Abstract- A novel quasi-MLD solution denoted Directional Lattice Descent (DLD in short) is proposed in this paper to the MIMO and ISI detection problems, which exhibits quadratic time complexity. We formulate the detection problem as that of finding the closest point in a lattice. This is solved by a proposed discrete space analogy to the continuous-space gradient descent method: an initial lattice point is selected and subsequent lattice points are iteratively calculated so that each resides at a smaller distance from the received point than its predecessor, till a termination point. The problem and its solution are formulated within a broader general framework. Implementation aspects and comparative simulation results are presented.

I. INTRODUCTION

A class of problems can be characterized by: a discrete ‘input’ space of regular structure; a finite size symbol set bijectively mapped to a constrained region of this discrete input space; a continuous, Euclidean metrics ‘output’ space; a finite dimension linear input-output mapping; a normally distributed, reasonably weak, non-correlated noise, additive to the result of said mapping; and a wish to recover or detect, as reliably as possible, the source symbol out of the linear mapping-distorted and noise-perturbed image vector.

Two exemplary problems that belong to this class are the detection of symbol-vectors in a Multi-Input-Multi-Output transmission scheme (the ‘MIMO problem’, [3]) and the detection of symbols in the presence of Inter-Symbol Interference (the ‘ISI problem’, [1], [2]). It is well known [3] that the Maximum Likelihood Decision (MLD) solution to the MIMO problem is of $O(NM)$ time complexity where $M$ is the number of spatial sub-streams and $N$ is the symbol set cardinality (i.e. the ‘constellation’ size). A more efficient MLD method, nominated Sphere Decoding (SD, [5]) has been proposed which models the MIMO problem as that of finding the closest point in a lattice. This limits the closest point search to a relatively small region of the lattice defined by a sphere centered at the received point. The time complexity of the SD method appears to be $O(M^2)$ time complexity have also been studied [3] like Zero Forcing (ZF) or Minimum Mean Square Estimation (MMSE) but their link performance is significantly inferior to MLD. The ISI problem is predominantly solved by applying the Viterbi Algorithm (VA, [4]) to asymptotically achieve its Maximum Likelihood Sequence Detection (MLSD) solution; the time complexity of the VA algorithm is $O(N^{L+1})$ where $N$ is the number of symbols in the constellation set and $L$ is the channel memory length [1]. It is the purpose of this paper to propose a novel, quasi-MLD solution, denoted Directional Lattice Descent (DLD in short) to this broader aforementioned class of problems, of quadratic complexity, insensitive to the constellation size, and able to seamlessly generate approximate soft decision results. This paper is organized as follows: in Section II we re-iterate the MIMO and ISI problems models arriving at a common governing equation. In Section III we present some basic Lattice concepts and notation. In Section IV our proposed detection method is presented. Link performance simulation results, are presented in Section V. We provide concluding remarks in Section VI.

II. THE MIMO AND ISI PROBLEMS MODEL

Refer to Figure 1a, which depicts a simplified complex base-band equivalent MIMO model with $Q=M_c$ transmitting and $R=M_c$ receiving antenna elements. The governing equation of this MIMO model is

$$y_c = H_c s_c + n_c$$  \hspace{1cm} (1)

where $H_c \in C^{M \times M_c}$ denotes the propagation channel matrix, where $s_c \in C^{M_c}$ denotes the transmitted random vector with assumed i.i.d., uniformly distributed elements, drawn from a ‘popular’ possibly complex constellation (such as 4-PAM, 64-QAM, etc.), and where $y_c \in C^{M_c}$ is the received vector, distorted by the channel $H_c$ and interfered by the channel additive noise $n_c \in C^{M_c}$. In the context of the MIMO problem it is usually and herein assumed that the channel matrix elements are frequency non-selective, complex Gaussian random variables. For purpose of notational convenience and with no other material significance we prefer to represent all complex matrices and vectors hereon by their well known (e.g. [7]) all-real equivalent representations of double sized
dimensions, so that \( M=2M_c \) and \( L=2L_c \) and so that (1) is reformulated as
\[
y = H s + n
\]
with \( M=2M_c \), \( H \in R^{M \times M} \), etc.

