Qualitative choice logic

Gerhard Brewka a,*, Salem Benferhat b, Daniel Le Berre b

a Universität Leipzig, Institut für Informatik, Augustusplatz 10–11, 04109 Leipzig, Germany
b CRIL-CNRS, Université d’Artois, Rue Jean Souvraz, SP 18, 62307 Lens Cedex, France

Received 28 December 2002; accepted 14 April 2004

Abstract

Qualitative choice logic (QCL) is a propositional logic for representing alternative, ranked options for problem solutions. The logic adds to classical propositional logic a new connective called ordered disjunction: $A \triangleright B$ intuitively means: if possible $A$, but if $A$ is not possible then at least $B$. The semantics of qualitative choice logic is based on a preference relation among models. Consequences of QCL theories can be computed through a compilation to stratified knowledge bases which in turn can be compiled to classical propositional theories. We also discuss potential applications of the logic, several variants of QCL based on alternative inference relations, and their relation to existing nonmonotonic formalisms.1

© 2004 Elsevier B.V. All rights reserved.

Keywords: Preference handling; Nonmonotonic reasoning; Qualitative decision making

1. Introduction

For many AI applications, e.g., in design or configuration, it is necessary to represent intended properties of a particular problem solution. For instance, if we want to book a hotel for a trip to a conference, we intend properties such as being close to the conference site, being close to potential sight-seeing objects, and we want the hotel reasonably priced. Most often not all of the intended properties can be satisfied, that is, we have to make some

* Corresponding author.
E-mail addresses: brewka@informatik.uni-leipzig.de (G. Brewka), benferhat@cril.univ-artois.fr (S. Benferhat), leberre@cril.univ-artois.fr (D. Le Berre).


0004-3702/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
sort of compromises. To do so it is very convenient to be able to express alternative, second (or third, etc.) best options one would like to be satisfied if the best option is unavailable. For the hotel booking example, for instance, we may want to express that we prefer to stay within walking distance of the conference site; if that is not possible transportation provided by the hotel should be available; if this is still not possible, we want at least public transportation (taxis are not being reimbursed according to our university’s travel refund policy).

To represent options of this kind we introduce in this paper a new nonmonotonic propositional logic for representing qualitative choices—hence the name qualitative choice logic (QCL). The logic is different from existing nonmonotonic logics in the way nonmonotonicity is introduced: we do not use non-standard inference rules, as in Reiter’s default logic [32], modal operators expressing consistency or belief, as in autoepistemic logic [30], or abnormality predicates whose extensions are minimized, as in circumscription [27,28]. The non-standard part of our logic is a new logical connective \( \vec{\times} \) which is fully embedded in the logical language. Intuitively, if \( A \) and \( B \) are formulas then \( A \vec{\times} B \) says: if possible \( A \), but if \( A \) is impossible then (at least) \( B \).

The intended use of this logic can be illustrated using the hotel booking example. Assume we want to represent the options concerning the location. Using mnemonic variable names we express the options as follows:

\[
\text{walking} \vec{\times} \text{hotel-transport} \vec{\times} \text{public-transport}.
\]

Assume there are 4 hotels available out of which we have to pick one:

\[
\text{hotel}_1 \lor \text{hotel}_2 \lor \text{hotel}_3 \lor \text{hotel}_4.
\]

We have the following information about the hotels

\[
\text{hotel}_1 \rightarrow \text{walking},
\]

\[
\text{hotel}_2 \rightarrow \neg\text{walking} \land \text{hotel-transport},
\]

\[
\text{hotel}_3 \rightarrow \neg\text{walking} \land \neg\text{hotel-transport} \land \text{public-transport},
\]

\[
\text{hotel}_4 \rightarrow \neg\text{walking} \land \neg\text{hotel-transport} \land \neg\text{public-transport}.
\]

Given these propositional formulas QCL will give us the conclusion \( \text{hotel}_1 \) since this is the only hotel satisfying our most intended option. Now assume that, after calling the hotel we find out that it is fully booked, that is, we have the additional information \( \neg \text{hotel}_1 \). This means that our most favoured property, being within walking distance of the conference site, cannot be satisfied. We now obtain the conclusion \( \text{hotel}_2 \) which is not exactly what we wanted but better than nothing.

Our new connective \( \vec{\times} \) can be viewed as a kind of disjunction. Classical disjunction allows us to represent alternatives. The new connective uses the order in which options are written down to express additional preference information: \( A \vec{\times} B \) is very different form \( B \vec{\times} A \). We therefore call the connective ordered disjunction.

The semantics of the new logic will be defined in terms of preferred models. The definition of preferred models will proceed in two steps:

1. each formula of the logic leads to a ranking of models, based on how well the models satisfy the formula,
(2) a global preference relation on models is defined on the basis of the rankings given by the single formulas.

The rest of the paper is organized as follows: we first introduce syntax and semantics for QCL and give a few motivating examples. We then consider aspects of computation. It turns out that the logic has a special normal form. Theories in this normal form can be translated to stratified knowledge bases. These in turn can be transformed into a classical propositional theory [5]. We thus can compute consequences of QCL through a compilation process in which the formulas are first translated into propositional logic. Section 4 describes potential applications of QCL. In Section 5 we present several alternative definitions of the consequence relation. In Section 6 we investigate the relationship between QCL and circumscription, in Section 7 that between QCL and possibilistic logic. Section 8 shows how QCL and Reiter's default logic [32] can be combined. Section 9 discusses other related work and concludes the paper. Proofs of propositions are contained in Appendix A.

2. Syntax and semantics of QCL

2.1. Syntax

We start with standard propositional logic and add a new non-standard kind of disjunction: given formulas $A_1, \ldots, A_n$\footnote{Throughout the paper we use capital letters from the beginning of the alphabet to represent formulas.} we will use

$$A_1 \bar{\times} \cdots \bar{\times} A_n$$

to express: some $A_j$ must be true, preferably $A_1$, but if this is not possible then $A_2$, if this is not possible $A_3$, etc. Since $\bar{\times}$ is a binary operator, the formula should be read as shorthand for $(A_1 \bar{\times} (\cdots \bar{\times} A_n \cdots))$.\footnote{We will later show that $\bar{\times}$ is associative, so the brackets do not matter.} The idea is that

- from $A \bar{\times} B$ you get $A$,
- from $A \bar{\times} B$, $\neg A$ you get $B$, and
- $A \bar{\times} B$, $\neg A$, $\neg B$ is inconsistent.

Clearly, the order matters. We, therefore, call $\bar{\times}$ ordered disjunction.

Ordered disjunction is fully embedded in the logical language to make sure that context dependent options can be expressed, that is we may have formulas like $A \rightarrow (B \bar{\times} C)$ and $\neg A \rightarrow (C \bar{\times} B)$. The precise definition of the syntax is as follows:

**Definition 1.** Let $\mathcal{V}$ be a set of atoms. The set of well-formed formulas of QCL is inductively defined as follows:

1. every element of $\mathcal{V}$ is a well-formed formula,
(2) if $F_1$ and $F_2$ are well-formed formulas, then $(\neg F_1), (F_1 \lor F_2), (F_1 \land F_2)$ and $(F_1 \vec{\times} F_2)$ are well-formed formulas.

As usual we use $A \rightarrow B$ as an abbreviation for $\neg A \lor B$ and $A \leftrightarrow B$ as an abbreviation for $(A \rightarrow B) \land (B \rightarrow A)$. $\top$ represents a (classical) tautology, $\bot$ a (classical) contradiction. We omit unnecessary brackets assuming that all classical connectives have stronger bindings than $\vec{\times}$.

2.2. Semantics

The semantics of $QCL$ is based on the degree of satisfaction of a formula in a particular (classical) model. Intuitively, the degree can be viewed as a measure of disappointment: the higher the degree the more disappointing the model (or the farther away from complete satisfaction). As in standard propositional logic, an interpretation $I$ is an assignment of the classical truth values $true$ and $false$ to the atoms. We identify $I$ with the subset of $true$ atoms.

Interpretations can satisfy formulas to a certain degree. For instance, if $A$ and $B$ are atoms and $I$ contains $A$ then $I$ satisfies the formula $A \vec{\times} B$ as good as possible. If $I$ does not contain $A$ but $B$, then it also satisfies $A \vec{\times} B$, but only in a suboptimal way: the second best option is now satisfied. We will say that a formula is satisfied to degree $1$ if it is satisfied as good as possible, to degree $2$ if it is satisfied in the second best way, etc. For classical formulas without ordered disjunction no suboptimal way of satisfaction exists. Hence, such formulas can only be satisfied to degree $1$ or not satisfied at all.

For formulas containing ordered disjunction let us first consider a simple special case. Let $F$ be a formula of the form

$$A_1 \vec{\times} \cdots \vec{\times} A_n$$

where each $A_i$ is a classical propositional formula without $\vec{\times}$ and $n > 1$. We will see later that arbitrary $QCL$ formulas can be equivalently transformed into formulas of this kind. In this case the satisfaction degree of $F$ in an interpretation $I$ is simply the smallest $k$ such that $I$ satisfies $A_k$. If none of the ordered disjuncts is satisfied, then also $F$ is not satisfied. More formally, using an index to express the degree of satisfaction, we define the satisfaction relation for formulas of this type as ($\models$ is classical propositional satisfaction):

$$I \models_k (A_1 \vec{\times} \cdots \vec{\times} A_n) \iff I \models (A_1 \lor \cdots \lor A_n) \text{ and } k = \min\{j \mid I \models A_j\}.$$

For arbitrary formulas determining the degree of disappointment is somewhat more involved. Consider an ordered disjunction $F = (F_1 \vec{\times} F_2)$ where $F_1$ and $F_2$ are complex formulas containing $\vec{\times}$. Assume that $F_1$ is not satisfied by $I$, and that $F_2$ is satisfied to degree $k$. How do we determine the satisfaction degree for $F$ in this case? This degree depends on how many options or, in other words, possible satisfaction degrees $F_1$ admits. Assume there are $j$ such options for $F_1$ all of which are, as we said, not satisfied. The satisfaction degree then will be $j + k$ since $F$ is satisfied in the $(j + k)$th best possible way.

We call the number of possible satisfaction degrees of a formula its optionality:
Definition 2. The optionality of a formula is defined as follows:

\[ \text{opt}(A) = \begin{cases} 1 & \text{if } A \text{ is an atom,} \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{opt}(\neg F) = 1, \]

\[ \text{opt}(F_1 \lor F_2) = \max\{\text{opt}(F_1), \text{opt}(F_2)\}, \]

\[ \text{opt}(F_1 \land F_2) = \max\{\text{opt}(F_1), \text{opt}(F_2)\}, \]

\[ \text{opt}(F_1 \rightarrow F_2) = \text{opt}(F_1) + \text{opt}(F_2). \]

Intuitively, if the optionality of \( F \) is \( n \), then there may be a best way, a second best way, etc. and an \( n \)th best way of satisfying \( F \). For classical formulas there is only one way to satisfy them, hence they all have optionality 1.

The optionality of a negated formula may seem puzzling at first, but there is not more than one way of making \( \neg(A \rightarrow B) \) true: you must make \( A \) and \( B \) false. There is no second best solution for this, in a sense negation transforms nested ordered disjunctions into standard disjunctions. Hence \( \neg F \) behaves like a classical formula which explains why \( \text{opt}(\neg F) = 1 \).\(^4\) For conjunction and disjunction we obtain the maximum optionality of the subformulas. For instance, if \( \text{opt}(F_1) = j \) and \( \text{opt}(F_2) = k \) with \( j < k \), then an interpretation which does not satisfy \( F_1 \) but satisfies \( F_2 \) to degree \( k \) will satisfy \( F_1 \lor F_2 \) to degree \( k \). Similarly, an interpretation which does satisfy \( F_1 \) to some degree and satisfies \( F_2 \) to degree \( k \) will satisfy \( F_1 \land F_2 \) to degree \( k \).

If \( F = A \rightarrow B \), where \( A \) and \( B \) are classical, 2 degrees are needed. The best way of satisfying \( F \) is making \( A \) true, but if this does not work there is still the second best option, namely making \( B \) true. This generalizes to the sum of optionals if \( A \) and \( B \) are arbitrary formulas. Note that for formulas of the form \( A_1 \rightarrow \cdots \rightarrow A_n \) where all \( A_i \) are classical the optionality is \( n \).

