left[ G_i \right]$. In [18], such a limiting
matching condition has been removed and a solution
for the case $p = 2$ has been presented along with a conjecture that such a solution can be generalized to
the case $p = \infty$ (of interest here).

Finally, an example is reported where the proposed
strategy is contrasted, in terms of performance reg-
ulation, with a method based on the classical notion of
positive d-invariance presented in [2]. It turns out that
the proposed control strategy, although still presenting
a quite consistent level of conservativeness, remarkably
outperforms the competitor in many critical cases.

2. Preliminary and Problem Formulation

Consider system (1) rewritten in a more compact form
\[ x(t+1) = \Phi x(t) + G u(t) + \tilde{G} w(t), \quad x(0) = x, \quad (2) \]
where $t \in Z_+: = \{0, 1, \ldots\}$; $x(t) \in \mathbb{R}^n$ is the state
vector; $u(t) \in \mathbb{R}^m$ is the manipulable input; $w(t) \in \mathbb{R}^n$ an
exogenous persistent disturbance. In what follows we
assume (see (1)) that
\[ \tilde{G}_u = 0 \quad (3) \]
and restrict ourselves to ANCBI systems, that is,
\[ (\Phi, G) \text{ stabilizable}, \]
\[ |\lambda| \leq 1, \quad \forall \lambda \in \text{sp}\{\Phi\}, \quad (4) \]
where sp\{\Phi\} denotes the spectrum of $\Phi$, viz. the set of
all its eigenvalues. It is also assumed that the dis-
turbance is bounded
\[ w(t) \in \Omega_w, \quad \forall t \in \mathbb{Z}_+, \quad (5) \]
and the plant input is subject to the following
saturation-type constraint
\[ u(t) \in \Omega_u, \quad \forall t \in \mathbb{Z}_+, \quad (6) \]
where $\Omega_u \subset \mathbb{R}^n$ and $\Omega_u \subset \mathbb{R}^m$ define compact pol-
yhedral regions
\[ \Omega_u := \{ u \in \mathbb{R}^n : -u^- \leq u \leq u^+ \}, \]
\[ \Omega_w := \{ w \in \mathbb{R}^n : -w^- \leq w \leq w^+ \}, \quad (7) \]
where $u^- := [u_1^-, u_2^- \ldots u_m^-] \in \mathbb{R}^m$ and $u^+ :=
[u_1^+, u_2^+ \ldots u_m^+] \in \mathbb{R}^m$, $\mathbf{u}_i^- > 0,$ $\mathbf{u}_i^+ > 0, \quad \forall i \in \mathbb{N} :=
\{1, 2, \ldots, m\}$, the same vectorial notations holding for
\[ w^- \in \mathbb{R}^w \text{ and } w^+ \in \mathbb{R}^w \] and the vector inequalities in (7)
interpreted in a componentwise sense.

The aim is to find a state-feedback regulation
\[ u(t) = g(x(t)), \quad (8) \]
such that the regulated system subject to (5)–(6)
is simultaneously globally internally asymptotically
stable (in the absence of disturbances) and semi-
globally externally finite-gain BIBO stable, viz. there
exists a compact set containing the origin, whose size
may depend on the given disturbance magnitude,
which is globally attractive for all state trajectories of
the closed-loop system which ultimately enter this set
without constraints violation.

Conditions under which a control law (8) with the
prescribed properties can be determined for ANCBI
systems have been given [17].

Lemma 1. [17,18]. Consider (2) satisfying (4) with
disturbances (5) and saturation constraint (6). Then, a
control law (8) exists which achieves simultaneously
global internal asymptotic stability and semiglobal
external $L_\infty$ finite-gain stability if and only if the dis-
turbance enters the asymptotically stable subsystem
$(\Phi_u, G_s)$ of (2), that is (3) holds true.