We refer now to the ISI problem. Following [1] and [2] we formulate this problem by means of a discrete-time model, equivalent to a continuous-time, ISI affected, base-band (or base-band equivalent) sampled model, as described in Figure 1b. A symbol sequence \( s_k \), drawn again at random from a base-band equivalent (or sampled) model, as described in Figure 1b. An ISI discrete-time equivalent model

\[
h_k = [w(t) * c(t) * g(t)]_{t=kT} \quad \text{(4)}
\]
where impulse responses are reasonably assumed to be causal and time-finite so that \( h_k = 0 \) for \( L_c < k < 0 \) (\( L_c \) is commonly called the channel memory length) and where \( n_k \) is a discrete sequence of non-correlated, normally distributed noise samples defined by
\[
n_k = [w(t) * n(t)]_{t=kT} \quad k = 1,2,\ldots \quad \text{(5)}
\]
Equation (3) can be reformulated to represent the reception of successive element batches (rather than the reception of successive individual elements) in matrix form as follows
\[
\begin{pmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_{M_c}
\end{pmatrix} =
\begin{pmatrix}
  h_{1c} & \cdots & h_{1L_c} & 0 & \cdots & 0 \\
  0 & \cdots & h_{2L_c} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & h_{L_c} & \cdots & h_{L_c+1} \\
  0 & \cdots & 0 & 0 & \cdots & h_{L_c+1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & \cdots & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
  s_1 \\
  s_2 \\
  \vdots \\
  s_{L_c} \\
  s_{L_c+1} \\
  \vdots \\
  s_{L_c+L_c}
\end{pmatrix}
\begin{pmatrix}
  n_1 \\
  n_2 \\
  \vdots \\
  n_{L_c} \\
  n_{L_c+1} \\
  \vdots \\
  n_{L_c+L_c}
\end{pmatrix}
\quad \text{(6)}
\]
where by convention ‘earlier-in-time’ vector elements are denoted by means of lower indexes. Equation (6) represents an under-determined system, since in general \( L_c > 0 \) and \( M_c < (M_c+L_c) \); this can be easily resolved by feeding back the \( L_c \) oldest decided symbols, \( \hat{s}_1 \) to \( \hat{s}_{L_c} \), effectively implementing a decision-feedback scheme. Thus, we can concisely rewrite
\[
y_c = H_c s_c + H_{cd} s_{cd} + n_c
\]
where \( H_c \in C^{M \times M_c} \) and \( H_{cd} \in C^{M \times L_c} \) are the system and decision-feedback Toeplitz matrices respectively, where \( s_c=(s_{L_c+1}, s_{L_c+2}, \ldots, s_{L_c+L_c}) \) and \( s_{cd}=(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{L_c}) \) are the corresponding unknown and decided-fed-back symbol vectors respectively, and where \( n_c \in C^M \) represents the noise vector (‘c’ and ‘d’ subscripts respectively denote ‘complex’ and ‘decision-feedback’; (.)’ denotes the transpose operator). In analogy to (1) above an all-real representation of (2) would then result in
\[
y = H s + H_d s_d + n
\]
with \( H \in R^{M \times M} \), etc, and where our previously used subscript ‘c’ (standing for ‘complex’) is now absent.

It is also assumed in the sequel, applicable to both the ISI and MIMO models, that the channel state matrix, received power, symbol timing, etc. are perfectly estimated during an implicit training stage, that the mean transmission power and channel power transfer are normalized and that the mean noise power is calibrated (so that for example in our MIMO model case \( E[s^1]\|s\|_2^2=1 \), and \( E[n^2]\|n\|_2^2=1/\text{SNR} \) where SNR is the mean receiver Signal to Noise Ratio [2]).