We now define the satisfaction relation. The relation is indexed according to the degree of satisfaction of a formula in a model.

Definition 3.

1. \( I \models_k A \) if \( k = 1 \) and \( A \in I \) (for propositional atoms \( A \));
2. \( I \models_k P \land Q \) if \( I \models_n P \) and \( I \models_n Q \) and \( k = \max(m, n) \);
3. \( I \models_k P \lor Q \) if \( I \models_m P \) or \( I \models_n Q \) and \( k = \min\{r \mid I \models_r P \text{ or } I \models_r Q\} \);
4. \( I \models_k \neg P \) if \( k = 1 \) and for no \( m \): \( I \models_m P \);
5. \( I \models_k P \rightarrow Q \) if \( I \models_k P \) or \( \{I \models_1 \neg P, I \models_m Q, \text{ and } k = m + \text{opt}(P)\} \).

Let us illustrate the satisfiability relation using a simple example. Let \( F = A \lor (B \rightarrow C) \) where \( A \), \( B \) and \( C \) are atoms. Now all interpretations \( I \) containing \( A \) as well as those containing \( B \) satisfy \( F \) with degree 1, that is, we have \( I \models_1 F \). Now assume \( I \) is an interpretation which contains \( C \) but makes \( A \) and \( B \) false. In this case \( F \) is satisfied with degree 2, that is \( I \models_2 F \). Interpretations satisfying none of the three atoms do not satisfy \( F \) to any degree; they satisfy \( \neg F \) with degree 1.

---

\(^4\) As a consequence the formula \( \neg \neg F \) is not equivalent to \( F \), see Section 2.3.
The use of optionalities in (5) can be illustrated using the formula
\[
F = (\text{walking} \times \text{hotel-transport}) \times \text{public-transport}
\]
Assume \(I \models \text{public-transport}, I \not\models \text{walking} \) and \(I \not\models \text{hotel-transport}\). \(I\) satisfies public-transport with degree 1. Since \(\text{opt(walking} \times \text{hotel-transport)} = 2\) we obtain satisfaction degree 3 for \(F\). This seems intuitive since the 3rd-best option is obtained.

The following lemmata can easily be proven by induction on the structure of formula \(F\):

**Lemma 1.** \(I \models_k F\) and \(I \models_j F\) implies \(k = j\).

We use \(\text{deg}^I(F)\) to denote the satisfaction degree of \(F\) in interpretation \(I\), that is \(\text{deg}^I(F) = k\) whenever \(I \models_k F\). If \(I \models \neg F\) we let \(\text{deg}^I(F) = 0\).

**Lemma 2.** \(I \models_k F\) implies \(k \leq \text{opt}(F)\).

The following proposition relates ordered and classical disjunction:

**Proposition 1.** Let \(F\) be a formula. There is a \(k\) such that \(I \models_k F\) iff \(I \models F^*\) where \(F^*\) is obtained from \(F\) by replacing each occurrence of \(\times\) with standard disjunction.

**Definition 4.** Let \(T\) be a set of formulas. An interpretation \(I\) is a model of \(T\) if it satisfies each formula in \(T\) to some degree.

The satisfaction degrees of formulas help us to determine preferred models. There are different ways of doing this. We will use here a lexicographic ordering of models based on the number of formulas satisfied to a particular degree. In Section 5 alternative preference relations will be discussed. The lexicographic ordering is defined as follows:

**Definition 5.** Let \(M^k(T)\) denote the subset of formulas of \(T\) satisfied by a model \(M\) to degree \(k\). A model \(M_1\) is \(T\)-preferred over a model \(M_2\) if there is a \(k\) such that \(|M_1^k(T)| > |M_2^k(T)|\) and for all \(j < k\): \(|M_1^j(T)| = |M_2^j(T)|\). \(M\) is a preferred model of \(T\) iff \(M\) is a maximally \(T\)-preferred model.

Intuitively, a preferred model of \(T\) is a model of \(T\) which satisfies the maximal number of best options of choice logic formulas. Note that \(QCL\) is a conservative extension of classical propositional logic: for a set of formulas \(T\) without appearance of \(\times\) the preferred models of \(T\) coincide with the classical models of \(T\) since all classical formulas must be satisfied to degree 1.

We next define a consequence relation based on preferred models. We will restrict our attention here to classical conclusions, that is, the consequence relation we are interested in will be a relation between sets of formulas and classical formulas. The justification for this restriction lies in the intended use of \(QCL\): we want to be able to derive properties describing a problem solution based on background knowledge, knowledge about the case at hand and choice formulas describing intended properties. Classical formulas are the formulas describing the intended problem solutions (e.g., the chosen hotel in the booking example).
Definition 6. Let $T$ be a set of formulas, and let $A$ be a classical formula. $T \models A$ iff $A$ is satisfied in all preferred models of $T$.

This inference relation is obviously nonmonotonic. As an example consider $T = \{A \times B\}$. We have three models $\{A\}$, $\{A, B\}$, $\{B\}$ with satisfaction degree 1, 1, 2, respectively. This means that $\{A\}$ and $\{A, B\}$ are maximally preferred and we have $A \times B \models A$. If we add $\neg A$ then the single model, and thus the single preferred model, is $\{B\}$. We thus obtain $\{A \times B, \neg A\} \models \neg A$ and $A$ is no longer a conclusion.

2.3. Properties of QCL

We first define a notion of equivalence:

Definition 7. Let $F_1, F_2$ be formulas. $F_1$ is strongly equivalent to $F_2$, denoted $F_1 \equiv F_2$, iff $\text{opt}(F_1) = \text{opt}(F_2)$ and for all interpretations $I$ and integers $k$ we have $I \models_k F_1$ iff $I \models_k F_2$.

The optionality of the formulas must be the same in order to guarantee that subformulas can be replaced. For instance, although $A \times A$ and $A$ have the same satisfaction degree in all interpretations, $A \times B$ and $A \times A \times B$ clearly have not. Indeed, we have the following substitution result:

Lemma 3. Let $F(A)$ be a QCL formula containing a subformula $A$. Let $F(B)$ be obtained from $F(A)$ by substitution of the formula $B$ for an occurrence of $A$. If $A \equiv B$ then $F(A) \equiv F(B)$.

Note that for classical formulas without $\times$ all classical logical transformations can be performed, but for those with $\times$ some standard transformations are not valid: if $A$ is classical, then $\neg \neg A$ is strongly equivalent to $A$, but $\neg \neg (A \times B)$ is not strongly equivalent to $A \times B$ due to the different optionalities of the formulas. Indeed, $\neg \neg (A \times B)$ is strongly equivalent to $A \vee B$.

Another interesting feature of the logic is that conjunction and comma behave differently: the QCL theory $T_1 = \{F_1, F_2\}$ must be distinguished from $T_2 = \{F_1 \wedge F_2\}$. For instance, if $I_1 \models F_1$ and $I_1 \models F_2$, whereas $I_2 \models F_1$ and $I_2 \models F_2$, then $I_1$ is $T_1$-preferred over $I_2$, but not $T_2$-preferred. In our semantics, sets of formulas allow for more fine-grained distinctions than the conjunction of these formulas.

Proposition 2. Ordered disjunction is associative, that is for arbitrary formulas $F_1$, $F_2$ and $F_3$ we have $((F_1 \times F_2) \times F_3) \equiv (F_1 \times (F_2 \times F_3))$.

We now show that our inference relation satisfies properties usually considered intended in nonmonotonic reasoning.

Proposition 3. The inference relation $\vdash$ satisfies cautious monotony, that is, $T \vdash A$ and $T \vdash B$ implies $T \cup \{A\} \vdash B$.
**Proposition 4.** The inference relation \( \vdash \) satisfies cumulative transitivity, that is, \( T \vdash A \) and \( T \cup \{A\} \vdash B \) implies \( T \vdash B \).

In addition, we can show that \( \vdash \) is rational, in the sense that it satisfies all rules of System \( P \) [21] as well as rational monotony [25]. This follows from the results of Section 3 and from the fact that lexicographic inference satisfies these rules as shown in [4].

3. Computation

In this section we investigate ways to compute consequences of \( QCL \) theories. The basic idea is to take a set of arbitrary formulas \( T \) and to proceed in 3 steps:

1. translate \( T \) to \( \text{Norm}(T) \), a strongly equivalent normal form where all formulas containing \( \vec{x} \) are basic choice formulas (to be defined below),
2. construct a stratified knowledge base \( \text{Skb}(\text{Norm}(T)) \) such that a classical formula \( F \) is lexicographically entailed by \( \text{Skb}(\text{Norm}(T)) \) iff \( \text{Norm}(T) \vdash F \),
3. use the techniques developed in [5] to generate \( \text{Extract}(\text{Skb}(\text{Norm}(T))) \), a classical propositional knowledge base whose consequences are exactly the formulas lexicographically entailed by \( \text{Skb}(\text{Norm}(T)) \).

From the equivalence results described in the next subsections the following proposition is immediate:

**Proposition 5.** Let \( T \) be a consistent set of formulas. \( T \vdash A \) iff \( \text{Extract}(\text{Skb}(\text{Norm}(T))) \vdash A \).

3.1. Translation to normal form

We want to translate \( QCL \) bases into a normal form consisting only of classical formulas and basic choice formulas which are defined as follows:

**Definition 8.** A formula \( F \) is a basic choice formula if it is of the form

\[
A_1 \vec{x} \cdots \vec{x} A_n
\]

where each \( A_i \) is a classical propositional formula and \( n > 1 \).

For the translation to normal form we need the following strong equivalences:

**Proposition 6.** Let \( A_i, B_j, C_k, \ldots \) be formulas without \( \vec{x} \). Then the following strong equivalences hold:

\[
(1) \, (A_1 \vec{x} \cdots \vec{x} A_n) \lor (B_1 \vec{x} \cdots \vec{x} B_m) \equiv (C_1 \vec{x} \cdots \vec{x} C_k) \quad \text{where} \quad k = \max(m, n) \quad \text{and}
\]

\[
C_i = \begin{cases} 
(A_i \lor B_i) & \text{if } i \leq \min(m, n), \\
A_i & \text{if } m < i \leq n, \text{ and} \\
B_i & \text{if } n < i \leq m.
\end{cases}
\]
Example:
\[(A_1 \times A_2) \lor (B_1 \times B_2 \times B_3) \equiv (A_1 \lor B_1) \times (A_2 \lor B_2) \times B_3.\]

(2) \[(A_1 \times \cdots \times A_n) \land (B_1 \times \cdots \times B_m) \equiv (C_1 \times \cdots \times C_k) \text{ where } k = \max(m, n) \text{ and } C_i = \begin{cases} [(A_1 \lor \cdots \lor A_i) \land B_i] \lor [A_i \land (B_1 \lor \cdots \lor B_i)] & \text{if } i \leq \min(m, n), \\ [(A_1 \lor \cdots \lor A_m) \land B_i] & \text{if } n < i \leq m, \\ [A_i \land (B_1 \lor \cdots \lor B_m)] & \text{if } m < i \leq n. \end{cases} \]

Example:
\[(A_1 \times A_2) \land (B_1 \times B_2 \times B_3) \equiv (A_1 \land B_1) \times [(A_1 \lor A_2) \land B_2] \lor [(A_1 \lor A_2) \land B_3].\]

(3) \[\neg(A_1 \times \cdots \times A_n) \equiv \neg(A_1 \lor \cdots \lor A_n).\]

Repeated application of these transformation rules moves \(\times\) outside (or eliminates it) until we obtain a classical formula or a basic choice formula:5

**Proposition 7.** Every formula \(F\) can be translated to a strongly equivalent formula \(F'\) which is either classical or a basic choice formula.

We next show how \(QCL\) bases in normal form can be translated to stratified knowledge bases. For the results of the following section we need an additional condition: we say a set \(T\) of formulas is standardized iff different basic choice formulas do not possess syntactically **identical** prefixes6 (they may, of course, be logically equivalent). This condition considerably simplifies the discussion in the next subsection where we will construct certain sets of prefixes of formulas. In our cardinality based approach it is important to distinguish prefixes coming from different formulas. Without standardization we would have to deal with multi-sets rather than ordinary sets in the following definitions.