Before continuing, we recall some properties of the
LQ output disturbance-free regulation (LQR) pro-
blem which are relevant for our subsequent develop-
ments. To this end, consider, for $\rho \in (0, \infty)$, the
following LQR cost
\[ J_\rho(x, u(\cdot)) = \sum_{i=0}^{\infty} \{ |x(t)|_{\Psi_x}^2 + \rho |u(t)|_{\Psi_u}^2 \} \]
(9)
where $\Psi_x = \Psi_x > 0, \quad \Psi_u = \Psi_u > 0$, and $|v|_{\Psi}^2 := v^T \Psi v$.
It is well known that, under the first assumption in (4),
the LQR state-feedback control
\[ u(t) = F_\rho x(t), \]
\[ F_\rho = - (\rho \Psi_u + G' P_\rho G)^{-1} G' P_\rho \Phi \]
(10)
minimizes (9) and asymptotically stabilizes (2) [11]. In
(10), $P_\rho = P_\rho > 0$ is the unique positive definite
solution of the following algebraic Riccati equation (ARE)
\[ P_\rho = \Phi' P_\rho \Phi - \Phi' P_\rho G (\rho \Psi_u + G' P_\rho G)^{-1} \]
\[ \times G' P_\rho \Phi + \Psi_x. \]
(11)
Moreover, the minimal cost equals
\[ \min_{u(\cdot)} J_\rho(x, u(\cdot)) = x' P_\rho x. \]
(12)
We recall also here that (11) can be rewritten in term of the closed-loop system matrices \( \Phi_\rho := \Phi + GF_\rho \) as
\[
P_\rho = \Phi'_\rho P_\rho \Phi_\rho + \rho F'_\rho \Psi_u F_\rho + \Psi_x.
\] (13)

A key property of (10) is that it makes the ellipsoidal set
\[
C(P_\rho, \sigma^2) := \{ x \in \mathbb{R}^n : x'P_\rho x \leq \sigma^2 \}
\] (14)
positively invariant for any arbitrary positive real \( \sigma^2 > 0 \). In fact, \( x(t + k) = \Phi^k x(t), \forall k \in \mathbb{Z}_+ \) along the optimal trajectory and (13) implies that
\[
\Phi'_\rho P_\rho \Phi_\rho - P_\rho \leq 0 \quad \text{and} \quad P_\rho \geq \Psi_x
\] (15)

We recall further the following results on the monotonic properties of \( P_\rho \), and \( \bar{P}_\rho := P_\rho / \rho \), whose proof can be found in [5].

**Lemma 2.** Let (2) satisfy the first assumption in (4) and \( \rho_1, \rho_2 \) be two positive reals. Then, \( \rho_2 > \rho_1 \) implies
\[
P_{\rho_2} > P_{\rho_1}, \quad \bar{P}_{\rho_2} < \bar{P}_{\rho_1}.
\] (16)

Further, \( \lim_{\rho \to \infty} \bar{P}_\rho = \bar{P}_\infty \geq 0 \), with \( \bar{P}_\infty \) the unique solution of the ARE (with \( \rho = \infty \)) for which \( \Phi_\infty := \lim_{\rho \to \infty} \Phi + GF_\rho \) has eigenvalues on the closed unit disk. Further,
\[
\bar{P}_\infty = 0_{n \times n}, \quad F_\infty = 0_{n \times n}
\] (17)
provided that the plant is ANCBI, viz. also the second assumption in (4) is satisfied.

Based on the invariance property (15), the control action (10) gives rise to feasible trajectories in the disturbance-free case provided that
\[
x \in C(\bar{P}_\rho, z^2),
\] (18)
where \( z^2 := \lambda(\Psi_u)^2 \), with \( v := \min \{ \min_{i \in \mathbb{M}} \{ u^*_i \}, \min_{i \in \mathbb{M}} \{ u^+_i \} \} \) and \( \lambda(\Psi_u) \) denoting the smallest eigenvalue of \( \Psi_u \). In fact, (18) implies (6) because, for the optimal input sequence, \( \sum_{i=0}^{\infty} |u(t)|^2 \leq x'P_\rho x \leq z^2 \Rightarrow |u(t)| \leq v. \) Notice that \( z^2 \) does not depend on \( \rho \).

Further, thanks to the monotonic properties of Lemma 2, admissible closed-loop evolutions for arbitrarily large initial states \( x \) can be ensured under the LQ state-feedback control law (10) by selecting a sufficiently large value for \( \rho \). On the other hand, observe that there is not limitation in selecting \( \rho \) as small as possible, as long as (18) is satisfied.

