ON NECESSARY CONDITIONS FOR CONVERGENCE OF STATIONARY ITERATIVE METHODS FOR HERMITIAN DEFINITE AND SEMIDEFINITE LINEAR SYSTEMS*

ANDREAS FROMMER† AND DANIEL B. SZYLD‡

Abstract. In an earlier paper [SIAM J. Matrix Anal. Appl. vol. 30 (2008), 925–938] we gave sufficient conditions in terms of an energy seminorm for the convergence of stationary iterations for solving singular linear systems whose coefficient matrix is Hermitian and positive semidefinite. In this paper we show in which cases these conditions are also necessary, and show that they are not necessary in others. We present similar results for the nonsingular case, thus providing some converse results for theorems on P-regular splittings.

Key words. linear systems, singular systems, stationary iterative methods, seminorm, convergence analysis

AMS subject classifications. 65F10, 65F20.

1. Introduction. We consider the linear system

\[ Ax = b, \tag{1.1} \]

where the coefficient matrix \( A \in \mathbb{C}^{n \times n} \) is assumed to be Hermitian and either positive definite or singular and Hermitian positive semidefinite.

If \( A \) is large and sparse, iterative methods for solving (1.1) are necessarily the standard approach. In this paper, we focus on stationary iterative methods, including, for example, certain algebraic multigrid methods and additive and multiplicative Schwarz methods. Sometimes, these iterations are accelerated by using them as preconditioners to Krylov subspace methods like Conjugate Gradients. While we do not consider the latter aspect in any detail in this work, let us just mention that one usually assumes convergence of the preconditioner as a prerequisite in this context, so our work is relevant in this case as well.

For a stationary iterative method, one usually considers a splitting \( A = M - (M - A) \) with \( M \) nonsingular, a resulting iteration matrix \( H = I - M^{-1}A \), together with the iteration

\[ x^{k+1} = Hx^k + M^{-1}b. \tag{1.2} \]

As is well known, when \( A \) is nonsingular, the iteration (1.2) converges to the unique solution of (1.1) for any initial vector \( x^0 \) if and only if \( \rho(H) < 1 \), where \( \rho(H) \) is the spectral radius of \( H \), and such matrix \( H \) is termed convergent\(^1\); see, e.g., [3], [18]. We mention in passing that in this case of nonsingular \( A \) it holds that for a given matrix \( A \) and a given convergent matrix \( H \), there exists a unique corresponding nonsingular matrix \( M \) such that \( H = I - M^{-1}A \) [9].

A sufficient condition for the convergence of the iteration (1.2) when \( A \) is Hermitian and positive definite is given by the following result in [19, Satz 1, p. 156]
(see also [6, p. 111], [7, p. 21]), now commonly called the *P-regular splitting theorem*. A splitting of (the Hermitian positive definite matrix) \( A = M - (M - A) \) is called *P-regular* if \( M + M^H - A \) is positive definite [13].

**Theorem 1.1.** Let \( A \in \mathbb{C}^{n \times n} \) be Hermitian positive definite, and \( A = M - (M - A) \) be a P-regular splitting. Then \( \rho(H) < 1 \), with \( H = I - M^{-1}A \).

The classical proof uses Stein’s theorem (see, e.g., [13, 7.1.8]). Here we offer *essentially the same proof* but highlight the fact that one not only has \( \rho(H) < 1 \) but even \( \| H \|_A < 1 \) with \( \| H \|_A \) the operator norm induced by the energy norm

\[
\| x \|_A = \left< x, x \right>_A^{1/2}, \quad \text{where} \quad \left< x, x \right>_A = \left< Ax, x \right> = \left< x, Ax \right> \tag{1.3}
\]

\( \langle x, y \rangle \) denoting the standard Euclidian inner product. Moreover, the converse, i.e. \( \| H \|_A < 1 \) implies that the splitting is P-regular then also holds.

**Theorem 1.2.** Let \( A \in \mathbb{C}^{n \times n} \) be Hermitian positive definite. Then, \( A = M - (M - A) \) is a P-regular splitting if and only if \( \| H \|_A < 1 \).