Of course, any theory \(T\) can be transformed into standardized form. This can simply be achieved by replacing identical formulas with some logically equivalent but syntactically different formulas. For instance, a standardized form of \(T = \{A \times B, A \times C\}\) can be \(\{A \land A \times B, A \times C\}\).

A set of \(QCL\) formulas \(T\) is in normal form if it consists of classical or basic choice formulas only, and if it is standardized.

---

5 Due to (2), our translation as it stands is exponential. Jérôme Lang (personal communication) has pointed out that a quadratic translation exists. Each \(QCL\) formula \(F_1 \times \cdots \times F_n\) is equivalent to \(F_1 \times (F_1 \lor F_2) \times \cdots \times (F_1 \lor \cdots \lor F_n)\). This transformation (with quadratic increase in size) needs to be done only once in the beginning. Then, under the assumption that \(F_1 \times F_2\) implies \(F_1 \models F_2\), we can reformulate the translation of a conjunction of ordered disjunctions as follows: \((A_1 \times \cdots \times A_k) \land (B_1 \times \cdots \times B_m) \equiv (C_1 \times \cdots \times C_k)\) where \(k = \max(m, n)\) and \(C_i = (A_i \land B_i)\) if \(i \leq \min(m, n), C_i = (A_i \lor B_i)\) if \(n < i \leq m,\) and \(C_i = (A_i \land B_i)\) if \(m < i \leq n.\) The total increase in size is quadratic.

6 \(A_1 \times \cdots \times A_k\) is a prefix of \(A_1 \times \cdots \times A_n\) whenever \(k \leq n.\)
3.2. Compilation to a stratified knowledge base

We recall the definition of a stratified knowledge base.

**Definition 9.** A stratified knowledge base is a sequence \((K, S_1, \ldots, S_n)\) of sets of classical propositional formulas.

Since we need \(K\) to be a non-default level we slightly modify the definition of lexicographically preferred subbases from [1,24].

**Definition 10.** Let \(KB = (K, S_1, \ldots, S_n)\) be a stratified knowledge base. A maximal consistent subset \(S\) of \((K \cup S_1 \cup \cdots \cup S_n)\) is a lexicographically preferred subbase of \(KB\) iff

1. \(K \subseteq S\), and
2. if \(S'\) is a maximal consistent subset of \((K \cup S_1 \cup \cdots \cup S_n)\) containing \(K\) and for some \(k \in \{1, \ldots, n\}\): \(|S' \cap S_k| > |S \cap S_k|\) then there is \(j < k\) such that \(|S \cap S_j| > |S' \cap S_j|\).

The only difference between this definition and the original one [1,24] is that there is no lexicographically preferred subbase if \(K\) is inconsistent.

**Definition 11.** Let \(KB = (K, S_1, \ldots, S_n)\) be a stratified knowledge base. A formula \(F\) is lexicographically entailed by \(KB\), denoted \(KB \vdash_{\text{lex}} F\), iff \(F\) is entailed by all lexicographically preferred subbases of \(KB\).

We now define the translation:

**Definition 12.** Let \(T\) be a QCL base in normal form. The stratified knowledge base associated with \(T\), denoted \(Skb(T)\), is

\[Skb(T) = (T^*, T_1, \ldots, T_n)\]

where \(n = \max\{k \mid F \in T, \text{opt}(F) = k\} - 1\), \(T^*\) is obtained from \(T\) by replacing each occurrence of \(\vec{\times}\) by \(\lor\), and

\[T_i = \{A_1 \lor \cdots \lor A_i \mid 1 \leq i < k, A_1 \vec{\times} \cdots \vec{\times} A_k \in T\}.

The translation is obviously polynomial in time and size.

**Proposition 8.** \(T \vdash F\) iff \(Skb(T) \vdash_{\text{lex}} F\).

Note that in the definition of \(T_i\) the case \(i = k\) is not needed since the corresponding disjunctions are already in \(T^*\). It turns out that there is a second, equivalent translation. In Definition 12 we can equivalently define

\[T_i = \{\neg A_1 \land \cdots \land \neg A_{i-1} \land A_i \mid 1 \leq i < k, A_1 \vec{\times} \cdots \vec{\times} A_k \in T\}.

To illustrate the translation consider the hotel example given in the introduction (after ¬hotel₁ was learned). The stratified knowledge base associated with \( T \) is \( \text{Skb}(T) = (T^*, T₁, T₂) \) where \( T^* \) consists of:

\[
\neg \text{hotel₁}, \\
\text{walking} \lor \text{hotel-transport} \lor \text{public-transport}, \\
\text{hotel₁} \lor \text{hotel₂} \lor \text{hotel₃} \lor \text{hotel₄}, \\
\text{hotel₁} \rightarrow \text{walking}, \\
\text{hotel₂} \rightarrow \neg \text{walking} \land \text{hotel-transport}, \\
\text{hotel₃} \rightarrow \neg \text{walking} \land \neg \text{hotel-transport} \land \text{public-transport}, \\
\text{hotel₄} \rightarrow \neg \text{walking} \land \neg \text{hotel-transport} \land \neg \text{public-transport},
\]

and

\[
T₁ = \{\text{walking}\}, \\
T₂ = \{\text{walking} \lor \text{hotel-transport}\}.
\]

There is exactly one lexicographically preferred subbase, namely \( T^* \cup T₂ \). We thus have

\[
\text{Skb}(T) \models_{lex} \text{hotel₂}
\]
as intended.

Note that a translation also exists in the opposite direction: we can translate a stratified knowledge base \( KB = (K, S₁, \ldots, Sₙ) \) to a \( QCL \)-theory \( T \) contains \( K \), and for each formula \( F \in S_i \) the formula \( \perp \times \cdots \times \perp \times F \times \top \) with \( i \) occurrences of \( \perp \).

### 3.3. Compilation to a classical KB

This step is described in [5]. To make this paper somewhat more self-contained we briefly describe the main idea underlying the compilation. Lexicographic entailment from the original base is replaced by classical entailment from a compiled base, which contains either formulas from the original base or formulas subsumed by the original ones, obtained from the disjunction of some of the original formulas. This idea has several implementations, namely Disjunctive Maxi-Adjustment (DMA), Iterative DMA and Whole-DMA, which produce logically equivalent results but with different spatial and computational needs.

- DMA replaces formulas involved in a conflict by disjunctions that restore consistency involving a minimum of these formulas (if any). Here the conflicting formulas are only computed once.
- Iterative DMA only adds pairwise disjunctions to the KB and iterates the detection of conflicting formulas in order to stay as close as possible to the original KB. Unfortunately, the approach is computationally costly since determining formulas involved in a conflict is on the second level of the polynomial hierarchy in complexity theory.
• In contrast, Whole-DMA does not detect formulas involved in conflicts, but works with disjunctions built from all the formulas. The advantage is that the complexity of the approach is on the first level of the polynomial hierarchy (SAT), and that in practice SAT solvers are now able to solve instances of the problem with several thousands of variables. The disadvantage is that the size of the compiled KB is likely to explode exponentially.

Note that for the hotel example discussed above the three approaches yield the classical knowledge base $T^* \cup \{\text{walking} \lor \text{hotel-transport}\}$.

For the details we refer to [5].

4. Applications

Qualitative choice logic has a number of possible applications.

• In design and configuration, intended properties can be described and ranked according to their desirability.
• In agent systems, intended actions to be performed by agents can be specified together with backup actions covering situations where the standard actions are inappropriate.
• In database and web applications, prioritized queries can be expressed which describe suboptimal results. For instance, one may start a query like: find a used convertible less than 2 years old in red, if red is unavailable then in black. This may be very useful for e-commerce applications.

We will focus on configuration and prioritized queries in this paper.

The configuration task can informally be described as follows: given a set of components $\text{Comp}$ determine a set $C \subseteq \text{Comp}$ such that $C$ is a complete solution (no necessary component is lacking), contains no unnecessary elements, and satisfies certain requirements depending on the case at hand as much as possible. More precisely the following pieces of information are relevant:

• A description of the available components which can be chosen for a particular configuration or subconfiguration.
• A description of additional components which are necessary if some component/subconfiguration is chosen.
• A description of the properties of the components.
• A description of the particular configuration task at hand.
• A description of the desired properties respectively intended components.

Let us illustrate this using a trip planning example. The available main options are going by $\text{plane}$, by $\text{train}$ or by $\text{car}$. If $\text{plane}$ is chosen it is necessary to organize transportation to and from the airport. If $\text{train}$ is chosen one has to get to the station and from the station to the final destination. For both cases taxis and buses are available. Here is a formalization of this information. It is convenient to use exclusive or ($\oplus$) in configuration examples.
$\text{trip} \leftrightarrow \text{plane} \otimes \text{train} \otimes \text{car},$

$\neg \text{plane} \lor \neg \text{train} \lor \neg \text{car},$

$\text{train} \leftrightarrow \text{toStation},$

$\text{train} \leftrightarrow \text{fromStation},$

$\text{plane} \leftrightarrow \text{toAirport},$

$\text{plane} \leftrightarrow \text{fromAirport},$

$\text{toStation} \leftrightarrow \text{taxi-toStation} \otimes \text{bus-toStation},$

$\text{fromStation} \leftrightarrow \text{taxi-fromStation} \otimes \text{bus-fromStation},$

$\text{toAirport} \leftrightarrow \text{taxi-toAirport} \otimes \text{bus-toAirport},$

$\text{fromAirport} \leftrightarrow \text{taxi-fromAirport} \otimes \text{bus-fromAirport}.$

Note that we use $\leftrightarrow$ rather than $\rightarrow$ to describe the available alternatives. This is necessary to make sure that a component is contained in a configuration only if it is necessary.7

We next describe our preferences. For short trips we prefer $\text{car}$ over $\text{train}$, for medium trips $\text{train}$ over $\text{car}$. We never use the plane for such trips. Also, if we have heavy luggage we never take the train. For long distance trips our first preference is $\text{plane}$, followed by $\text{train}$ and $\text{car}$. We also need to represent the background knowledge that a trip belongs to exactly one of the three categories:

$\text{short} \rightarrow \neg \exists \text{train},$

$\text{medium} \rightarrow \neg \exists \text{car},$

$\text{long} \rightarrow \neg \exists \text{flight} \times \text{train} \times \text{car},$

$\text{luggage} \rightarrow \neg \exists \text{train},$

$\neg \text{short} \lor \neg \text{medium} \lor \neg \text{long}.$

The preferences for traveling to and from the airport, respectively station, are as follows:

$\text{toStation} \leftrightarrow \neg \exists \text{taxi-toStation} \times \text{bus-toStation},$

$\text{fromStation} \leftrightarrow \neg \exists \text{taxi-fromStation} \times \text{bus-fromStation},$

$\text{toAirport} \leftrightarrow \neg \exists \text{bus-toAirport} \times \text{taxi-toAirport},$

$\text{fromAirport} \leftrightarrow \neg \exists \text{bus-fromAirport} \times \text{taxi-fromAirport}.$

Now given a description of the trip to be planned and the relevant requirements we obtain a suitable configuration. For instance, if in addition to the formulas above we have $\text{trip, short}$ then the preferred model contains $\text{car}$ and no other component. If we have $\text{trip, short, } \neg \text{car}$ we get $\text{taxi-toStation, train, taxi-fromStation}$. In each case we obtain a configuration which best satisfies our preferences.