Different ways of dealing with persistent disturbances have been proposed in literature, all of them consisting of exploiting a robust variant of positive invariance, viz. positive invariance with respect to all possible disturbance realizations. In [6], the notion of controlled disturbance (cd) invariance was used. A set \( \Sigma \) is cd-invariant if there exists a closed-loop input function \( u(x) \) such that all related closed-loop evolutions remain in \( \Sigma \) regardless of any possible disturbance realization. When \( u(x) = 0 \), the cd-invariance reduces to d-invariance considered in [7]. In particular

**Definition 1.** A set \( \Sigma \) containing \( 0_x \) is said to be positively d-invariant with respect the closed-loop system (2) and (10) if \( x \in \Sigma \) implies \( \Phi_\rho x + \bar{G}w \in \Sigma, \forall w \in \Omega_w \).

In order to exploit the positive d-invariance in a global setting, it is necessary to ensure that the state trajectories under the effect of disturbances remain bounded as \( \rho \to \infty \). To this end, consider for any finite \( \rho > 0 \) the following family of sets
\[
\mathcal{R}_\rho(T) := \{ x \in \mathbb{R}^n : x \in \sum_{t=0}^{T-1} \Phi'_\rho \bar{G} \Omega_w \}.
\] (19)

It represents the set of all states of the closed-loop system \( x(t + 1) = \Phi_\rho x(t) + \bar{G}w(t) \) reachable from the origin in \( T \) steps under any possible \( \Omega_w \)-valued disturbance sequence \( w(t) \). By virtue of the asymptotic stability of \( \Phi_\rho \), each element \( \mathcal{R}_\rho(T), T \in \mathbb{Z}_+ \), is compact and the sequence converges (in the Hausdorff sense) to a limit compact set \( \mathcal{R}_\rho \) which enjoys the following properties [10]: (1) \( \mathcal{R}_\rho \subset \mathcal{R}_\rho, \forall T \in \mathbb{Z}_+ \); (2) for every \( \varepsilon < 0 \) there exists a \( T \in \mathbb{Z}_+ \) such that \( \mathcal{R}_\rho \subset \mathcal{R}_\rho(T + \varepsilon)B(0_x) \), where \( B(0_x) \) denotes the unitary sphere of \( \mathbb{R}^n \) centered in \( 0_x \); (3) \( \mathcal{R}_\rho \) is the smallest d-invariant set, viz. the set where the closed-loop state trajectories remain ultimately trapped. Then, let \( G_\rho \) denote the \( l_\infty \)-induced gain from \( w \) to \( x \), viz.
\[
G_\rho := \sum_{i=0}^{\infty} \sigma \{ \Phi'_\rho \bar{G} \}
\] (20)
with \( \sigma \{ \} \) denoting the largest singular value, and \( W \in \mathbb{R}^n \) be a disturbance vector with components \( W_i := \max \{ w_i^-, w_i^+ \} \). It follows that along the closed-loop trajectories one has that \( \max |x(t)| \leq G_\rho |W| \) and hence \( \mathcal{R}_\rho \subset G_\rho |W|B(0_x) \). Moreover, by direct manipulations involving the fact that \( x'P_\rho x \leq \bar{\lambda}(\bar{P}_\rho)|x|^2 \), one finds that
\[
\mathcal{R}_\rho \subset C(\bar{P}_\rho, \bar{\lambda}(\bar{P}_\rho)G_\rho^2 |W|^2), \quad \forall \rho > 0
\] (21)
where \( \bar{\lambda}(\cdot) \) denotes the largest eigenvalue. Moreover, because of the decoupling condition (3), the open-loop \( l_\infty \)-induced gain \( \bar{G} := \sum_{i=0}^{\infty} \sigma \{ \Phi'_i \bar{G} \} \) from \( w \) to \( x \) is
finite and a corresponding compact reachable set $\mathcal{R}$, there exists. Then, problem (2)–(8) is solvable via (10) if $\mathcal{R} := \bigcup_{\rho > 0} \mathcal{R}_\rho$ is a bounded region of $\mathbb{R}^n$. This is the case if