### III. LATTICE BASIC CONCEPTS

We now turn our attention to basic lattice related concepts. A lattice (e.g. [5]) is essentially an infinite enumerable set of regularly spaced points in a multi-dimensional discrete space. Any lattice point \( s \in R^M \) can be described by its associated lattice generator matrix \( A \in R^{M \times M} \), namely
\[
s = a_0 + A m
\]
where \( m \in Z^M \) is the lattice point descriptor, where the M column vectors of \( A \) are the basis vectors of the lattice, and where \( a_0 \in R^M \) is an inconsequential translation vector. In a typical N-PAM/QAM based communication system, transmitted vectors (spatial in the MIMO model, temporal in the ISI model) can be represented by such a lattice with \( A=2I \) where \( I \) is the identity matrix, with \( a_0=(-1, -1, \ldots, -1) \)’ and with the constraint
\[
m_{\min} \leq m \leq m_{\max} \quad i = 1, 2, \ldots, M
\]
so that the transmitted vectors are actually confined to a bounded region of the transmission lattice. Such a transmission lattice, for the simplistic real 2-dimensional 4-PAM case is shown in Figure 2a, where \( a_0=(2, 0) \), \( a_0=(0, 2) \), \( a_0=(-1, -1) \), \( m_{\max}=2 \) and \( m_{\min}=-1 \). It should be noted that in the case of such commonplace constellations the transmission lattice basis vectors, i.e. the columns of \( A \), are mutually

![Figure 1a. A simplified base-band MIMO model](image1.png)

![Figure 1b. An ISI discrete-time equivalent model](image2.png)
orthogonal. The reception lattice, shown in Figure 2b, is affected by the scaling and rotation effects caused by the channel matrix $\mathbf{H}$. Substituting (9) into (2) or (8) we see that any received point $\mathbf{y} \in \mathbb{R}^M$ is expressed by

$$\mathbf{y} = (\mathbf{b}_0 + \mathbf{B} \mathbf{m}) + \mathbf{n} = \mathbf{p} + \mathbf{n}$$  \hspace{1cm} (11)

where $\mathbf{B}=(\mathbf{H} \mathbf{A})$ is the reception lattice generator matrix, where $\mathbf{b}_0=(\mathbf{H} \mathbf{a}_0)$ and $\mathbf{b}_0=(\mathbf{H} \mathbf{a}_0 + \mathbf{H}_2 \mathbf{s}_2)$ are the rotated-scaled translation vectors for the MIMO and ISI models respectively, and where $\mathbf{p}=(\mathbf{b}_0 + \mathbf{B} \mathbf{m})$ is the scaled-rotated and translated (but non-perturbed) reception lattice point. We note then that both the MIMO and ISI problems are represented by (11) whereby the spatial dimension of MIMO is replaced by the temporal dimension in ISI. We note also that the same descriptive vector $\mathbf{m}$ and the same bounded region constraint (10) are preserved at the reception lattice as can be observed in Figure 2b. The optimal MLD receiver consists then of solving

$$\mathbf{m}_{\text{MLD}} = \underset{\mathbf{m}}{\arg \min} \| \mathbf{y} - (\mathbf{b}_0 + \mathbf{B} \mathbf{m}) \|^2$$  \hspace{1cm} (12)

where the minimization is carried out over an enumerable constraint set which consists of all the feasible descriptor vectors $\mathbf{m}$. We note that the received lattice basis vectors, columns of $\mathbf{B}$, are not orthogonal any more, as exemplified by $\mathbf{b}_1$ and $\mathbf{b}_2$ in Figure 2b. In the context of our proposed detection method it is important to describe the reception lattice by means of a lattice generator matrix $\mathbf{C}$ with basis vectors which are more mutually orthogonal to each other than the original reception lattice basis vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. Such basis vectors usually exist, and are commonly called lattice reduced basis vectors; computationally efficient Lattice Reduction algorithms for the calculation of such reduced basis vectors set have been developed, including the so called LLL Lattice Reduction algorithm (e.g. [8]). Since said reduced basis vectors depend on the channel matrix $\mathbf{H}$, but not on the transmitted vector $\mathbf{s}$, they can be calculated only once (per data burst transmission) at a Channel Processing stage. Since both the original and reduced reception lattice generator matrices $\mathbf{B}$ and $\mathbf{C}$ describe the same lattice, then any lattice point $\mathbf{p}$ can be equivalently expressed by

$$\mathbf{p} = (\mathbf{b}_0 + \mathbf{B} \mathbf{m}) = (\mathbf{b}_0 + \mathbf{C} \mathbf{k})$$  \hspace{1cm} (13)

where $\mathbf{k} \in \mathbb{Z}^M$ is a lattice point descriptive integer column vector, from which the following significant transformations are immediately derived:

$$\mathbf{m} = (\mathbf{B}^{-1} \mathbf{C}) \mathbf{k} = \mathbf{W} \mathbf{k}$$  \hspace{1cm} (14)

and

$$\mathbf{k} = (\mathbf{C}^{-1} \mathbf{B}) \mathbf{m} = \mathbf{V} \mathbf{m}$$  \hspace{1cm} (15)

assuming $\mathbf{B}$ and $\mathbf{C}$ are invertible and where $\mathbf{W}=(\mathbf{B}^{-1} \mathbf{C})$ and $\mathbf{V}=(\mathbf{C}^{-1} \mathbf{B})$ can be shown [8] to be integer matrices.

IV. PROPOSED DETECTION METHOD

It is the main idea of our paper to solve the MIMO or ISI detection problem (12) by DLD, a discrete version of the gradient descent method. This solution incurs the selection of an Initial Lattice Point $\mathbf{p}_0$, followed by an iterative Lattice Descent Stage and an occasional Outlier Stage.
parallelepiped projection of our received vector $\mathbf{y}$, i.e. the Initial Lattice Point $\mathbf{p}_0$ described by $\mathbf{k}_0$ is then

$$\mathbf{k}_0 = \text{round}(\mathbf{h})$$

where round(.) is the rounding (to the nearest integer) operator. The similarity (and difference) between (16), (17) and the Zero Forcing solution of (2) is evident. It is readily seen that in our illustrative example $\mathbf{p}_0 = \mathbf{p}$, that is no more Lattice Descent Stage iterations will be required to achieve a minimum point. Although this is the (happy) case in the majority of instances, there are cases (not shown in Figure 3) where the resulting rounded parallelepiped projection $\mathbf{k}_0$ differs from the MLD solution. Even in these cases simulation shows that the calculated $\mathbf{p}_0$ is, in the average, very close to the final Lattice Descent Stage convergence point, independently of the constellation size. The average number of iterations is close to 1.

**Figure 3:** Initial Lattice Point calculation by means of parallelepiped projection

b. **Lattice Descent**: following $\mathbf{p}_0$ calculation, subsequent ‘adjacent’ lattice points $\mathbf{p}_i, i=1, 2, \ldots$ are iteratively selected if their (Euclidean) distance from the received point $\mathbf{y}$ is smaller than the distance of the previous lattice point $\mathbf{p}_{i-1}$ from $\mathbf{y}$. The iteration is terminated whenever the attempt to identify such a closer lattice point fails. The points $\mathbf{p}_i$ are preferably expressed (for reasons that will be momentarily clarified) by $\mathbf{C}$ and $\mathbf{k}_i$ as in (13) above. For each point $\mathbf{p}_{i-1}$ the number of checked candidate next points $\mathbf{p}_i$ is at most $2M$, simply calculated by

$$\mathbf{p}_i(j,t) = \mathbf{p}_{i-1} + \mathbf{t C} \mathbf{e}_j \quad j = 1, 2, \ldots M, \quad t = +/- 1$$

where $\mathbf{e}_j = (0 \ 0 \ \ldots \ 0 \ 1 \ 0 \ \ldots \ 0)$. Thus, each candidate lattice point $\mathbf{p}_i(j,t)$ is calculated by moving from the present point $\mathbf{p}_{i-1}$ in the direction (either positive or negative) and magnitude of a selected lattice basis vector. At each test $\{j, t\}$ the evaluated Euclidean distance is

$$d(\mathbf{p}_i(j,t), \mathbf{y}) = (\mathbf{y} - \mathbf{p}_i(j,t))^T (\mathbf{y} - \mathbf{p}_i(j,t))$$

and a next lattice point $\mathbf{p}_i = \mathbf{p}_i(j^*, t^*)$ is selected which satisfies