7 The second formula is necessary because plane $\otimes$ train $\otimes$ car is true also if all three alternatives are true.
The next example involves prioritized queries. Assume a database \( DB \) of certain items, say cars, with a description of their properties is given:

\[
\begin{align*}
\text{car}_1 & \rightarrow \text{BMW} \land \text{red} \land \neg \text{convertible}, \\
\text{car}_2 & \rightarrow \text{BMW} \land \text{green} \land \text{convertible}, \\
\text{car}_3 & \rightarrow \text{VW} \land \text{blue} \land \neg \text{convertible}, \\
\text{car}_4 & \rightarrow \text{VW} \land \text{blue} \land \text{convertible}.
\end{align*}
\]

We assume \( DB \) also contains background knowledge of the kind

\[
\text{BMW} \rightarrow \neg \text{VW}, \quad \text{red} \rightarrow \neg \text{blue}, \quad \text{red} \rightarrow \neg \text{green},
\]

etc. together with information that exactly one car is to be chosen:

\[
\text{car}_1 \lor \text{car}_2 \lor \cdots
\]

and the formulas \( \{ \text{car}_i \rightarrow \neg \text{car}_j \mid i < j \} \). Now a prioritized query is just a set of QCL formulas \( Q \). An answer \( \text{Ans} \) is a disjunction of items such that \( DB \cup Q \models \neg \text{Ans} \). In the example we might have

\[
Q = \{ \text{red} \times \text{blue}, \text{convertible} \times \neg \text{convertible} \}.
\]

This query would lead to the answer \( \text{car}_1 \lor \text{car}_4 \). \( \text{car}_1 \) is contained in the answer since it has the most preferred colour, \( \text{car}_4 \) since it is a convertible. If we add to \( Q \) the formula \( \text{BMW} \times \neg \text{VW} \) the answer becomes \( \text{BMW} \). This does not exclude blue cars from being answers in case no red car is available.

5. Alternative definitions of entailment

In Section 2 we defined entailment for QCL in terms of a lexicographic ordering on models, based on the number of formulas satisfied to a certain degree. This leads to an approach where solutions are preferred when they contain the highest number of most preferred options. For instance, if there are three choices with three options each, say

\[
A_1 \times A_2 \times A_3, \quad B_1 \times B_2 \times B_3, \quad C_1 \times C_2 \times C_3
\]

then a model \( M_1 \) satisfying \( A_1, B_1 \) and \( C_3 \) is preferred over a model \( M_2 \) satisfying \( A_2, B_2 \) and \( C_1 \) because the number of formulas satisfied in \( M_1 \) with degree 1 is 2, the number in \( M_2 \) is 1. This may not be wanted for all applications. In our example we might consider \( M_2 \) a reasonable alternative: although it gives us only one best choice, we still get two second best options.

In this section we will discuss alternative ways to define preferences on models based on the satisfaction degrees of the premises.
5.1. Inclusion based preference

The following strengthening of the preference relation is based on subsets rather than the number of formulas satisfied with a particular degree.

**Definition 13.** Let $M^k(T)$ denote the set of formulas of $T$ satisfied by a model $M$ of $T$ to degree $k$. A model $M_1$ of $T$ is $T$-inclusion-preferred over a model $M_2$ if there is a $k$ such that $M^k_1(T)$ is a strict superset of $M^k_2(T)$ and for all $j < k$: $M^j_1(T) = M^j_2(T)$.

Note that inclusion preference implies the cardinality based preference introduced earlier but not vice versa. In the example above, $M_1$ is not inclusion-preferred over $M_2$. We therefore get in general more maximally inclusion-preferred models and thus fewer conclusions, that is, the inference relation $\models_{inc}$ (defined as $T \models_{inc} F$ iff $F$ is true in all inclusion preferred models of $T$) is more cautious than $\models$.

It turns out that a result corresponding to Proposition 8 can be established for inclusion based preference using the same translation. The definition of an inclusion preferred subbase of a stratified knowledge base is obtained from Definition 10 by replacing clause 2 with

if $S'$ is a maximal consistent subset of $(K \cup S_1 \cup \ldots \cup S_n)$ containing $K$ and for some $k \in \{1, \ldots, n\}$: $S \cap S_k$ is a proper subset of $S' \cap S_k$ then there is $j < k$ such that $S' \cap S_j$ is a proper subset of $S \cap S_j$.

We have the following proposition:

**Proposition 9.** $T \models_{inc} F$ iff $Skb(T) \models_{inc} F$.

A recent complexity result from Costé-Marquis and Marquis [14] is relevant here: they prove that there is no way to compile any stratified knowledge base under inclusion preference in polynomial space whereas such a translation is possible under lexicographic preference by adding new propositional variables (the compiled base is “query equivalent” to the original one). This is an additional argument in favour of lexicographic preference.

5.2. Preference based on ranking functions

Another possibility for defining preferred models is to use ranking functions. Ranking functions assign an integer rank to a model $M$ of a set of formulas

$T = \{f_1, \ldots, f_n\}$

based on the vector of satisfaction degrees

$(deg^M(f_1), \ldots, deg^M(f_n))$
of the formulas in $T$. More precisely, let $\ominus$ be a function from a vector of integers to an integer. $\ominus$ will be called a ranking function if it satisfies the following requirements:

Unanimity: If $\forall i = 1, \ldots, n, j_i \geq k_i$ then: $\ominus(j_1, \ldots, j_n) \geq \ominus(k_1, \ldots, k_n)$.

Model preservation: $\ominus(j_1, \ldots, j_n) = 0$ iff $j_i = 0$ for some $i = 1, \ldots, n$.

Intuitively, the second requirement means that a given interpretation should not be considered as a model of $T$ if and only if it falsifies some formula of $T$. Ranking functions induce preferences on models in a straightforward way:

**Definition 14.** Let $T = \{f_1, \ldots, f_n\}$ be a set of QCL formulas, $\ominus$ a ranking function. A model $M_1$ of $T$ is a $\ominus$-preferred model of $T$ iff there is no model $M_2$ of $T$ such that $\ominus(\deg_{M_2}(f_1), \ldots, \deg_{M_2}(f_n)) < \ominus(\deg_{M_1}(f_1), \ldots, \deg_{M_1}(f_n))$.

The inference relation induced by this preference relation will be denoted $\models \ominus$.

As an example consider the ranking function

$$\ominus(j_1, \ldots, j_n) = \begin{cases} 0 & \text{if } j_k = 0 \text{ for some } k \in \{1, \ldots, n\}, \\ \sum_{1 \leq k \leq n} j_k & \text{otherwise}. \end{cases}$$

This ranking function just adds up the degrees of (dis)satisfaction of all formulas in $T$ for each model and prefers those models whose sum is minimal, that is, whose overall dissatisfaction is minimal. In the example from the beginning of this section, both $M_1$ and $M_2$ have overall degree 5 (1 + 1 + 3 and 2 + 2 + 1, respectively), thus none of the models is preferred to the other.

Even more fine grained distinctions could be introduced by adding to each option in an ordered disjunction some integer (increasing in value from left to right) which could be used as a kind of penalty to compute the overall dissatisfaction degree of models.

The relationship between preference based on ranking functions and possibilistic logic will be further investigated in Section 7.

**6. Relation to circumscription**

Since its invention in the seventies, circumscription [26–28] was certainly one of the most influential nonmonotonic formalisms. In the first order case, circumscription allows the extensions of certain predicates to be minimized. Propositional circumscription makes certain atoms false whenever possible. Semantically, this is achieved by defining a preference relation on models and reasoning from most preferred models. The preference relation depends on the predicates, respectively atoms, chosen for minimization.

Since QCL is a propositional logic, we consider propositional circumscription for the comparison in this section. Let us first discuss a short example. According to the

---

8 For the reader’s convenience we recall the definition of $\deg_M$ given after Lemma 1: $\deg_M(f) = k$ iff $M \models_k f$ and $\deg_M(f) = 0$ iff $M \models_{k+1} \neg f$. 
methodology proposed by McCarthy, defaults can be represented using $ab$ atoms which are then circumscribed. For instance, circumscribing $ab_1$ in

1. $\text{penguin} \land \neg ab_1 \rightarrow \neg \text{flies},$
2. $\text{penguin},$

yields the conclusion $\neg \text{flies}$. In $QCL$ there is a simple way to model this: we just have to add

$$\neg ab_1 \bowtie ab_1$$

and obtain exactly the same conclusions.

For our formal result we will consider one of the most general forms of circumscription, prioritized circumscription with fixed atoms. The priorities are used to handle potential conflicts between minimized atoms, the fixed atoms are not allowed to vary during the minimization.

Assume that, in addition to formulas (1) and (2), we have

3. $\text{bird} \land \neg ab_2 \rightarrow \text{flies},$
4. $\text{bird}.$

Minimizing the two $ab$-atoms now yields two minimal models. Intuitively, we would expect (1) to be preferred over (3) for reasons of specificity. In circumscription this can be achieved by minimizing $ab_1$ with higher priority than $ab_2$. This can be modeled in $QCL$ by adding to the four premises the formulas

$$\neg ab_1 \bowtie ab_1, \quad \bot \bowtie \neg ab_2 \bowtie ab_2.$$

Adding an unsatisfiable option to the first choice formula has the desired effect: making $ab_1$ false is more important than making $ab_2$ false.

It is also not difficult to handle fixed atoms. Assume $b$ is fixed. We have to make sure that models which differ in the truth value of $b$ become incomparable. If inclusion based preference is used this can be achieved by adding:

$$\neg b \bowtie b, \quad b \bowtie \neg b.$$

Here are the formal definitions needed for our result:

**Definition 15.** Let $T$ be a propositional theory, $V_1, \ldots, V_n$ sets of atoms to be circumscribed, $W$ a set of fixed atoms such that $(V_1 \cup \cdots \cup V_n) \cap W = \emptyset$. A formula $F$ is a consequence of the prioritized circumscription of $V_1, \ldots, V_n$ in $T$ with fixed $W$, denoted $\text{Circ}(T; V_1, \ldots, V_n; W) \models F$, iff $F$ is true in all $<v_1, \ldots, v_n; W>$-preferred models of $T$, where $M_1 \preceq_{V_1, \ldots, V_n; W} M_2$ if$^9$

$^9$ As usual, we identify models with the set of their true atoms.
(1) there is \( i \in \{1, \ldots, n\} \) such that \( M_1 \cap V_i \subset M_2 \cap V_i \), and for all \( j < i \): \( M_1 \cap V_j = M_2 \cap V_j \), and
(2) \( M_1 \cap W = M_2 \cap W \).

Since the preference criterion used for circumscription is based on subsets, the inclusion based variant of QCL is the natural candidate to capture circumscription. Indeed, we have the following proposition:

**Proposition 10.** Let \( T \) be a propositional theory, \( V_1, \ldots, V_n \) sets of atoms to be circumscribed, \( W \) a set of fixed atoms such that \((V_1 \cup \cdots \cup V_n) \cap W = \emptyset\). Circ\((T; V_1, \ldots, V_n; W)\) \(|\vdash F\) iff \( T'|_{\text{inc}} F \) where

\[
T' = T \cup \{ \perp \times \cdots \times \perp \times \neg v \times v \mid v \in V_i, \perp \text{ appears } i - 1 \text{ times}\}
\cup \{v \times \neg v \mid v \in W\}
\cup \{\neg v \times v \mid v \in W\}.
\]

We have seen that QCL can quite easily capture propositional circumscription. How about the converse? Of course, since circumscription by definition is a minimization technique whereas QCL allows us to specify arbitrary preferences we cannot expect modular translations from QCL to circumscription which give us exactly the same preferred models.\(^\text{10}\) However, if we admit additional symbols in the language we can use circumscription to generate models corresponding to the inclusion preferred QCL models up to the additional atoms.

We assume that all QCL formulas are in normal form, i.e., of the form \( F_j = A_1 \times \cdots \times A_n \). Using additional abnormality atoms this formula can be represented as the set of formulas:

1. \( \neg ab_{j,1} \rightarrow A_1 \),
2. \( ab_{j,1} \land \neg ab_{j,2} \rightarrow A_2 \),
   
   : 
   
(n) \( ab_{j,1} \land \cdots \land ab_{j,n-2} \land \neg ab_{j,n-1} \rightarrow A_{n-1} \),
(n) \( ab_{j,1} \land \cdots \land ab_{j,n-1} \rightarrow A_n \).