$$\lim_{\rho \to \infty} G_{\rho} < \infty \quad (22)$$

In fact, in such a case a suitable upper-bound $G = \sup_{\rho > 0} G_{\rho}$ exists finite. Notice that (22) is satisfied by all asymptotically stable open-loop plants and l.o.g. may occur only for strictly ANCBI plants. Henceforth,

$$\mathcal{R}_\rho \subset \text{int}(\mathcal{P}_\rho, \lambda(\mathcal{P}_\rho)G^2|W|^2), \quad \forall \rho > 0 \quad (23)$$

Hereafter, we will assume that condition (22) holds true. Notice that the latter would be satisfied if

$$\lim_{\rho \to \infty} G_{\rho} = \bar{G} \quad (24)$$

because of boundedness of the the open-loop $L_\infty$-gain $\bar{G}$, but this need not hold true for arbitrary feedback structure satisfying the structural conditions (1) and (3) even if $\lim_{\rho \to \infty} \Phi_{\rho} = \Phi$. In fact, consider the following counterexample.

\textbf{Example 1.} Consider the following closed-loop system

$$x_1(t + 1) = 0.5x_1(t) + w(t),$$

$$x_2(t + 1) = \frac{\rho}{1 + \rho} x_2(t) + \frac{1}{2\sqrt{1 + \rho}} x_1(t),$$

where $w(t)$ is the disturbance acting only on the stable subsystem. It results that $\Phi_{\rho} = \begin{bmatrix} 0.5 & 0 \\ \frac{\rho}{1 + \rho} & \frac{1}{2\sqrt{1 + \rho}} \end{bmatrix}$ is continuous in $\rho$, the open-loop $L_\infty$-induced gain is given by $\bar{G} = 2$ but $G_{\rho} = \sum_{i=0}^{\infty} \sigma\left(\begin{bmatrix} (0.5)^i & 0 \\ \alpha_{\rho}(i) & (\frac{\rho}{1 + \rho})^i \end{bmatrix} \right)$ becomes unbounded for $\rho \to \infty$. In fact, simple algebra shows that the closed-loop dc gain $G^\text{de}_\rho := (I - \Phi_{\rho})^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ equals the vector $\begin{bmatrix} \frac{2}{\rho} \end{bmatrix}$. Then, the result follows because $\sigma(G^\text{de}_\rho) \leq G_{\rho}$.

The standard d-invariance property hinges upon the “worst-case” paradigm, viz. the control action is selected by considering always the worst possible disturbance sequence even if the actual disturbance is zero. A control methodology based on the above notion of d-invariance has been recently presented in [2] for the control problem at hand and will be used in the final example for the sake of comparisons. It usually leads to bad regulation performance due to conservative approximations made in the derivation of the control design methodology. A possible way to mitigate the situation is to exploit a weaker concept of robust positive invariance, referred next to as positive quasi $d$-invariance.

\textbf{Definition 2.} A set $\Sigma$ is said to be positively quasi $d$-invariant with respect the closed-loop system (2) and (10) if there exists a nonempty proper subset $\Sigma_1 \subset \Sigma$, $0 \in \Sigma_1$, such that for all $x \in \Sigma_1$ it results that $\Phi^\rho_{\rho} x + \sum_{i=0}^{\rho-1} \Phi^i_{\rho} \bar{G}w(i) \in \Sigma$, $\forall w(0), w(1), \ldots, w(\rho-1) \in \Omega_w$, $\forall \rho > 0$.

Clearly, $\Sigma$ quasi d-invariant w.r.t. (2) and (10) implies that $\mathcal{R}_\rho \subset \Sigma$.

\section{The Quasi d-Invariance Approach}

In order to exploit the quasi-d-invariance (QDI) property, one needs to determine QDI restrictions of $C(\mathcal{P}_\rho, z^2)$ for a given $\rho$. In particular, we are interested in determining for the $\rho$-parameterized family $C(\mathcal{P}_\rho, z^2)$ a corresponding $\rho$-parameterized family of ellipsoidal subsets which yield $C(\mathcal{P}_\rho, z^2)$ QDI under $F_{\rho}$.