**Proof.** We write

\[
H^H AH = (I - M^{-1}A)^H A(I - M^{-1}A) = A - AM^{-H}(M + M^H - A)M^{-1}A.
\]

Then, \( \| Hu \|_A^2 = u^H HAHu < u^H Au = \| u \|_A^2 \) for all vectors \( u \neq 0 \) if and only if \( M + M^H - A \) is positive definite. \( \square \)

In this paper we investigate in detail the role of the \( A \)-norm (or a corresponding seminorm if \( A \) is semidefinite) in convergence theorems for iterations based on splittings. For example, we show that the converse of Theorem 1.1 is not true in general, i.e., we may have \( \rho(H) < 1 \), with \( H = I - M^{-1}A \), but \( M + M^H - A \) may not be positive definite. On the other hand, with the additional hypothesis of \( M \) being Hermitian, this converse of Theorem 1.1 holds. In other words, given Theorem 1.2, if \( M \) is Hermitian, \( H \) convergent implies \( \| H \|_A < 1 \). We show this in Section 2.

We mention that other converses of Theorem 1.1 are possible. A typical result is that for a given P-regular splitting and \( A \) Hermitian, \( \rho(H) < 1 \) implies that \( A \) is positive definite, see [13, E71.9], and further results of this kind, see, e.g., [1], [6, p. 111], and references therein.

We turn our attention now to the case where \( A \) is singular and Hermitian positive semidefinite. Denoting by \( \text{Null}(A) \) the nullspace of \( A \) and by \( \text{Range}(A) \) its range, we assume that \( b \in \text{Range}(A) \). This implies that the solution set of (1.1) is nonempty and it is given as an affine space \( x^* + \text{Null}(A) \) for some \( x^* \in \mathbb{C}^n \) solution of (1.1).

Following [5], we consider the very general situation in which we are given an iteration matrix \( H \) for (1.1) of the form

\[
H = I - \tilde{M} A, \tag{1.4}
\]

where \( \tilde{M} \in \mathbb{C}^{n \times n} \) is a matrix which might be singular but is injective on \( \text{Range}(A) \), i.e.,

\[
\text{Null}(\tilde{M} A) = \text{Null}(A). \tag{1.5}
\]

For a given matrix \( H \), the matrix \( \tilde{M} \) is not necessarily unique; cf. [2]. The matrices \( H \) and \( \tilde{M} \) induce the iteration

\[
x^{k+1} = Hx^k + \tilde{M} b. \tag{1.6}
\]

\[2\]
Convergence of the iteration \((1.6)\) is equivalent to \(H\) being semiconvergent according to the following definition\(^2\); see, e.g., [3], [4], [16].

**Definition 1.3.** A matrix \(H \in \mathbb{C}^{n \times n}\) is called semiconvergent, if \(\rho(H) = 1\), \(\lambda = 1\) is the only eigenvalue of modulus 1 and \(\lambda = 1\) is a semisimple eigenvalue of \(H\), i.e., its geometric multiplicity is equal to its algebraic multiplicity.

The general form of the iteration operator from \((1.4)\) applies in particular to iterations induced by splittings of the form \(A = M - (M - A)\), \(M\) nonsingular, in which case \(\tilde{M}\) is taken to be \(M^{-1}\). Then condition \((1.5)\) is automatically satisfied. There are iterations which can be interpreted as being of the form \((1.4)\) with \(\tilde{M} = M^\dagger\), the Moore-Penrose pseudoinverse of some singular matrix \(M\); see [4], [10], [11], where such iterations are studied. This situation occurs in particular in the analysis of Schwarz iterations where the artificial boundary conditions between subdomains are of Neumann or Robin type; see, e.g., [12], [15], [17].

Since we are considering \(A\) semidefinite, then, \((1.3)\) is a seminorm, and not a norm. The following sufficient condition for the convergence of the iteration \((1.6)\) was proved in [5].