To translate a formula of optionality \( n \) in normal form we thus need \( n - 1 \) new \( ab \)-atoms not appearing anywhere else in the premises and in the translation of other formulas, and we generate \( n \) implications as illustrated above. Intuitively, \( ab_{j,i} \) says: option \( i \) of formula \( j \) is impossible. We have the following result:

**Proposition 11.** Let \( T = \{F_1, F_2, \ldots\} \) be a set of QCL formulas in normal form and \( n = \max\{j \mid F_i \in T, \text{opt}(F_i) = j\} - 1 \). Let \( T' \) be the translation of \( T \), and for \( i \in \{1, \ldots, n\} \)

\(^{10}\) A translation \( \text{Trans} \) is modular iff for arbitrary sets \( S, S' \) we have \( \text{Trans}(S \cup S') = \text{Trans}(S) \cup \text{Trans}(S') \).
let $AB_i = \{ab_{j,i} \mid F_j \in T\}$ the set of newly introduced abnormality atoms with second index $i$. Moreover, let $F$ be a formula not containing any atom in $AB_1 \cup \cdots \cup AB_n$. Then

$$T \models \text{inc}\ F \iff \text{Circ}(T', AB_1, \ldots, AB_n; \emptyset) \models F.$$ 

This result shows that in principle prioritized circumscription is able to express $QCL$ under inclusion based preference. However, for each formula with optionality $n$ we need $n - 1$ new abnormality atoms, and the representation is quadratic in the size of the original $QCL$ theory. We also consider it as an advantage that in $QCL$ the knowledge is completely represented by means of formulas whereas circumscription needs the additional specification of a circumscription policy. Moreover, $QCL$ offers alternative preference criteria which are not captured by standard versions of circumscription. 11

7. Possibilistic logic and $QCL$

In Section 5 several alternative definitions of entailment have been proposed. Among them, we defined preferred models based on ranking functions $\ominus$. This section investigates relationships between possibilistic logic and a class of $QCL$ based on ranking functions $\ominus$. In particular, we show that (i) any basic choice logic formula can be viewed as a possibilistic knowledge base, and (ii) for each ranking function $\ominus$, the entailment $\models \ominus$ can be recovered in a possibility theory framework. A corollary of this result is that, for any ranking function, any $QCL$ theory can be equivalently transformed into a basic choice logic formula. The following first provides a brief reminder on possibilistic logic.

7.1. A brief reminder on possibilistic logic

7.1.1. Guaranteed possibilistic knowledge bases

Possibilistic logic provides a tool for performing uncertainty reasoning, where uncertain information is semantically represented by means of possibility distributions. Possibility distributions are means to rank order different interpretations of a language. More precisely, a possibility distribution, denoted by $\pi$, is a function from a set of mutually exclusive situations (solutions or interpretations) to the interval $[0, 1]$. By convention $\pi(I) = 1$ means that $I$ is among the most normal (or preferred) situations, $\pi(I) = 0$ means that $I$ is impossible or excluded as a possible solution. More generally, $\pi(I) \geq \pi(I')$ means that $I$ is at least as preferred as $I'$. There are several compact (or syntactic) encodings of possibility distributions [2]: necessity based knowledge bases, min-based possibilistic graphs, product-based possibilistic graphs, etc. Recently, another type of compact representation, called guaranteed possibilistic knowledge bases or simply $\Delta$-knowledge bases, has been investigated [3,16]. It is based on the notion of guaranteed possibility measures, which are defined on formulas from a possibility distribution $\pi$ in the following way:

$$\Delta(\phi) = \min \{\pi(I) \mid I \models \phi\}.$$ 

11 For a cardinality based treatment of circumscription see [29].
A \(\Delta\)-knowledge base is composed of a set of weighted formulas of the form \([\phi_i, \alpha_i]\), where \(\phi_i\) denotes a propositional formula, and \(\alpha_i\) is a real number between 0 and 1. The pair \([\phi_i, \alpha_i]\) means that any model of \(\phi_i\) is satisfactory to a degree at least equal to \(\alpha_i\), namely:

\[ \Delta(\phi_i) \geq \alpha_i \]

or

(1) \(\forall I \models \phi_i, \pi(I) \geq \alpha_i\).

**Definition 16.** Each \(\Delta\)-knowledge base \(\Delta\) induces a unique possibility distribution \(\pi\) defined by:

\[ \forall I, \pi(I) = \begin{cases} 0 & \text{iff } I \text{ falsifies all formulas of } \Delta, \\ \max\{\alpha \mid [\phi, \alpha] \in \Delta, I \models \phi\} & \text{otherwise}. \end{cases} \]

In [16] it has been shown that this possibility distribution corresponds to the most specific possibility distribution\(^{12}\) satisfying (1) for each weighted formula in \(\Delta\). As we will show later, any basic choice logic formula can be equivalently represented by a \(\Delta\)-knowledge base. However, in order to show the encoding of a set basic choice logic formulas, we need to use possibilistic fusion operators which are recalled in next subsection.

### 7.1.2. Possibilistic fusion

In [3] several possibilistic fusion operators have been proposed to merge a set \(\Delta\)-knowledge bases. In this section we restrict the class of merging operators to those which are useful for establishing relationships between \(QCL\) and possibilistic logic. More precisely, let \(\oplus\) be a function from a vector of real numbers in \([0, 1]\) to a real number between \([0, 1]\). \(\oplus\) will be called a \([0, 1]\)-based merging operator. The requirements for \(\oplus\) are similar to those defined for \(\ominus\) in Section 5.2:

- **Unanimity:** If \(\forall i = 1, \ldots, n, \alpha_i \geq \alpha'_i\) then: \(\oplus(\alpha_1, \ldots, \alpha_n) \geq \oplus(\alpha'_1, \ldots, \alpha'_n)\).
- **Model preservation:** \(\oplus(\alpha_1, \ldots, \alpha_n) = 0\) iff \(\alpha_i = 0\) for some \(i = 1, \ldots, n\).

The second requirement is stronger than the one used in [3] which simply requires that \(\oplus(0, \ldots, 0) = 0\). It is strengthened here in order to have an easier connection with \(QCL\). Let \(\Delta_1, \ldots, \Delta_n\) be the \(\Delta\)-knowledge bases to merge. We denote by \([\phi_{ij}, \alpha_{ij}]\) the \(j\)th weighted formula in \(\Delta_i\).

**Definition 17.** The result of merging \(\Delta_1, \ldots, \Delta_n\), denoted by \(\Delta_\oplus\), is defined as

\[ \Delta_\oplus = \{[\phi_{1i} \land \cdots \land \phi_{ni}, \oplus(\alpha_{1i}, \ldots, \alpha_{ni})] \mid [\phi_{1i}, \alpha_{1i}] \in \Delta_1, \ldots, [\phi_{ni}, \alpha_{ni}] \in \Delta_n\} \]

It can be checked that the possibility distribution associated to \(\Delta_\oplus\) using the above definition can be characterized as follows:

\(^{12}\) \(\pi\) is said to be more specific than \(\pi'\), if \(\forall I, \pi(I) \leq \pi'(I)\).
Lemma 4. Let $\Delta_1, \ldots, \Delta_n$ be a set of $\Delta$-knowledge bases. Let $\pi_1, \ldots, \pi_n$ be their associated possibility distributions, respectively. Let $\oplus$ be a $[0,1]$-based merging operator, and $\Delta_\oplus$ be the $\Delta$-knowledge base given by Definition 17. Then $\forall I, \pi_{\Delta_\oplus}(I) = \oplus(\pi_1(I), \ldots, \pi_n(I))$.

7.2. Encoding ranking functions-based QCL in possibilistic logic

We restrict ourselves to sets of basic choice formulas. This is not a limitation since a classical propositional formula $p$ can be represented as a basic choice formula $p \times \bot$ without changing the ranking of interpretations. The following lemma is immediate, noticing that models of $\{ p_i \times \bot \mid p_i \text{ is a classical formula of } T \}$ are exactly the same as the ones of $\{ p_i \mid p_i \text{ is a classical formula of } T \}$.

Lemma 5. Let $T$ be a set of formulas. Let $T'$ be obtained from $T$ by replacing each classical formula $p$ in $T$ by a new basic choice formula $p \times \bot$. Then $T$ and $T'$ induce the same ranking on the set of interpretations.

The following lemma establishes a first connection between QCL and possibilistic logic when we only have one basic choice formula.

Lemma 6. Let $F = A_1 \times \cdots \times A_n$ be a basic choice formula. Let $\alpha_i = \varepsilon_i$ for $i = 1, \ldots, n$, where $0 < \varepsilon < 1$. Let $\Delta = \{ [A_1, \alpha_1], \ldots, [A_n, \alpha_n] \}$ be the $\Delta$-knowledge base associated to $F$. Then $\forall I, I \models_k F$ if and only if $\pi(I) = \alpha_k$ and $I \models_1 \neg F$ iff $\pi(I) = 0$.

Hence, each basic choice logic formula can be represented by a $\Delta$-knowledge base. As a corollary of Lemmas 4 and 6, it is possible to provide a possibilistic encoding of a general QCL $\ominus$ theory:

Lemma 7. Let $T = \{ F_1, \ldots, F_n \}$ be a QCL theory, and let $\ominus$ be a ranking function. Let us again denote by $\alpha_i = \varepsilon_i$ for $i \in \mathbb{N}$. Let $\Delta_i$ be the $\Delta$-knowledge base associated with $F_i$ using Lemma 6. Then the $\Delta$-knowledge base associated with $T$ is the one obtained from Definition 17 by merging $\Delta_i$’s with $\ominus(\alpha_1, \ldots, \alpha_k) = \varepsilon^{\ominus(i, \ldots, k)}$.

Lemma 7 is interesting since it means that, for any ranking function $\ominus$, any QCL theory can be transformed into one basic choice formula. Indeed, let $T = \{ F_1, \ldots, F_n \}$ be a set of basic choice formulas. Again, we denote by $A_{ij}$ the propositional formula which is in position $j$ (namely after $j - 1$ occurrences of $\times$) in the basic choice formula $F_i$. We denote by

$$C = A_{i1} \land \cdots \land A_{ik}$$

any conjunction of exactly one $A_{ij}$ from each $F_i$. $C$ is called a complex cube. We denote by $\text{degree}(C)$ the rank associated to $C$ defined as equal to $\ominus(i, \ldots, k)$. Then, it can be checked

$^{13}$ In fact, any function such that $\alpha_i > \alpha_j$ if and only if $i < j$ is appropriate.
that for each ranking function \( \ominus \), it is possible to replace \( T \) by one basic choice formula defined as \( B_1 \times \cdots \times B_m \) where:

\[
m = \ominus(\text{opt}(F_1), \ldots, \text{opt}(F_n)),
\]

and

\[
B_i = \begin{cases} 
\bot & \text{if there is no } C \text{ such that } \text{degree}(C) = i, \\
\bigvee_{\text{degree}(C) = i} C & \text{otherwise}.
\end{cases}
\]

Example 1. Let us assume that \( T \) is composed of two basic choice formulas \( F_1 = A_{11} \times A_{12} \) and \( F_2 = A_{21} \times A_{22} \). Then we have 4 complex cubes:

\[
A_{11} \land A_{21}, \quad A_{11} \land A_{22}, \quad A_{12} \land A_{21}, \quad A_{12} \land A_{22}.
\]

Let us assume that the ranking function \( \ominus \) yields the sum of the satisfaction degrees of the single formulas (as described in Section 5.2). Then, we have:

\[
\text{degree}(A_{11} \land A_{21}) = 2, \\
\text{degree}(A_{11} \land A_{22}) = 3, \\
\text{degree}(A_{12} \land A_{21}) = 3, \\
\text{degree}(A_{12} \land A_{22}) = 4.
\]

Therefore, it can be checked that if we rank-order interpretations with respect to sum of degree then this ranking can be recovered from the basic choice formula \( C_1 \times C_2 \times C_3 \times C_4 \) with:

\[
C_1 = \bot, \\
C_2 = A_{11} \land A_{21}, \\
C_3 = (A_{12} \land A_{21}) \lor (A_{12} \land A_{21}), \quad \text{and} \\
C_4 = A_{12} \land A_{22}.
\]

8. Combining QCL and default logic

QCL allows us to specify preferences and to reason from most preferred models. So far, we haven’t said much about the very nature of the preferences. A model \( M_1 \) may be preferred to another model \( M_2 \) because it better satisfies some desires, intentions or norms. But it may also be preferred because it describes more normal states of the world, in other words, because it is more in accordance with our expectations about what is true in the world.