To this end, let us introduce $\rho$-parameterized ellipsoidal set restrictions $C(\mathcal{P}_\rho, w^2_{\rho})$ such that

$$C(\mathcal{P}_\rho, w^2_{\rho}) \subset C(\mathcal{P}_\rho, z^2) \quad (25)$$

$$C(\mathcal{P}_{\rho_1}, w^2_{\rho_1}) \subset C(\mathcal{P}_{\rho_2}, w^2_{\rho_2}) \quad \text{for } \rho_1 < \rho_2 \quad (26)$$

with $w^2_{\rho}$ to be determined. With respect to Definition 2, the sets $C(\mathcal{P}_\rho, w^2_{\rho})$ and $C(\mathcal{P}_\rho, z^2)$ will play respectively the role of $\Sigma_1$ and $\Sigma$. The basic idea here is to determine an interval $\rho \in (\rho, \infty)$ under which the inclusion (25) always holds true, viz. $w^2_{\rho} \leq z^2 \forall \rho \geq \rho$.

To this end, for a given $\rho$, the above sets are characterized by the property that for all $x \in C(\mathcal{P}_\rho, w^2_{\rho})$ it results $x(t) \in C(\mathcal{P}_\rho, z^2)$, $\forall w(0) \in \Omega_w$, $\forall t \in \mathbb{Z}_+$, where

$$x(t) := \Phi^\rho_{\rho} x + \sum_{i=0}^{\rho-1} \Phi^i_{\rho} \bar{G}w(i) \quad (27)$$

By direct manipulations, recalling that the following inequality (vectors $x$, $w$ and square matrix $P = P^T \geq 0) (x + w)' P(x + w) \leq (1 + \frac{1}{\epsilon_1}) x' P x + (1 + \epsilon_1) w' P w$ holds true for any arbitrary real $\epsilon_1 > 0$, one finds that

$$x'(t)P_{\rho} x(t) \leq (1 + \frac{1}{\epsilon_1}) |\Phi^\rho_{\rho} x|^2 + (1 + \epsilon_1) \sum_{i=0}^{\rho-1} \Phi^i_{\rho} \bar{G}w(i) \leq (1 + \frac{1}{\epsilon_1}) x' P_{\rho} x + (1 + \epsilon_1) \bar{G}w(i)^2 \quad (28)$$
with $G^2$ and $|W|^2$ defined in (19)–(23). Then, for a
chosen constant $\varepsilon_1$ (typically $\varepsilon_1 = 1$ but different
values may vary the level of conservativeness), one can
ensure that $x(t) \in C(P_{\rho}, z^2)$ for all $t$, if there exists at
least a $\rho$ such that
\[
\left(1 + \frac{1}{\varepsilon_1}\right)x^T P_{\rho} x + (1 + \varepsilon_1)\hat{\lambda}(P_{\rho}) G^2 |W|^2 \leq z^2
\]
Equivalently, (29) can be rewritten in the form
\[
x^T \hat{P}_{\rho} x \leq w_{\rho}^2,
\]
making sense only if positive. In such a case, it characterizes the desired family of ellipsoidal restrictions
(25). Notice in particular that $w_{\rho}^2$ is a continuous monotonically increasing function of $\rho$ and it is
positive for all $\rho \in (\hat{\rho}, \infty)$ with
\[
\hat{\rho} := \max\{\rho > 0 : w_{\rho}^2 = 0\}
\]
The existence of such a $\hat{\rho}$ follows easily by the
monotonic properties of Lemma 2 for any arbitrarily large value of $W$ if the system is ANCBI. Moreover, the free parameter $\varepsilon_1$ could be chosen in order to make $\hat{\rho}$ as small as possible. As a conclusion, in the case of persistent disturbances acting on the plant, the control action (10) gives rise to feasible closed-loop trajectories, that is
\[
x(t) \in C(P_{\rho}, w_{\rho}^2) \Rightarrow x(t) \in C(P_{\rho}, z^2), \quad \forall t \quad (32)
\]
for any $\rho > \hat{\rho}$. Notice that in this case no limitations exist in selecting $\rho$ as large as required for satisfying the left-hand side of (32) due to arbitrarily large initial states $x$. \forall x, \exists \hat{\rho} : x^T \hat{P}_{\rho} x \leq w_{\rho}^2, \forall \rho \geq \hat{\rho}$. However, unlike the disturbance-free case, the presence of persistent bounded disturbances poses an upper-bound on the gain of the state-feedback control laws compatible with the saturation constraint.

