Critical phenomena in evolving networks: Ferromagnetic phase transition in evolving network of stochastic cellular automata

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It is shown by MC simulations that ferromagnetic system modeled by cellular automata with synchronously rewired interconnections changes critically when preference in rewireing is added — the system becomes resistant against stochastic noise.

1 Motivation
The real-world systems: technological, biological or social networks are networks where connections between elements change in time. In the following we consider how changes in network connections influence properties of the system of ferromagnetically interacting elements living on the evolving network. In particular we concentrate on critical phenomena properties related to both topological changes and ferromagnetic phase transition. As a tool we use equilibrium statistical mechanics which is a thermodynamic formalism providing understanding critical phenomena in complex networks [1].

2 The model
Let us consider a network of interacting two-state systems called spins: \(\sigma_i \in \{-1, 1\}\) placed on a network and \(i = 1, 2, \ldots, N\) describes the network index. Let \(n(i)\) denotes a set of spins with which the \(i\)th spin interacts. Each step of evolution is defined as the following two-fold synchronized procedure:

I stochastic changes of network of interactions which intensity are measured by \(p\).
Two dynamics of the network reorganization are considered and both dynamics are performed with the constrains: loops are excluded and disconnection of the network is prohibited,:

SR stochastic rewiring: for each vertex \(i\), an \((i, j)\)-edge, \(j \in n(i)\), is chosen at random with probability \(p\) and the edge end \(j\) is rewired to some other randomly chosen vertex \(k\).

PR preference rewiring with threshold \(T\): for each vertex \(i\), an \((i, j)\)-edge, \(j \in n(i)\), is chosen with the preference: if \(\deg(j) > 1\) then probability to unlink is \(p\frac{T}{\deg(j)}\), and then the edge end \(j\) is rewired to a vertex \(k\) chosen according to the following preference \(\frac{\deg(k)}{T}\).

PR demands reading the vertex degree table. To amplify synchronization effect we assume that the degree table is updated once per time step. One can say that if \(p\) increases then information about vertex degrees becomes less accurate.

II stochastic evolution of spin states with noise intensity measured by \(\varepsilon\)

\[
\sigma_i(t+1) = \begin{cases} 
\text{sgn} \sum_{j \in n(i,t)} \sigma_j(t) & \text{with probability } 1 - \varepsilon \quad \text{n}(i,t) \text{ set of neighbors of } i \text{ spin at time } t \\
-\text{sgn} \sum_{j \in n(i,t)} \sigma_j(t) & \text{with probability } \varepsilon \quad \text{n}(i,t) \text{ set of neighbors of } i \text{ spin at time } t
\end{cases}
\]

(1)

If \(\text{sgn} \sum_{j \in n(i,t)} \sigma_j(t) = 0\) then we assume \(\text{sgn} \sum_{j \in n(i,t)} \sigma_j(t) \equiv \sigma_i(t)\).

Our study is numerical. We start with a square lattice with periodic boundary conditions and all spins set in +1 state. Then for a give \((p, \varepsilon)\) we apply the above defined dynamics. It appears that stationary spin states emerge in the considered system. We believe that these stationary states determine the equilibrium statistical ensemble and therefore we can apply the equilibrium statistical mechanics understanding of criticality to the observed phenomena. In computer simulations we calculate the magnetization (the normalized sum of all spin states) and susceptibility (the variation of the magnetization) to find the critical exponents what determine the universality class of the thermodynamic ferromagnetic phase transition.

3 Results
In Figs 1–4 we present magnetization and susceptibility found in stationary states for different \(\varepsilon\) and \(p\). Results obtained by others [2] suggest that we should observe the mean-field universality class. We found that in case of SR independently of \(p\) the magnetization and susceptibility change critically their values. The transition singularities are characterized by the power-laws with the magnetization exponent \(\beta \approx 0.5\) and susceptibility exponent \(\gamma \approx 1\), hence equal to the exponents of the mean-field class, see Fig.1. When \(p\) increases the value of the critical noise moves from \(\varepsilon_{\text{crit}} \approx 0.14\) for \(p = 0\) to \(\varepsilon_{\text{crit}} \approx 0.20\)

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for $p = 1$. Applying mean-field approximation one can say that rewiring of interactions is felt by spins as the average number of neighbors increases. In Fig.2 we show the distribution of vertex degree of stationary states. It appears that all stationary states distributions are Poissonian with mean vertex value $<k> = 4$.

When considering PR then one meets some difficulty in reaching stationarity if $p$ is small. It appears that if $p \leq 0.01$ then the influence of imperfect information about vertex degrees can be neglected and therefore qualitative changes to network topology are present until the same stationary limit network is reached. The network ensemble reached at $p = 0.01$ can be considered as the representative case of all these systems [3]. The limit ensemble can be characterized as the sum of a random subgraph consisting of few vertices with large number of multiple edge connections — the frozen core of the network, and star-like subgraphs attached to each vertex of the core, see Fig.3. The network is frozen since PR changes edges from the core only. When $p$ is increased the frozen core becomes even more condensed (see $p = 0.1$). However if the intensity of rewiring is high ($p = 1.0$) then the frozen core does not emerges at all. The network undergoes a topological phase transition at about $p = 0.2$ [3]. When the network is in the frozen state then the stationary magnetization is low and only slightly depends on $\varepsilon$, see Fig.4. But the singular features of magnetization and susceptibility are observed at $\varepsilon_{\text{crit}} \approx 0.34$ with the exponents: $\beta \approx 0.28$, $\gamma \approx 0.57$ which are far from mean-field values. If PR goes with $p = 1$ then the ferromagnetic phase transition takes place at $\varepsilon_{\text{crit}} \approx 0.27$ with the mean-field exponents. Notice that when $p$ increases then $\varepsilon_{\text{crit}}$ decreases what is opposite to the feature observed in SR.

**Summary.** Cellular automata with stochastic majority rule on the evolving network restores thermodynamic properties known for the Ising model on static complex networks. Majority rule gains profit from network changes, namely, the ferromagnetic system becomes more resistant to stochastic noise. Synchronous rewiring, seen as the evolution with limited information, influences critically the network ensemble when preference rewiring is applied. The network heterogeneity changes ferromagnetic critical behavior — the transition is less sharp and the critical noise increases.

**References**