In situations where different types of preferences play a role, e.g., desires as well as default information, it may be useful to have different formal tools available to represent them. In this section we discuss how QCL and Reiter’s default logic [32], one of the standard logics for representing defeasible information, can be combined. The motivation for this section is twofold: (1) we want to show that ordered disjunction cannot only be introduced in classical propositional logic, but also in a non-classical logic, and (2) we
want to propose a formalism where the mechanisms of default logic are used to determine what is normally the case, in other words, what is expected to be true, and the mechanisms of QCL to determine what is desired to be true. We assume some familiarity with default logic and refer the reader to Reiter’s original paper for more details.

In Reiter’s approach a default theory \((D,W)\) is a pair consisting of a set of classical formulas \(W\) representing what is known to be true and a set of default rules \(D\). Default theories induce extensions which can be viewed as sets of acceptable beliefs a reasoner may adopt based on the default theory. The extensions are logically closed superset sets of \(W\) which are closed under the default rules and contain only formulas possessing a non-circular derivation. A (propositional) default rule is of the form \(A : B_1, \ldots, B_n / C\) where \(A\), the prerequisite, \(B_i\), the consistency conditions, and \(C\), the consequent (also called head of the default rule), are formulas. The rule is applicable with respect to a set of formulas \(S\) iff \(A \in S\) and for no \(i \in \{1, \ldots, n\}\) \(\neg B_i \in S\).

We will extend default logic in two ways: we will admit ordered disjunction in \(W\) and in the head of default rules. This will allow us to represent desires in \(W\) and to derive preferences by default. The general principle is the same as in QCL, but instead of preferred models of the premises we have to consider preferred models of extensions. To define the preference relation between models a satisfaction degree of default rules needs to be defined.

**Definition 18.** A (propositional) choice default theory is a pair \((D,W)\), where \(W\) is a set of QCL formulas and \(D\) is a set of rules of the form \(A : B_1, \ldots, B_n / C\) where \(A\) and the \(B_i\) are classical formulas, \(C\) is a QCL formula.

To define extensions we will simply consider the default theories obtained by replacing ordered disjunction with ordinary disjunction:

**Definition 19.** Let \(\Delta = (D,W)\) be a choice default theory. \(E\) is an extension of \(\Delta\) iff \(E\) is an extension of \((D^*, W^*)\) where \(D^*\) and \(W^*\) are the propositional counterparts of \(D\), respectively \(W\), obtained by replacing ordered disjunction with ordinary disjunction.

Ordered disjunction is now used to determine the most preferred models of extensions. We will use \(\text{Ext}(\Delta)\) to denote the set of extensions of \(\Delta\). To compare the models of extensions we have to define a satisfaction degree not only for formulas, but also for the rules in \(D\). The satisfaction degree of a rule will not only depend on the satisfaction degree of its consequent, but also on the applicability of the rule within the extension. Since there is no need to punish a rule for being inapplicable we will say that a rule is satisfied to the best possible degree 1 whenever its prerequisite is underivable or one of its consistency conditions is violated.

**Definition 20.** Let \(E\) be an extension of a choice default theory \((D,W)\), \(M\) a model of \(E\) and \(r = A : B_1, \ldots, B_n / C\) a rule in \(D\). The \(E\)-satisfaction degree of \(r\) in \(M\), denoted \(\text{deg}_E^M(r)\), is defined as follows:

\[
\text{deg}_E^M(r) = \begin{cases} 
1 & \text{if } A \notin E \text{ or } \neg B_i \in E \text{ for some } i \in \{1, \ldots, n\}, \\
\text{deg}_M(C) & \text{otherwise.}
\end{cases}
\]
Here \( \deg^M_C \) denotes the degree of satisfaction of \( C \) in \( M \). \( \deg^M_E(r) \) is well-defined since the consequent of an applicable rule must be in the extension. \( M \) must therefore be a model of \( C \).

The definition of the satisfaction degree of a QCL formula in a model is independent of a particular extension and remains unchanged. Given the satisfaction degrees of rules and formulas, we can define a preference ordering on the models of an extension \( E \) using any of the methods discussed earlier for QCL. For instance, in the cardinality based approach we can count the formulas in \( W \) and rules in \( D \) satisfied to a certain degree, etc.

Assume a preference ordering on the models of each extension \( E \) is fixed that way. Let \( \text{Pref}(E) \) denote the maximally preferred models of \( E \) based on this ordering.

**Definition 21.** A formula \( F \) is a consequence of a choice default theory \( \Delta = (D, W) \) iff \( F \) is true in each model of the set \( \bigcup_{E \in \text{Ext}(\Delta)} \text{Pref}(E) \).

This definition generalizes both QCL and default logic (under sceptical inference): if \( D = \emptyset \) the consequences coincide with the QCL consequences of \( W \), and if the default theory does not contain ordered disjunction then the consequences are the formulas contained in all extensions.

Here is a small example illustrating what can be expressed in choice default logic. Assume you prefer having a Porsche over having a BMW over having a VW, but you also know that you cannot have an expensive car, and that normally a Porsche is expensive. Moreover, you prefer a convertible, unless you live in Germany where it rains quite often:

\[
\begin{align*}
\text{have}(\text{Porsche}) & \prec \text{have}(\text{BMW}) \prec \text{have}(\text{VW}), \\
\text{expensive}(\text{Porsche}) & \rightarrow \neg \text{have}(\text{Porsche}), \\
\text{true} : \text{expensive}(\text{Porsche})/\text{expensive}(\text{Porsche}), \\
\text{true} : \neg \text{residence}(\text{Germany})/\text{convertible} \prec \neg \text{convertible}.
\end{align*}
\]

The single extension contains \( \text{expensive}(\text{Porsche}) \) and thus \( \neg \text{have}(\text{Porsche}) \). Independently of whether a cardinality or an inclusion based preference criterion is used, the most preferred models of the extension contain \( \text{have}(\text{BMW}) \) and \( \text{convertible} \) which are therefore consequences of the choice default theory. In general, the conclusions obtained this way describe what is true in the most desired worlds which are considered plausible.

9. Discussion

We proposed in this paper a new nonmonotonic propositional logic for representing ranked options. The logic has a new connective called ordered disjunction. Since ordered disjunction is fully embedded in the language, the ranking of the options may depend on the particular context, that is, it may depend on what else is true in the current situation.

We investigated computational aspects of QCL and showed how sets of formulas can be translated to stratified knowledge bases and, by results in [5], to classical propositional knowledge bases. We indicated a number of potential applications of the logic, and we presented alternative definitions of the consequence relation for applications where the
lexicographic ordering based on the number of best possible options is not adequate. Moreover, we investigated the relationship between \(\text{QCL}\) on one hand and circumscription respectively possibilistic logic on the other. We also proposed a combination of \(\text{QCL}\) and Reiter’s default logic.

In the next subsection we clarify the role of formulas in \(\text{QCL}\). We then discuss related work.

### 9.1. Beliefs and desires in \(\text{QCL}\)

The reader will have noticed that in \(\text{QCL}\) there is no syntactic distinction between formulas representing beliefs and formulas representing desires or intentions. For the applications discussed in Section 4 this did not pose any problems. In contexts where the distinction is important it may be useful (and necessary to avoid wishful thinking, see [36]) to split the premises \(T\) into two subsets, a set \(K\) of classical formulas representing beliefs about the real world and a set \(D\) of \(\text{QCL}\) formulas representing desires.

We call \(BD = (K, D)\) where \(K\) is a set of propositional formulas and \(D\) a set of \(\text{QCL}\) formulas a belief-desire theory. For simplicity, we will assume that \(D\) is in normal form, that is \(D = \{F_1, \ldots, F_n\}\) where \(F_i = C_{i,1} \times \cdots \times C_{i,k_i}\). Let \(S\) be a set of classical formulas. A classical formula \(F\) is called BD-belief iff \(K \vdash F\). \(F\) is called conditional desire given \(S\) iff 

\[
\bigvee_{M \in \text{Pref}(S \cup D)} \bigwedge_{1 \leq i \leq n} C_{i, \deg^M(F_i)} \vdash F.
\]

Here \(\text{Pref}(S \cup D)\) denotes the preferred models of \(S \cup D\), \(\deg^M(F_i)\) is the satisfaction degree of \(F_i\) in \(M\). The disjunction used here can be viewed as a representation of the desires which can be satisfied in the most preferred models satisfying \(S\).

An unconditional desire is a conditional desire given \(\emptyset\). A BD-desire is a conditional desire given \(K\). Obviously, each belief and each BD-desire is a \(\text{QCL}\) consequence of \(K \cup D\).

Using this terminology it is possible to explain the behaviour of a “flat” \(\text{QCL}\) theory \(T\) which does not distinguish between beliefs and desires. Let \(BD_T = (K_T, D_T)\) be an arbitrary partition of \(T\) into beliefs and desires. Then there are three categories of conclusions of \(T\): beliefs, \(BD_T\)-desires, and “mixed” formulas containing, for instance, conjunctions of beliefs and desires. Moving a classical formula from \(D\) to \(K\), or vice versa, may change the category of a conclusion, but never the set of conclusions. Thus, whenever it is unimportant to distinguish between beliefs and desires in the conclusions flat \(\text{QCL}\) can be used.

### 9.2. Related work

Preference handling in nonmonotonic reasoning and logic programming has received considerable attention in recent years. For an overview of some of the existing approaches see the discussion in [12] or the more recent [33]. Only few approaches allow for context dependent preferences. Existing work on context dependent preferences in nonmonotonic logics is based on explicit representations of a preference ordering together with names for default rules and sophisticated reformulations of the acceptable belief sets [9,10] or
makes heavy use of meta-predicates and compilation techniques [15,20]. The availability of ordered disjunction in QCL allows context dependent preferences among properties of a problem solution to be expressed much more conveniently.

QCL is also closely related to approaches in qualitative decision making [17]. Poole [31] aims at a combination of logic and decision theory. His approach incorporates quantitative utilities whereas our preferences are qualitative. Interestingly, Poole uses a logic without disjunction (“rather than using disjunction ... we want to use probability and decision theory to handle uncertainty”, Section 1.5) whereas we enhance disjunction.

In [8] CP-networks are introduced, together with corresponding algorithms. These networks are a graphic representation, somewhat reminiscent of Bayes nets, for conditional preferences among feature values under the ceteris paribus principle. Our approach differs from CP-networks in at least two respects:

- Since ordered disjunction is fully embedded in the logic, we are able to represent more general preferences. Preferences in CP-networks are always total orders of the possible values of a variable.
- The ceteris paribus interpretation of preferences is different from our interpretation. The former views the available preferences as (hard) constraints on a global preference order. A set of QCL formulas, on the other hand, is more like a set of different criteria in multi-criteria optimization. For example, the QCL theory

\[ \{ A \rightarrow (C \land D), B \rightarrow (D \land C), A, B \} \]

is not inconsistent. There is reason to prefer C over D, and reason to prefer D over C. In QCL such conflicting preferences may neutralize each other, but do not lead to inconsistency.

In a series of papers [22,23,37], originally motivated by [7], the authors propose viewing conditional desires as constraints on utility functions. Intuitively, \( D(a|b) \) stands for: the \( b \)-worlds with highest utility satisfy \( a \). Our interpretation of ranked options is very different. Rather than being based on decision theory our approach can be viewed as giving a particular interpretation to the ceteris paribus principle: a model \( M_1 \) is preferred over a model \( M_2 \) if there is a formula \( F \in T \) satisfied to degree \( j \) by \( M_1 \) and to degree \( k > j \) by \( M_2 \) provided \( M_1 \) satisfies the other formulas in \( T \) at least as well as \( M_2 \). The last phrase is made precise as follows: for each degree \( i \leq j \ M_1 \) satisfies at least as many formulas in \( T \setminus \{F\} \) as \( M_2 \) to degree \( i \).

Our work is also related to valued (sometimes also called weighted) constraint satisfaction [6,18,19,34]. A classical constraint problem is given by a set of variables \( V \), a domain \( D_v \) for each variable \( v \), and a set of constraints specifying conditions for solutions. A solution is an assignment of values from the respective domain to the variables satisfying all constraints.

A valued constraint, rather than specifying hard conditions, yields a ranking of solutions. A global ranking of solutions then is obtained from the rankings provided by the single constraints through some combination rule. In MAX-CSP [18], for instance, constraints assign penalties to solutions and solutions with the lowest penalty sum are
preferred. In fuzzy CSP [19] each solution is characterized by the worst violation of any constraint. Preferred solutions are those where the worst violation is minimal.