The conclusion is that, at each time $t$, given the actual state $x(t)$ and feedback control law $F_{\rho(-1)}$, the control performance could be improved by using from $t$ onward, if there exists, a new feedback gain $F_{\rho(t)}$ corresponding to the smallest $\rho(t)$ satisfying $\rho(t) < \rho(t(-1))$, capable of ensuring feasible system evolutions, viz. the smallest $\rho(t)$ such that $x(t) \in C(P_{\rho(t)}, w_{\rho(t)}^2)$. All the above discussion, can be summarized in the following main result.

Theorem 1. Consider the ANCBI system (2) satisfying conditions (3) and (22), with disturbances (5) and saturation constraints (6). Choose $\rho > \hat{\rho}$ as a

convenient level of performance in steady-state and consider the following $\rho(t)$-selection logic
\[
\rho(0) := \text{argmin}_{\rho > \hat{\rho}} \{x^T \hat{P}_{\rho} x \leq w_{\rho}^2\}, \quad (33)
\]
Then, the time-varying feedback control law $u(t) = F_{\rho(t)} x(t)$, $F_{\rho(t)}$ as in (10), is simultaneously globally internally asymptotically stabilizing and achieves semiglobal external $\ell_\infty$ stability with finite-gain. Moreover, let $\rho > \hat{\rho}$ be such that
\[
\hat{\lambda}(P_{\rho}) G^2 |W|^2 := \frac{\varepsilon_1}{(1 + \varepsilon_1)(1 + 2\varepsilon_1)} z^2.
\]
Then, if $\rho > \hat{\rho}$ the sequence of $\rho(t)$ converges to $\rho$ in
finite time and regardless of any possible disturbance realization. On the contrary, if $\rho \leq \hat{\rho}$ it converges asymptotically to some limit in the interval $[\rho, \hat{\rho}]$, the limit depending on the actual disturbance realization.

Proof. The proof follows by considering that the sequence of $\rho(t)$ is piecewise-constant, non increasing and bounded from the below. The convergence in finite time regardless of any possible disturbance sequence follows by the fact that $\mathcal{R}_\rho \subset \text{int} \ C(P_{\rho}, w_{\rho}^2)$ for all $\rho > \hat{\rho}$. In particular, the latter inclusion follows because
\[
\hat{\lambda}(P_{\rho}) G^2 |W|^2 \leq w_{\rho}^2
\]
results for all $\rho > \hat{\rho}$ with $\hat{\rho}$ in (35) determined by equating the left and right terms in (36).

Then, under a constant state-feedback $F_{\rho}$ and starting from any initial state $x \in C(P_{\rho}, w_{\rho}^2)$, the state trajectories will reach asymptotically $\mathcal{R}_\rho$ where will remain ultimately trapped. Then, after a finite time the state will be contained inside $C(P_{\rho}, w_{\rho}^2)$ and a lower value for $\rho$ can be chosen until $\rho$ is reached. As long as such a value has been reached, there is not guarantee of further decrements and the limit values for $\rho(t)$ cannot be specified, possibly depending on the actual sequence $w(t)$.