**Theorem 1.4.** Let \(A\) be Hermitian and positive semidefinite. Let \(H = I - \tilde{M}A\) be the iteration operator of the iteration \((1.6)\). Assume that the following holds:

\[
x \notin \text{Null}(A) \implies \|Hx\|_A < \|x\|_A.
\]  
(1.7)

Then,

(i) \(\text{Null}(\tilde{M}A) = \text{Null}(A)\), i.e., \(\tilde{M}\) is injective on \(\text{Range}(A)\).

(ii) \(H\) is semiconvergent.

**Remark 1.5.** Let \(\Pi\) denote the orthogonal projection onto \(\text{Range}(A)\). Since \(\mathbb{C}^n = \text{Range}(A) \oplus \text{Null}(A)\) we have \(\|x\|_A = \|\Pi x\|_A\) for all \(x\). Since \(Hx = x\) for \(x \in \text{Null}(A)\), we also have \(\Pi Hx = \Pi x\) for all \(x\). This implies that for any \(x\) we have \(\|x\|_A = \|\Pi x\|_A\) as well as \(\|Hx\|_A = \|H\Pi x\|_A\), and therefore we see that \((1.7)\) is equivalent to

\[
x \in \text{Range}(A) \implies \|Hx\|_A < \|x\|_A.
\]  
(1.8)

Let us define

\[
\|H\|_A = \sup_{x \in \text{Range}(A)} \frac{\|Hx\|_A}{\|x\|_A}.
\]

It follows that condition \((1.8)\), and thus condition \((1.7)\) is equivalent to \(\|H\|_A < 1\). As a consequence, Theorem 1.4 says in particular that \(\|H\|_A < 1\) implies that \(H\) is semiconvergent.

In summary, Theorems 1.1 and 1.4 present sufficient conditions for the \(A\)-norm or the \(A\)-seminorm of the iteration matrix is less than 1 which in turn implies the convergence of the iterations \((1.2)\) and \((1.6)\), respectively. There are several papers describing other sufficient conditions for such convergence, especially in the singular semipositive definite case; see [4], [5], [8], [10], [11], [14]. In this paper, for \(A\) assumed to be Hermitian positive definite or positive semidefinite we present necessary conditions on the splitting for such convergence, highlighting the role of the \(A\)-norm and \(A\)-seminorm. We formulate them in sections 2 and 3.

---

\(^2\)We note that in some papers such a matrix is simply called convergent.
We end this introduction by stating the counterparts to Theorems 1.1 and 1.2 in the semidefinite case, and by establishing a condition equivalent to (1.5). We say that a Hermitian matrix $B \in \mathbb{C}^{n \times n}$ is positive definite on a subspace $V$ of $\mathbb{C}^n$ if $\langle Bx, x \rangle > 0$ for all $x \in V, x \neq 0$.

**Theorem 1.6.** Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and positive semidefinite. Let $M$ be nonsingular, let $H = I - M^{-1}A$ and assume that $M + MH - A$ is positive definite on $\text{Range}(M^{-1}A)$. Then $H$ is semiconvergent, and the iteration (1.2) converges to a solution whenever $b \in \text{Range}(A)$.

This result is often attributed to Keller [8], but the conclusion that the iterates converge to a solution can actually already be found in [19, Hauptsatz, p. 160].

**Theorem 1.7.** Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and positive semidefinite. Let $M, \tilde{M} \in \mathbb{C}^{n \times n}$ satisfy

$$M\tilde{M}A = A$$

and let $H = I - \tilde{M}A$. Then $\|H\|_A < 1$ if and only if $M + MH - A$ is positive definite on $\text{Range}(\tilde{M}A)$.

This result was shown in [5]; see also [11] for a special case. In the light of Theorem 1.7 we see that in Theorem 1.6 we can not only conclude that $H$ is semiconvergent, but also that $\|H\|_A < 1$.

As we shall see, the sufficient conditions presented in Theorem 1.7 are also necessary when $\tilde{M}$ is Hermitian. When $\tilde{M}$ is not Hermitian, this is not always the case.

We now formulate a result which clarifies the relation between (1.9) of Theorem 1.7 and (1.5).