Determining a preferred QCL model can be viewed as a valued constraint problem where the variables are the atoms, the domains are the truth values, solutions are models, and the valued constraints are expressed as QCL formulas. It is a topic of further research whether combination rules used in constraint satisfaction have interesting applications in QCL, and vice versa.

In future work we want to investigate combinations of QCL and existing product configuration methodologies, e.g., [35], extensions of the inference relation to non-classical formulas, and the first order case. An application of the ideas underlying QCL to answer set programming is described in [11], an implementation of this approach based on the answer set solver Smodels is described in [13].

Acknowledgements

We would like to thank Jérôme Lang, David Makinson, Pierre Marquis, Ilkka Niemelä, Leon van der Torre and the anonymous referees for very helpful comments. The first author acknowledges support from DFG (Computationale Dialektik: BR 1817/1-5). The second and the third author were supported in part by the IUT de Lens, the Université d’Artois, the Nord/Pas-de-Calais Région under the TACT-TIC project and by the European Community FEDER Program.

Appendix A. Proofs of propositions

This appendix contains proofs of Propositions 2, 6, 8, 9, 10 and 11 and proofs of Lemmas 4 and 6. A few additional lemmas are proven which turn out to be useful for the main proofs. The proofs of Lemmas 1–3 and Proposition 1 are straightforward by induction on the structure of formula \( F \) and are therefore omitted. Proofs of Propositions 3 and 4 follow from Proposition 8 and the well-known properties of the lexicographic systems. Proposition 7 is a corollary of Propositions 2 and 6. Proposition 5 is a corollary of Proposition 7 and results in [5].

**Fact 1.** Let \( F_1, F_2 \) be two arbitrary choice formulas. Let \( I \) be an interpretation, and \( k \geq 1 \) be such that \( I \models_k F_1 \). Then we also have: \( I \models_k F_1 \times F_2 \).

**Proof of Proposition 2 (Ordered disjunction is associative).** First, it easy to check that the two formulas \( ((F_1 \times F_2) \times F_3) \) and \( (F_1 \times (F_2 \times F_3)) \) have the same optionality, namely:

\[
\text{opt}((F_1 \times F_2) \times F_3) = \text{opt}(F_1 \times (F_2 \times F_3)) = \text{opt}(F_1) + \text{opt}(F_2) + \text{opt}(F_3).
\]

Let \( I \) be an interpretation, let us consider different cases of satisfaction of the formulas \( F_i \) by \( I \):
Proof of Proposition 6 (Strong equivalences of formulas without $\times$). For the three equivalences, it is easy to check the equality of the optionality of the equivalent formulas.

(1) We assume for the sake of simplicity, and without loss of generality, that $m \leq n$. The equivalence becomes:

$$(A_1 \times \cdots \times A_n) \lor (B_1 \times \cdots \times B_m) \equiv (A_1 \lor B_1) \times \cdots \times (A_n \lor B_n)$$

where for $i > m$, $B_i = \bot$ (since $A_i$ is classically equivalent to $A_i \lor \bot$).

Let us consider different cases of satisfaction of $A_i$’s and $B_j$’s by $I$.

- There exists $i > 0$ and $j > 0$ such that $I \models \neg A_1 \land \cdots \land \neg A_{i-1} \land A_i$ and $I \models \neg B_1 \land \cdots \land \neg B_{j-1} \land B_j$.

This implies that:

- for $k < \min(i, j)$, $I \models \neg(A_k \lor B_k)$, and for $n \geq k \geq \min(i, j)$: $I \models A_k \lor B_k$,
- $I \models A_1 \times \cdots \times A_n$,
- $I \models B_1 \times \cdots \times B_m$.

This leads by definition to:

$$I \models_{\min(i, j)} (A_1 \times \cdots \times A_n) \lor (B_1 \times \cdots \times B_m), \quad \text{and}$$

$$I \models_{\min(i, j)} (A_1 \lor B_1) \times \cdots \times (A_n \lor B_n).$$

- $\forall i \leq n, I \models \neg A_i$ and there exists $j > 0$ such that $I \models \neg B_1 \land \cdots \land \neg B_{j-1} \land B_j$.

This implies that:

$$I \models \neg(A_1 \lor B_1) \land \cdots \land \neg(A_{j-1} \lor B_{j-1}) \land (A_j \lor B_j) \land \cdots \land (A_n \lor B_n),$$

$$I \models \neg(A_1 \times \cdots \times A_n),$$

$$I \models B_1 \times \cdots \times B_m.$$

Hence by definition, we get:

$$I \models_j (A_1 \times \cdots \times A_n) \lor (B_1 \times \cdots \times B_m), \quad \text{and}$$

$$I \models_j (A_1 \lor B_1) \times \cdots \times (A_n \lor B_n).$$
Let us assume that Lemma 8.

The following lemma:

Proof of Proposition 8 (respectively. In a similar way, we assume for the sake of simplicity, and without loss of generality, that $m \leq n$.

The equivalence becomes:

$$(A_1 \times \cdots \times A_n) \land (B_1 \times \cdots \times B_m) \equiv (C_1 \times \cdots \times C_n)$$

where $C_i = [(A_1 \lor \cdots \lor A_i) \land B_j] \lor [A_i \land (B_1 \lor \cdots \lor B_l)]$, and $B_j = \bot$ for $m < i$.

Let us consider different cases of satisfaction of $A_i$'s and $B_j$'s by $I$.

- There exists $i > 0$ and $j > 0$ such that $I \models \neg A_1 \land \cdots \land \neg A_i \land A_j$ and $I \models \neg B_1 \land \cdots \land \neg B_{j-1} \land B_j$.

This implies that:

- for $k < \max(i, j)$, $I \models \neg C_i$, and for $n \geq k \geq \max(i, j)$, $I \models C_k$,
- $I \models A_1 \times \cdots \times A_n$,
- $I \models B_1 \times \cdots \times B_m$.

This leads by definition to:

$I \models_{\max(i, j)} (A_1 \times \cdots \times A_n) \land (B_1 \times \cdots \times B_m)$, and

$I \models_{\max(i, j)} C_1 \times \cdots \times C_n$.

- $\forall i \leq n$, $I \models \neg A_i$, or $\forall j \leq m$, $I \models \neg B_j$.

This implies that: $\forall i \leq n$, $I \models \neg C_i$. Hence:

$I \models \neg [(A_1 \times \cdots \times A_n) \land (B_1 \times \cdots \times B_m)]$, and

$I \models \neg [C_1 \times \cdots \times C_n]$.

(3) $I \models \neg (A_1 \lor \cdots \lor A_n)$ iff $\forall 0 \leq i \leq n$, $I \models \neg A_i$

iff $I \models \neg [(A_1 \times \cdots \times A_n) \land (B_1 \times \cdots \times B_m)]$, and

I $\models \neg [C_1 \times \cdots \times C_n]$.

Proof of Proposition 8 (Sketch). $T \models F$ iff $S(kb(T)) \models F$.

Let $S(kb(T)) = S = (T^*, T_1, \ldots, T_n)$ be the stratified base associated with $T$. In the following, $M_1$ and $M_2$ denote two models of $T$. $|M_1^p(T)|$ and $|M_2^p(T)|$ denote the number of simple choice logic formulas from $T$ satisfied to a degree $p$ by $M_1$ and $M_2$, respectively. In a similar way, $|M_1^S(S)|$ and $|M_2^S(S)|$ denote the number of propositional formulas from $T$ satisfied by $M_1$ and $M_2$, respectively. The idea of the proof is to show that the lexicographic ordering between models of $T$ which is based on the number of satisfied choice formulas from $T$ is the same as the one based on the number of satisfied propositional formulas from Skb(T).

Fact 2. $|M_1^1(S)| \geq |M_2^1(S)|$ iff $|M_1^1(T)| \geq |M_2^1(T)|$.

Given, this fact, the proof of Proposition 8 follows immediately by applying iteratively the following lemma:

Lemma 8. Let us assume that:

$\forall j = 1, \ldots, i$, $|M_1^j(T)| = |M_2^j(T)|$ iff $|M_1^j(S)| = |M_2^j(S)|$.

Then:
\begin{itemize}
  \item $|M_{i+1}^1(T)| = |M_{i+1}^2(T)|$ if and only if $|M_{i+1}^1(S)| = |M_{i+1}^2(S)|$, and
  \item $|M_{i+1}^1(T)| > |M_{i+1}^2(T)|$ if and only if $|M_{i+1}^1(S)| > |M_{i+1}^2(S)|$.
\end{itemize}

**Proof.** We will only show the first item. The other case follows similarly by replacing the symbol $=$ by $>$. 

First note that:

\[
|M_{i+1}^1(S)| + |M_{i+1}^1(T)| = |\{A_1 \check{x} \cdots \check{x} A_n \in T: M_1 \models A_1 \lor \cdots \lor A_i, \text{ and } n \geq i+1\}|
\]

\[
+ |\{A_1 \check{x} \cdots \check{x} A_n \in T: M_1 \models \neg A_1 \land \cdots \land \neg A_i \land A_{i+1}, \text{ and } n \geq i+1\}|
\]

\[
= |\{A_1 \check{x} \cdots \check{x} A_n \in T: M_1 \models A_1 \lor \cdots \lor A_i \lor A_{i+1}, \text{ and } n \geq i+1\}|
\]

Hence:

\[
(1) \quad |M_{i+1}^1(S)| = |M_i^1(S)| + |M_{i+1}^1(T)| - |T^{i+1}|
\]

where $T^{i+1} = \{A_1 \check{x} \cdots \check{x} A_n \in T: n = i+1\}$.

It is clear that any model $M$ of $T$ satisfies $A_1 \lor \cdots \lor A_n$ where $A_1 \check{x} \cdots \check{x} A_n \in T^{i+1}$.

Given (1) the proof follows straightforwardly. Indeed,

\begin{itemize}
  \item if $|M_i^1(S)| = |M_i^1(T)|$ and $|M_{i+1}^1(S)| = |M_{i+1}^1(T)|$ then using (1) we also have $|M_{i+1}^1(S)| = |M_{i+1}^2(S)|$;
  \item if $|M_i^1(S)| = |M_i^2(S)|$ and $|M_{i+1}^1(S)| = |M_{i+1}^2(S)|$ then using again (1) we also have $|M_{i+1}^1(T)| = |M_{i+1}^2(T)|$.
\end{itemize}

**Proof of Proposition 9 (Sketch).** $T \vdash_{inc} F$ iff $\text{Skb}(T) \vdash_{inc} F$.

In the following, $M_i^p(T)$ and $M_i^s(T)$ denote the set of simple choice logic formulas from $T$ satisfied to a degree $p$ by $M_1$ and $M_2$, respectively. In a similar way, $M_i^p(S)$ and $M_i^s(S)$ denote the set of propositional formulas from $T_i$ satisfied by $M_1$ and $M_2$, respectively. We define $\text{Skb}$-inclusion-preference between models of $T$ exactly like in Definition 13, by replacing $M_i^p(T)$ and $M_i^s(T)$ by $M_i^p(S)$ and $M_i^s(S)$, respectively. The idea of the proof is then to show that the $T$-inclusion-based ordering coincides with $\text{Skb}$-inclusion-based ordering. We first give two facts and show a lemma.

**Fact 3.** If $M \models_k A_1 \check{x} \cdots \check{x} A_n$ then for any $j = k, \ldots, n$ we have $M \models A_1 \lor \cdots \lor A_j$.

**Fact 4.** If $M \models A_1 \lor \cdots \lor A_j$ then there exists some $k \leq j$ such that $M \models_k A_1 \check{x} \cdots \check{x} A_n$.

**Lemma 9.** If there exists some $k$ such that for all $j = 1, \ldots, k$ we have $M_i^j(T) = M_i^j(S)$ then for all $j = 1, \ldots, k$ we have $M_i^j(S) = M_i^j(S)$. The converse is also true.

**Proof.**

\begin{itemize}
  \item Assume that for all $j = 1, \ldots, k$ we have
    \[
    M_i^j(T) = M_i^j(S)
    \] 

    and there exists some $i \leq k$ such that $M_i^j(S) \neq M_i^j(S)$.
\end{itemize}
Lemma 10. \( M_1 \) is \( \text{Skb}\)-inclusion-preferred to \( M_2 \) iff \( M_1 \) is \( T \)-inclusion-preferred to \( M_2 \).