$$d(\mathbf{p}_i(j^*, t^*), \mathbf{y}) < d(\mathbf{p}_{i-1}, \mathbf{y})$$

for some $\{j, t\} = \{j^*, t^*\}$. If at some point $\mathbf{p}_{i-1}$ no such point $\mathbf{p}_i$ exists then the iterative process is terminated. This process is shown in Figure 4 where, for illustrative purposes, an arbitrary Initial Lattice Point $\mathbf{p}_0$ has been assumed. At termination, the final selected point can be the true minimal distance point; in such case the solution achieved is identical to the MLD solution; or it can be a false minimum point. Such a false minimum point (denoted $\mathbf{p}_i$) event means that another lattice point exists, say $\mathbf{p}_m$, for which $d(\mathbf{p}_m, \mathbf{y}) < d(\mathbf{p}_i, \mathbf{y})$. It can be shown by simulation that by conducting the Lattice Descent Stage using the reduced lattice generator matrix $\mathbf{C}$ (rather than the original channel transformed lattice generator matrix $\mathbf{B}$) the probability of false minima events is reduced. This lower probability of false minima events is reflected in significantly superior link performance. Other means to resolve this false minima effect were devised; the interested reader is referred to [11]. The metrics gathered along this process can be utilized to effectively calculate approximate bit-LLRs; their application in soft-decoding typically yield 1-2 db gain in excess of hard decoding gain.

**Figure 4:** Lattice Descent process illustration. Fine arrows indicate candidate, non-selected directions, bold arrows indicate the selected directions. An arbitrary Initial Lattice Point $\mathbf{p}_0$ is assumed. An exception Outlier event case, $\mathbf{p}_e$, is also shown.

c. **Outliers**: The aforementioned Lattice Descent Stage termination lattice point may occasionally reside out of the lattice bounded region as defined by (10) above. This is illustrated in Figure 4 by the lattice point $\mathbf{p}_e$. We denote these, for shorthand, ‘Outlier’ events. By checking the condition (10) at the end of the Lattice Descent Stage these exceptions may be detected. Once detected such an exception may be resolved by a variety of methods, one of which will be briefly described. This is similar to the aforementioned Lattice Descent Stage as described above, with the distinction that termination occurs whenever either no lattice point is found closer to $\mathbf{y}$ than the current point (as in the Lattice Descent Stage) or when such a point is found but it resides out of the lattice bounded region defined by (10). While this variant yields a lower link performance than our originally proposed Lattice Descent Stage (and hence it was not proposed for that

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3 In fact should lattice reduction yield perfect mutually orthogonal vectors (which it does not) then no false minima events would occur.

4 Details omitted.
stage) it guarantees that the terminal lattice point indeed belongs to the constraint set (10). It is not necessary to recalculate the Initial Point \( p_0 \), and \( k_0 \) of (17) may be used as the starting point for this stage as well\(^5\).

To wrap up this Section we briefly describe our proposed detection method as depicted in the block diagram of Figure 5 for our MIMO problem example (the scheme can be trivially adapted to the ISI case). Upon burst reception, channel processing tasks take place; these include traditional channel estimation, followed by Lattice Reduction and calculation of its associated products resulting in \( b_0 \), \( C \), \( C^{-1} \), and \( W \) as shown in Figure 5.

Vector processing commences thereafter. Upon each vector reception the Initial Processor calculates the Initial Lattice Point \( k_0 \), as per (16) and (17) above. The Descent Processor utilizes \( k_0 \) as its starting point and computes the Lattice Descent Stage terminal point \( k_d \) by iterative application of (18) and (19). Condition (10) is then checked for by the Outlier Processor; in most cases this processor is bypassed and \( k_d \) (or equivalently, \( m_d \)) will be the final result of our detector. In the exceptional Outlier events the Outlier Processor resolves the problem, for example by means of the iterative technique variant as described above, resulting in a final estimated lattice point \( m_d \neq m_d \).

The received vector processing time complexity of the proposed method consists of the sum of the individual Initial Point, Lattice Descent Stage and (occasional) Outlier processing complexities. The Lattice Initial Point calculation is readily verified, by inspection of (16) and (17) above, to be of \( O(M^2) \) complexity. The Lattice Descent Stage time complexity consists of (at most) \( 2M \) calculations of the metric (19), each of linear complexity in \( M \), iteratively repeated \( \sim 1 \) time, overall resulting, again, in \( O(M^2) \) complexity. Finally, the Outlier processing complexity is negligible during the majority of the received vectors. When such an Outlier event occurs and is solved by the aforementioned iterative scheme, then its incremental complexity is readily seen to be quadratic by the same reasoning we have applied during the Lattice Descent Stage analysis. This justifies our claim for total quadratic vector processing time complexity of the proposed method.