**Proposition 1.8.** A matrix $M$ such that $M\tilde{M}A = A$ exists if and only if $\tilde{M}$ is injective on $\text{Range}(A)$. In this case, all such matrices $M$ induce an identical quadratic form $\langle (M + MH - A)x, x \rangle$ on $\text{Range}(\tilde{M}A)$.

**Proof.** If $\tilde{M}$ is injective on $\text{Range}(A)$, the images $\tilde{M}x_i$ of a basis $\{x_1, \ldots, x_\ell\}$ of $\text{Range}(A)$ are linearly independent, and any matrix $M$ which maps these images back to $x_i, i = 1, \ldots, \ell$ fulfills $M\tilde{M}A = A$. Conversely, if there exists $M$ such that $M\tilde{M}A = A$, we trivially have that $\tilde{M}$ is injective on $\text{Range}(A)$. Finally, if two matrices $M$ and $\overline{M}$ differ only by their action on a space complementary to $\text{Range}(M\tilde{M}A)$, then $\langle Mx, x \rangle = \langle \overline{M}x, x \rangle$ for $x \in \text{Range}(M\tilde{M}A)$, and, similarly, $\langle MHx, x \rangle = \langle x, Mx \rangle = \langle x, \overline{M}x \rangle = \langle MHx, x \rangle$ for $x \in \text{Range}(\overline{M}A)$. $\square$

Proposition 1.8 says in particular that if $\tilde{M}$ is injective on $\text{Range}(A)$, then either for all $M$ with $M\tilde{M}A = A$ the matrix $M + MH - A$ is positive definite on $\text{Range}(\tilde{M}A)$ or $M + MH - A$ is positive definite for no such $M$.

2. **The positive definite case.** We begin by considering necessary and sufficient conditions for convergence of the iteration (1.2), i.e., in the case of $A$ Hermitian positive definite. These will serve as models for the conditions in the case of $A$ singular and Hermitian semipositive definite, which are discussed in the next section.

**Theorem 2.1.** Let $A \in \mathbb{C}^{n \times n}$ be Hermitian positive definite, and let $M$ be invertible. Let $H = I - M^{-1}A$. Then,

(i) $\|H\|_A < 1$ if and only if $M + MH - A$ is positive definite.

(ii) $M + MH - A$ positive definite implies that $\rho(H) < 1$.

(iii) $\rho(H) < 1$ does not necessarily imply that $M + MH - A$ is positive definite.

(iv) If $M$ is Hermitian, then $\rho(H) < 1$ implies that $M + MH - A = 2M - A$ is positive definite (or, by (i), equivalently $\|H\|_A < 1$).
To prove (i), we first note that since \( \text{Null}(A) \perp \text{Range}(A) \), it follows immediately from (i); see Theorem 1.1.

(iii) This was shown already with an example in [1]. For completeness, we reproduce it here. Let \( A = I_2 \) (the \( 2 \times 2 \) identity matrix),

\[
M = \begin{bmatrix}
1 & 3/2 \\
0 & 1
\end{bmatrix}, \quad M^{-1} = \begin{bmatrix}
1 & -3/2 \\
0 & 1
\end{bmatrix}, \quad H = I - M^{-1}A = \begin{bmatrix}
0 & 3/2 \\
0 & 0
\end{bmatrix},
\]

so that \( \rho(H) = 0 < 1 \). But

\[
M + M^H - A = \begin{bmatrix}
1 & 3/2 \\
3/2 & 1
\end{bmatrix},
\]

with eigenvalues \(-1/2\) and \(5/2\). As it was made clear in [1], it is worth mentioning here that in this example the matrix \( M + M^H \) is positive definite.