Proof.

- Let \( M_1 \) be \( \text{Skb}\)-inclusion-preferred to \( M_2 \) but \( M_1 \) is not \( T \)-inclusion-preferred to \( M_2 \). By definition, \( M_1 \) is \( \text{Skb}\)-inclusion-preferred to \( M_2 \) implies that there exists some \( i \) such that for all \( j < i \) we have \( M_1^j(S) = M_2^j(S) \) and \( M_2^j(S) \subseteq M_1^j(S) \). From Lemma 5, we also get: for all \( j < i \) we have \( M_1^j(T) \subseteq M_2^j(T) \).

Moreover, \( M_2^j(S) \subseteq M_1^j(S) \) means that there exists a formula \( A_1 \lor \cdots \lor A_j \) which is satisfied by \( M_1 \) but not by \( M_2 \). This means that there exists a choice logic formula \( A_1 \times \cdots \times A_n \) which is satisfied to degree \( i \) by \( M_1 \) but not by \( M_2 \) (which means that \( M_2^j(T) \) is not included in \( M_1^j(T) \)).

Now, assume that there exists a simple choice logic formula \( A_1 \times \cdots \times A_n \) which is satisfied by \( M_2 \) to a degree \( i \) but not by \( M_1 \). From Fact 3, this implies that \( M_2 \models A_1 \lor \cdots \lor A_j \), which implies \( M_1 \models A_1 \lor \cdots \lor A_j \) (since \( M_2^j(S) \subseteq M_1^j(S) \)) and this contradicts the fact that \( A_1 \times \cdots \times A_n \) is not satisfied by \( M_1 \) to a degree \( i \).

- The proof is symmetric. Let \( M_1 \) be \( T \)-inclusion-preferred to \( M_2 \) but \( M_1 \) is not \( \text{Skb}\)-inclusion-preferred to \( M_2 \). By definition, \( M_1 \) is \( \text{Incl}\)-\( T \)-preferred to \( M_2 \) implies that there exists some \( i \) such that for all \( j < i \) we have \( M_1^j(T) = M_2^j(T) \) and \( M_2^j(T) \subseteq M_1^j(T) \). From Lemma 5, we also get: for all \( j < i \) we have \( M_1^j(S) = M_2^j(S) \).

Moreover, \( M_2^j(T) \subseteq M_1^j(T) \) means that there exists a simple choice logic formula \( A_1 \times \cdots \times A_n \) which is satisfied to degree \( i \) by \( M_1 \) but not by \( M_2 \). This means that there exists a propositional formula \( A_1 \lor \cdots \lor A_i \) in \( S_i \) which is satisfied by \( M_1 \) but not by \( M_2 \) (which means that \( M_2^j(S) \) is not included in \( M_1^j(S) \)). Now, assume that there exists a formula \( A_1 \lor \cdots \lor A_i \) in \( S_i \) which is satisfied by \( M_2 \) but not by \( M_1 \). Using Fact 4, this implies that \( M_2 \models A_1 \times \cdots \times A_n \), which implies \( M_1 \models A_1 \times \cdots \times A_n \) (since \( M_2^j(T) \subseteq M_1^j(T) \)) and this contradicts the fact that \( A_1 \lor \cdots \lor A_i \) is not satisfied by \( M_1 \).
Proof of Proposition 10 (Representing circumscription in QCL). We first observe that \( T \) and \( T' \) have exactly the same models since the propositional counterpart of each formula in \( T' \setminus T \) (obtained by replacing \( \times \) with \( \lor \)) is a tautology. Now assume \( M \) is not \( \prec_{V_1,..,V_r;W} \)-preferred. Then there exists a model \( M' \) of \( T \) and an \( i \) such that \( M' \) agrees with \( M \) on atoms in \( W \) and on atoms in \( V_1,..,V_{i-1} \) and makes fewer atoms in \( V_i \) true than \( M \). Thus the formulas in \( T' \setminus T \) satisfied to degree \( 1,..,n-1 \) by \( M' \) are the same as those satisfied to degree \( 1,..,n-1 \) by \( M \). Moreover, \( M' \) satisfies a superset of those satisfied by \( M \) to degree \( i \) and thus \( M \) is not an inclusion preferred model of \( T' \).

Conversely, let \( M \) be a non-preferred model of \( T' \). Then there is a model \( M' \) and an \( i \) such that the formulas in \( T' \setminus T \) satisfied to degree \( i \) by \( M' \) are a superset of those satisfied to degree \( i \) by \( M \), and those satisfied to any degree \( j < i \) by the two models coincide. Since changing the truth value of a variable in \( W \) always changes the satisfaction degree of some formula in \( T' \) from 1 to 2, \( M \) and \( M' \) agree on atoms in \( W \) and there must be a variable in \( V_i \) false in \( M' \) and true in \( M \). Since all variables in \( V_i \) which are false in \( M \) are also false in \( M' \), it follows that \( M \) is not \( \prec_{V_1,...,V_r;W} \)-preferred. \( \Box \)

Proof of Proposition 11 (Representing QCL in circumscription). The proof is based on the following

Lemma 11. Let \( F_{i,k} \) denote the \( k \)th ordered disjunct of formula \( F_i \in T \). If \( M \) is a model of \( T \), then \( M' = M \cup \{ab_{i,h} \mid h \leq j, M \models F_i \} \) is a model of \( T' \). Vice versa, if \( M' \) is a model of \( T' \), then \( M' \setminus AB_1 \cup \cdots \cup AB_n \) is a model of \( T \).

To prove the lemma consider a model \( M \) of \( T \). \( M \) satisfies each formula \( F_i \) to some degree \( j \). Implications \((1),\ldots,(j-1)\) of the translation of \( F_i \) are satisfied in \( M' \) because \( M' \) contains \( ab_{i,1},\ldots,ab_{i,j-1} \). Implication \((j)\) is satisfied because \( M \) and thus \( M' \) satisfies \( F_{i,j} \). Implications \((k)\) for \( k > j \) are satisfied because \( M' \) does not contain \( ab_{i,j} \).

Conversely, let \( M' \) be a model of \( T' \), \( F_j \) a formula in \( T \) with optionality \( m+1 \). \( M' \) contains an arbitrary subset \( S \) (possibly empty) of \( \{ab_{j,1},\ldots,ab_{j,m}\} \). \( S \) satisfies at least one of the preconditions of the implications obtained from translating \( F_j \) and thus the consequent of the implication which is one of the options of \( F_j \). Since \( F_j \) does not contain any of the new \( ab \)-atoms it follows that \( M \) satisfies \( F_j \). This concludes the proof of the lemma.

Now assume \( M_1 \) is a non-preferred model of \( T \), that is, there is a model \( M_2 \) inclusion preferred over \( M_1 \). Let \( j \) be the smallest satisfaction degree at which the two models differ, and let \( F_m \) be a formula satisfied to degree \( j \) in \( M_2 \), but not in \( M_1 \).

Consider a model \( M'_1 \) of \( T' \) such that \( M'_1 \setminus AB_1 \cup \cdots \cup AB_n = M_1 \). Let \( M'_2 \) be the model of \( T' \) constructed from \( M_2 \) according to the lemma. Since \( M_1 \) and \( M_2 \) agree on formulas satisfied to any degree smaller than \( j \) and \( M_2 \) satisfies more formulas to degree \( j \) we have, for each \( h \leq j \), \( ab_{r,h} \in M'_2 \) implies \( ab_{r,h} \in M'_1 \). Moreover, since \( M'_1 \) must contain \( ab_{m,j} \) whereas \( M'_2 \) does not, it is straightforward to show that \( M'_2 \prec_{AB_1,\ldots,AB_n;\emptyset} M'_1 \).

Conversely, assume \( M_1 \) is a maximally preferred model of \( T \). Let \( M'_2 \) be the model of \( T' \) obtained through the construction in the lemma. We show by contradiction that \( M'_1 \) is maximally \( \prec_{AB_1,\ldots,AB_n;\emptyset} \)-preferred. Assume for some model \( M'_2 \) of \( T' \) we have \( M'_2 \prec_{AB_1,\ldots,AB_n;\emptyset} M'_1 \). Since every model different from \( M'_1 \) which agrees with \( M'_1 \) on
atoms in $T$ must contain more $ab_{i,j}$ literals than $M_1'$, $M_2'$ cannot agree with $M_1'$ on atoms in $T$. Using the lemma we thus have that $M_2 = M_2' \setminus (AB_1 \cup \cdots \cup AB_n)$ is a model of $T$ different from $M_1$. Let $j$ be the smallest index such that, for some $k$, $ab_{i,j}$ is false in $M_2'$ but true in $M_1'$, $M_2$ satisfies all formulas $F \in T$ with $\deg_{M_1}(F) < j$ with a degree at least as good as $M_1$. Moreover, $M_2$ satisfies $F_k$ at least to degree $j$ whereas $M_1$ does not. Therefore $M_2$ is inclusion preferred to $M_1$, contrary to our assumption.

We thus have for each maximally inclusion preferred model of $T$ a corresponding model of $T'$ agreeing on $F$ and vice versa. □

**Proof of Lemma 4 (Possibility distribution associated with $\Delta_\oplus$).** We only show the case where two $\Delta$-knowledge bases are merged. The general case follows in a similar way.

Recall that

$$\pi_{\Delta_\oplus} = \left\{ [\phi_i \land \psi_j, \oplus(\alpha_i, \beta_j)] : [\phi_i, \alpha_i] \in \Delta_1 \text{ and } [\psi_j, \beta_j] \in \Delta_2 \right\}.$$  

Note first that if for some interpretation $I$, $\pi_1(I) = 0$ or $\pi_2(I) = 0$ then this means that $I$ falsifies all propositional formulas of $\Delta_1$, or $I$ falsifies all propositional formulas of $\Delta_2$. Hence, $I$ also falsifies all propositional formulas of $\Delta_\oplus$. Then, $\pi_{\Delta_\oplus} = 0 = \oplus(\pi_1(I), \ldots, \pi_n(I))$, since $\oplus(0, 0) = 0$.

Now assume that $I$ satisfies at least one propositional formula from $\Delta_1$ and at least one propositional formula from $\Delta_2$. Then $\pi_{\Delta_\oplus}$ is computed as follows:

$$\pi_{\Delta_\oplus}(I) = \max \{ \oplus(\alpha_i, \beta_j) : I \models [\phi_i \land \psi_j, \oplus(\alpha_i, \beta_j)] \in \Delta_\oplus \}$$

$$= \max \left\{ \oplus(\alpha_i, \beta_j) : I \models [\phi_i \land \psi_j, \oplus(\alpha_i, \beta_j)] \in \Delta_1 \text{ and } [\psi_j, \beta_j] \in \Delta_2 \right\}$$

$$= \max \left\{ \oplus(\alpha_i, \beta_j) : I \models [\phi_i, \alpha_i] \in \Delta_1 \text{ and } I \models [\psi_j, \beta_j] \in \Delta_2 \right\}.$$

Since $\oplus$ satisfies the unanimity condition then when $\alpha_i$ and $\beta_j$ are maximal then $\oplus(\alpha_i, \beta_j)$ is also maximal.

Hence,

$$\pi_{\Delta_\oplus}(I) = \oplus \left( \max \{ \alpha_i : I \models [\phi_i, \alpha_i] \in \Delta_1 \}, \max \{ \beta_j : I \models [\psi_j, \beta_j] \in \Delta_2 \} \right)$$

$$= \oplus(\pi_1(I), \pi_2(I)).$$

**Proof of Lemma 6 (Basic choice formulas as $\Delta$-knowledge bases).** The proof is immediate. Indeed, if for all $i$, $I \models A_i$, then from Definition 16 we have $\pi(I) = 0$, and from Definition 3 we have $I \models_1 \neg F$. Now, assume that $I$ satisfies some $A_i$. Then $\pi(I) = \alpha_k$ means that $I \models A_k$, and for $i = 1, \ldots, k-1$, we have $I \models A_i$, and this is exactly equivalent to $I \models_k F$. □

**References**