Notice that the selection logic (33)–(34) can directly be implemented via bisection. At each time $t$, the value $\rho(t-1)$ is tentatively decreased, and the ARE (11) correspondingly solved, until $\rho(t)$ is determined within the prescribed number of bisection’s steps.
Remark 1. In order to identify situations in which the problem of simultaneous stabilization is possible while the disturbance enters the system without restriction, it is worth pointing out that the key point underlying the previous solution is the fact that the \( l_\infty \)-induced gain of the closed-loop disturbance-to-input map monotonically goes to zero as \( \rho \to \infty \). This is ensured by the facts that \( F_\rho \to 0_{m \times n} \) as \( \rho \to \infty \) for ANCBI systems and that the \( l_\infty \)-induced gain of the closed-loop disturbance-to-state map remains bounded as \( \rho \to \infty \) thanks to assumptions (3) and (22). When the latter does not hold true the situation is trickier, because in open-loop the disturbance-to-state map becomes unbounded and the asymptotic properties of the closed-loop system for \( \rho \to \infty \) are difficult to analyze. Nevertheless, simultaneous stabilization, although in local sense w.r.t. to the disturbance, can be obtained in special cases, e.g. when the \( l_\infty \)-induced gain of the closed-loop disturbance-to-input map remains bounded for any \( \rho \). Such cases are under investigation.

Example 2. Consider the following hydraulic three tanks system:

\[
\begin{align*}
S_1 \hat{x}_1(t) &= -\alpha_1 x_1(t) + \alpha_3 x_3(t) + u(t) + \gamma w(t), \\
S_2 \hat{x}_2(t) &= -\alpha_2 x_2(t) + \alpha_3 x_3(t) + \gamma w(t), \\
S_3 \hat{x}_3(t) &= \alpha_1 x_1(t) + \alpha_2 x_2(t) - 2\alpha_3 x_3(t) - 2\gamma w(t)
\end{align*}
\]

with \( S_i \) and \( x_i \) representing the section and, respectively, the water level in the three tanks, \( u(t) \) a manipulable input flow (positive or negative) feeding the first tank and \( w(t) \) a persistent disturbance acting on the system. It is assumed that an auxiliary pump is used to feed tanks 1 and 2 from tank 3, with a flow proportional to \( x_3 \). Observe that \( w(t) \) enters on the system in such a way that the overall amount of water in the three tanks (for \( u(t) \equiv 0 \)) remains constant. It can be interpreted as a perturbation on the nominal flow rate of the auxiliary pump.

By setting \( S_i = 1, \alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.5 \gamma = 0.001 \), and discretizing (Zoh) the system with a sampling time of \( 10 \) s, one arrives to a discrete-time system that satisfies the condition \( \tilde{G}_u = 0 \) in the appropriate basis (see (1)). This simply follows by considering that the system matrix (continuous-time) has a unique eigenvalue in \( \lambda = 0 \) (being all others asymptotically stable) corresponding to the eigenspace \( x_1 + x_2 + x_3 = \int u \) which is unreachable from the disturbance \( w \).

It is further assumed that the input is constrained between \([-1,1]\) and we consider both the cases of low and high disturbances by letting them take values in \([-1,1]\) and, respectively, \([-1000,1000]\]. We run the simulations with initial condition \( x(0) = [10 0 5] \), cost weights \( \Psi_x = I, \Psi_u = 1 \) and a stochastic white noise as disturbance.

Figure 1 shows the state, input and \( \rho \) evolutions for the proposed quasi d-invariant approach under both low (continuous) and high (dashed) disturbances whereas Fig. 2 reports the same quantities pertaining to the d-invariant-based approach of [2]. For this plant it is found that both approaches behave comparably (the d-invariant approach being slightly better) under low disturbances but the quasi-d-invariant approach is less sensitive to disturbance and still works well under very high disturbance levels.

4. Conclusions

This paper has addressed the problem of simultaneous internal global asymptotic and external semiglobal finite-gain BIBO stabilization of discrete-time LTI systems subject to input saturations and persistent disturbances.
The main contribution of this paper is to have shown that the quasi d-invariance is instrumental to the achievement of global stabilizing control laws for input-saturated linear systems less affected by conservativeness than those based on the usual notion of positive d-invariance and related methodologies [2].

A final example has been presented showing that the QDI scheme exhibits in many critical cases less conservative regulation performance and disturbance sensitiveness and it is more easily implementable. The other scheme, due to very crude approximations, leads to very sluggish responses and very small control actions with respect to the disturbance level [2]. However, it also gives evidence that conservativeness is still present in the QDI-based control scheme and further research efforts are requested in order to ameliorate the solution.

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References

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