(iv) With \( M \) being Hermitian, consider

\[
\hat{H} = A^{1/2}HA^{-1/2} = I - A^{1/2}M^{-1}A^{1/2}, \quad (2.1)
\]

which is also Hermitian, and has the same spectrum as \( H \). This implies in particular that \( \rho(H) = \rho(\hat{H}) \). But for the Hermitian matrix \( \hat{H} \) we have \( \rho(\hat{H}) = \|\hat{H}\|_2 \), and therefore \( \rho(H) = \rho(\hat{H}) = \|\hat{H}_2\| = \|A^{1/2}HA^{-1/2}\|_2 = \|H\|_A \). \( \Box \)

3. The positive semidefinite case. In this section, we state and prove a counterpart to Theorem 2.1 when the matrix \( A \) is singular and Hermitian positive semidefinite. To that end, we note that a construction such as (2.1) is possible, in the sense that given \( H = I - MA \) we can write \( \hat{H} = I - A^{1/2}MA^{1/2} \), but since there is no inverse of \( A^{1/2} \), the fact that \( H \) and \( \hat{H} \) have the same spectra needs to be shown using other tools. We show this in the next result, together with the relation between the corresponding eigenvectors. Let \( \Lambda(Z) \) denote the spectrum of the matrix \( Z \).

**Lemma 3.1.** Let \( A, \tilde{M} \in \mathbb{C}^{n \times n} \) be Hermitian positive semidefinite. Let \( M, \tilde{M} \in \mathbb{C}^{n \times n} \) satisfy \( MMA = A \). Let \( H = I - MA \) and \( \tilde{H} = I - A^{1/2}MA^{1/2} \). Then,

(i) \( Hx = x \) if and only if \( Mx = x \) if and only if \( x \in \text{Null}(A) \).

(ii) For \( \lambda \neq 1 \), \( \tilde{H}x = \lambda x \), \( \tilde{x} \neq 0 \) if and only if \( \tilde{x} = A^{1/2}x \) such that \( Hx = \lambda x \), \( x \neq 0 \), and \( x \in \text{Range}(A) \).

Therefore, \( \Lambda(\tilde{H}) = \Lambda(H) \).

**Proof.** Since \( A \) is Hermitian positive semidefinite, we can write \( A = XAX^H \) with \( XH = I \) and \( \Lambda \) a nonnegative diagonal matrix. Then \( A^{1/2} = XA^{1/2}X^H \). It follows directly that \( \text{Null}(A^{1/2}) = \text{Null}(A) \) and that \( \text{Range}(A^{1/2}) = \text{Range}(A) \). It also holds that \( \text{Null}(\tilde{A}) \perp \text{Range}(A) \). Furthermore, since by assumption, \( \tilde{M} \) is injective on \( \text{Range}(A) \) and \( \tilde{M} \) is Hermitian, this implies that \( \tilde{M} \) is a bijective linear mapping of \( \text{Range}(A) \) onto itself.

To prove (i), we first note that since \( \text{Null}(A^{1/2}) = \text{Null}(\tilde{A}) \), \( Ax = 0 \) implies both \( Hx = x \) and \( \tilde{H}x = x \). Conversely, if \( Hx = x \), then \( \tilde{M}Ax = 0 \), but then \( Ax = M\tilde{M}Ax = 0 \). Similarly, if \( Hx = x \), then \( A^{1/2}\tilde{M}A^{1/2}x = 0 \), i.e., \( \tilde{M}A^{1/2}x \in \text{Null}(A^{1/2}) \).

Since \( \tilde{M} \) is bijective as a mapping of \( \text{Range}(A) = \text{Range}(A^{1/2}) \) onto itself, then \( \tilde{M}A^{1/2}x \in \text{Range}(A^{1/2}) \) as well, but since \( \text{Null}(A^{1/2}) = \text{Range}(A^{1/2}) \), this implies \( A^{1/2}x = 0 \), and thus \( Ax = 0 \).

One direction of (ii) is easy. Let \( x \) be an eigenvector of \( H \) with \( x \notin \text{Null}(A) \), i.e., \( \lambda x = Hx \), for \( \lambda \neq 1 \). Multiplying with \( A^{1/2} \) we obtain

\[
\lambda A^{1/2}x = A^{1/2}x - A^{1/2}\tilde{M}A^{1/2}A^{1/2}x = (I - A^{1/2}\tilde{M}A^{1/2})A^{1/2}x = \hat{H}A^{1/2}x,
\]