As a final remark, relevant to the ISI and possibly to other temporal problems, we note that since each detected vector \( m_d \) yields \( M \) symbols then the symbol processing time complexity for the ISI problem is merely \( O(M) \).

V. SIMULATION RESULTS

Most of the results presented in this section refer to the MIMO problem and the rest to the ISI problem; these latter deal in particular with the performance impact due to decision feedback, as per (7) above. The MIMO problem results shown herein pertain to configurations of \( M=4 \) and \( M=10 \), 64-QAM constellation and random channel matrices \( H \) of complex, frequency non-selective, quasi-static (no channel change during burst transmission), uncorrelated, normally distributed elements, normalized so that \( E[|h|^2]=1 \). An additive white, normally distributed complex noise vector of dimension \( M \) was generated, with i.i.d. elements, and calibrated variance so that prescribed mean channel SNR values were measured at the receiving antenna elements. A new channel \( H \) was picked at random for every new burst. Channel matrices were assumed to have been perfectly estimated. SER (BER) was estimated by counting symbol (bit) errors on a selected spatial sub-stream. Refer to Figures 6a and 6b which respectively present \( M=4 \), 64-QAM simulated link uncoded BER and soft and hard-coded BER performance plots. The results of the aforementioned SD are shown as reference for the uncoded BER. We note, in Figure 6a, that the proposed method is close up to 1db (for \( M=4 \), 64-QAM, at BER=10\(^{-3} \)) to the performance of the significantly more complex SD. In Figure 6b about 1.2 db DLD’s soft decision extra gain relative to hard decision is observed, for a half-rate convolution code with \( k=7 \) for the \( M=4 \) 64-QAM case. In Figure 6c (\( M=10 \), 64-QAM) the enormous gain of a variant of the DLD method (including enhancements as described in [11]) over the comparable complexity MMSE detector, is readily observed; SD complexity for such relatively high dimension (and 64-QAM constellation) would possibly result in unaffordable simulation run-time. The complexity of the basic DLD variant as described above, in terms of runtime counted and averaged MAC/vector (MAC operations are the main complexity contributor), is shown in Table 1 for several configurations, with 64-QAM constellation and soft and hard decision\(^6\). The claimed quadratic scalability of the proposed method can be verified. It should be also noted that DLD method complexity, apparently unlike SD and other near-MLD detectors, is insensitive to constellation size. Such low complexity features may make DLD especially attractive to high order MIMO, or joint MIMO-STBC [10], or ISI applications, and to high order constellations.

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\(^5\) Other methods have been devised by the author that resolve the Outlier event; description of these is omitted for the sake of brevity.

\(^6\) The complexity of DLD variant as per [11] would be somewhat higher than stated in Table 1.
Elementary simulations were run also on the ISI problem, in order to probe the performance degradation due to the application of decision feedback as per (7) above. For that purpose a 4-PAM symbol source was transmitted through a partial response discrete-time channel of memory length \( L=1 \), defined by \( h_0=\{-1, 1\} \) and interfered by additive white Gaussian noise.

Reference simulation runs with true MLD (exhaustive search as per (12) above) were also executed. Results show a gap of 0.7 db at \( \text{SER}=10^{-3} \) between our DLD proposed method run with ideal decision feedback and DLD with actual feedback and a gap of 0.1 db between this latter and true MLD (also run with actual decision feedback). As known and expected the gap between ideal and actual decision feedback decreases as SNR improves.

VI. SUMMARY AND CONCLUDING REMARKS

A class of discrete-input continuous-output, linear, perturbed systems includes the ISI and MIMO problems as important examples. A detection method has been proposed in this paper with vector processing time complexity which is quadratic in the dimension of the problem and independent of the constellation order. The claimed quadratic vector processing complexity of the proposed method was justified and simulated. The near-MLD link performance of this proposed method was demonstrated, for the two exemplar problems, by means of simulation. The proposed method could also be applied for other problems belonging to the discussed class.

REFERENCES