5
and \( \hat{x} = A^{1/2}x \neq 0 \). For the other direction, let \( \hat{H}\hat{x} = \lambda \hat{x}, \lambda \neq 1, \hat{x} \neq 0 \). Since \( \hat{H} \) is Hermitian, then from part (i), \( \hat{x} \perp \null(A) \), i.e., \( \hat{x} \in \text{Range}(A) \), i.e., \( \hat{x} = A^{1/2}x \) for some \( x = A^{1/2}y \neq 0 \), i.e., for some \( x \in \text{Range}(A^{1/2}) = \text{Range}(A) \). We then have

\[
A^{1/2}(\lambda x) = \lambda \hat{x} = \hat{H}A^{1/2}x = (I - A^{1/2}M_{A}^{1/2})A^{1/2}x = A^{1/2}Hx.
\]

Since \( A^{1/2} \) is a bijection on \( \text{Range}(A^{1/2}) \), then \( \lambda x = Hx \). \( \square \)

We are ready to state the main theorem of the paper.

**Theorem 3.2.** Let \( A \in \mathbb{C}^{n \times n} \) be Hermitian positive semidefinite.

(i) Let \( M \) be nonsingular and let \( H = I - M^{-1}A \). Then, \( M + M_{H} - A \) is positive definite on \( \text{Range}(M^{-1}A) \) if and only if \( \| H \|_A < 1 \) (and thus \( H \) is semiconvergent).

(ii) Let \( M, \tilde{M} \in \mathbb{C}^{n \times n} \) satisfy \( M\tilde{M}A = A \) and let \( H = I - \tilde{M}A \). Then, \( M + M_{H} - A \) is positive definite on \( \text{Range}(\tilde{M}A) \) if and only if \( \| H \|_A < 1 \) (and thus \( H \) is semiconvergent).

(iii) Let \( H = I - \tilde{M}A \) such that \( H \) is semiconvergent. This does not necessarily imply that there exists \( M \) with \( M\tilde{M}A = A \) such that \( M + M_{H} - A \) is positive definite on \( \text{Range}(\tilde{M}A) \).

(iv) Let \( H = I - \tilde{M}A \) such that \( H \) is semiconvergent, assume that there exists \( M \in \mathbb{C}^{n \times n} \) such that \( A = M\tilde{M}A \) and let \( \tilde{M} \) be Hermitian. Then \( \| H \|_A < 1 \) (and thus, by (ii), \( M + M_{H} - A \) is positive definite on \( \text{Range}(\tilde{M}A) \)).

**Proof.** (i) is [5, Theorem 3.1].

(ii) is Theorem 1.7.

(iii) We provide an example inspired by the proof of Theorem 2.1 (iii).

\[
A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{M} = \begin{bmatrix} 1 & 3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -3/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

It follows then that \( M\tilde{M}A = A \), and that

\[
H = I - \tilde{M}A = \begin{bmatrix} 0 & 3/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

is semiconvergent. But we have that

\[
M + M_{H} - A = \begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

is not positive definite on \( \text{Range}(\tilde{M}A) = \text{Range}(A) = \text{span}\{(1,0,0)^T, (0,1,0)^T\} \). By Proposition 1.8, for any matrix \( \overline{M} \) with \( M\tilde{M}A = A \) the quadratic forms \( \langle (M + M_{H} - A)x, x \rangle \) and \( \langle (\overline{M} + \overline{M}_{H} - A)x, x \rangle \) coincide on \( \text{Range}(\tilde{M}A) \), implying that for any such \( \overline{M} \) the matrix \( \overline{M} + \overline{M}_{H} - A \) is not positive definite on \( \text{Range}(\tilde{M}A) \).

We note that, in analogy to the example in the proof of Theorem 2.1(iii), the matrix \( M + M_{H} \) is positive definite on \( \text{Range}(\tilde{M}A) \).

(iv) Let \( H = I - A^{1/2}\tilde{M}A^{1/2} \) and note that

\[
A^{1/2}Hx = \hat{H}A^{1/2}x \quad \text{for all} \ x, \quad (3.1)
\]
and, as shown in Lemma 3.1, \( \text{Range}(A^{1/2}) = \text{Range}(A) \). This gives

\[
\|H\|_A = \sup_{x \in \text{Range}(A)} \frac{\|Hx\|_A}{\|x\|_A} = \sup_{x \in \text{Range}(A)} \frac{\|A^{1/2}Hx\|}{\|A^{1/2}x\|} = \sup_{x \in \text{Range}(A)} \frac{\|HA^{1/2}x\|}{\|A^{1/2}x\|} = \sup_{y \in \text{Range}(A)} \frac{\|\hat{H}y\|}{\|y\|}.
\]

(3.2)

Since \( H \) is semiconvergent, for \( \lambda \in \Lambda(H) \), \( \lambda \neq 1 \), we have \( |\lambda| \leq \gamma \), with \( \gamma < 1 \) the modulus of the second largest (in modulus) eigenvalue of \( H \). By Lemma 3.1, \( \Lambda(\hat{H}) = \Lambda(H) \), and \( \text{Range}(A) \) is the orthogonal sum of eigenspaces of \( \hat{H} \), each corresponding to eigenvalues of modulus not exceeding \( \gamma \), so that in (3.2) we have

\[
\sup_{y \in \text{Range}(A)} \frac{\|\hat{H}y\|}{\|y\|} = \gamma < 1.
\]

Remark 3.3. The reader may have noticed that the example used for the proof of part (iii) in Theorem 2.1 is \( 2 \times 2 \), while that used in part (iii) of Theorem 3.2 is \( 3 \times 3 \). It turns out that in the \( 2 \times 2 \) case, \( A \) Hermitian positive definite, and \( H = I - \tilde{M}A \) semiconvergent, with \( M \) such that \( M\tilde{M}A = A \), implies that \( M + M^H - A \) is positive definite on \( \text{Range}(\tilde{M}A) \), as we show next.

If \( A = 0 \), we have \( H = I \) which is semiconvergent, and for any \( M \) the matrix \( M + M^H - A \) is trivially positive definite on \( \text{Range}(\tilde{M}A) = \{0\} \).

Let us now turn to the case where the rank of \( A \) is 1. By a change of basis, without loss of generality, we can assume that

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Condition (1.5) implies that

\[
\tilde{M} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}
\]

for some nonzero \( a \). Thus, we have that

\[
H = I - \tilde{M}A = \begin{bmatrix} 1 - a & 0 \\ -b & 1 \end{bmatrix}.
\]

Since \( H \) is semiconvergent, this implies that \( 0 < a < 2 \). Condition (1.9) gives

\[
M = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix},
\]

if \( b \neq 0 \) (and zero in the (2,2) position otherwise). Thus

\[
M + M^H - A = \begin{bmatrix} 2/a - 1 & 0 \\ 0 & 2/b \end{bmatrix} \left( \text{or} \begin{bmatrix} 2/a - 1 & 0 \\ 0 & 0 \end{bmatrix} \right),
\]

which, since \( 2/a - 1 > 0 \), is positive definite on \( \text{Range}(\tilde{M}A) = \text{Range}(A) = \text{span}\{(1,0)^T\} \).
4. Conclusions. In earlier works, it was shown that for a linear system with a Hermitian positive semidefinite coefficient matrix $A$, a sufficient condition for the convergence of a stationary iteration with iteration matrix $I - \tilde{M}A$, with $\tilde{M}$ injective on Range($A$), is that the splitting $A = M - (M - A)$ be a $P$-regular splitting for a matrix $M$ with $M\tilde{MA} = A$. In this paper, we have shown that with $M$ Hermitian, this sufficient condition is also necessary and that convergence of the iteration matrix is then equivalent to its $A$-norm being less than 1. We have also shown that when $M$ is not Hermitian, $P$-regularity of the splitting is not always a necessary condition for convergence. Similar results were also shown for the case that $A$ is Hermitian definite positive, using the $A$-seminorm.

Acknowledgement. We thank Jinchao Xu who posed a question leading us to write this paper. We also thank Michele Benzi for his comments on an earlier draft, which helped us to improve our presentation.

REFERENCES
