Minimal cutwidth linear arrangements of abelian Cayley graphs

Daniel Berenda\textsuperscript{a}, Ephraim Korach\textsuperscript{b}, Vladimir Lipets\textsuperscript{c}

\textsuperscript{a}Departments of Mathematics and Computer Science, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
\textsuperscript{b}Department of Industrial Engineering and Management, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel
\textsuperscript{c}Department of Computer Science, Ben-Gurion University of the Negev, 84105 Beer-Sheva, Israel

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Abstract

We find the minimal cutwidth and bisection width values for abelian Cayley graphs with up to 4 generators and present an algorithm for finding the corresponding optimal ordering. We also find minimal cuts of each order.

Keywords: Combinatorial optimization; Graph algorithm; Cayley graph; MINCUT linear arrangement

1. Introduction

A large number of theoretical and practical problems in various areas may be formulated as graph layout problems. Such problems arise in connection with planar graphs, the optimization of networks for parallel computer architectures, VLSI circuit design, and numerous other problems. Many interesting graph layout problems are NP-hard, and thus a lot of work has been done on solving them for some structured graph families. Here we concentrate on the minimal cutwidth linear arrangement (MINCUT) problem, which may be posed in the following form. For a graph $G = (V, E)$ with $|V| = n$, and a placement of its vertices at positions $1, 2, \ldots, n$ on a line, the width of the cut (cutwidth) between positions $i$ and $i + 1$ (for $1 \leq i \leq n - 1$) is the number of edges, one of whose endpoints is placed between 1 and $i$ and the other between $i + 1$ and $n$.

\textbf{Problem.} Given a graph $G = (V, E)$, find a placing of the vertices for which the maximal cutwidth is as small as possible.

This problem is NP-hard in general [7], and even when restricted, for example, to polynomially (edge-)weighted trees or to planar graphs with maximum degree 3 [16]. In this paper, we provide a formula for the size of the optimal cutwidth for abelian Cayley graphs with up to 4 generators. Moreover, we obtain a tight upper bound on this size in terms of the order of the group only. Along with the formula, we give a linear time algorithm for finding the optimal ordering and minimal cut of given order. This result forms a generalization of some of the results for toroidal 2-meshes (see Table 1 below) in terms of the considered groups and the generators of the Cayley graph.
Table 1
Complexity of MINCUT for certain families of graphs

<table>
<thead>
<tr>
<th>Class of graph</th>
<th>Complexity</th>
<th>Formula</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trees</td>
<td>O(n log n)</td>
<td></td>
<td>[28]</td>
</tr>
<tr>
<td>Hypercubes</td>
<td>O(n)</td>
<td>+</td>
<td>[21]</td>
</tr>
<tr>
<td>d-dimensional c-ary cliques</td>
<td>O(n^2)</td>
<td>+</td>
<td>[20]</td>
</tr>
<tr>
<td>Max degree d and treewidth ≤ k</td>
<td>O(n^{d/2})</td>
<td></td>
<td>[3]</td>
</tr>
<tr>
<td>Ordinary two- and three-dimensional meshes</td>
<td>O(n)</td>
<td>+</td>
<td>[26]</td>
</tr>
<tr>
<td>Toroidal two- and three-dimensional meshes</td>
<td>O(n)</td>
<td>+</td>
<td>[26]</td>
</tr>
<tr>
<td>Cylindrical two-dimensional meshes</td>
<td>O(n)</td>
<td>+</td>
<td>[26]</td>
</tr>
<tr>
<td>Complete binary trees</td>
<td>O(n)</td>
<td>+</td>
<td>[14]</td>
</tr>
<tr>
<td>Complete p-partite graphs</td>
<td>O(n + p log p)</td>
<td></td>
<td>[18]</td>
</tr>
</tbody>
</table>

As a by-product of our proofs, we obtain minimal cuts of any order and a formula for computing their size for our family of graphs—abelian Cayley graphs with up to 4 generators. In particular, this provides a solution to the bisection width problem for the same family.

1.1. Historical perspective and applications

A layout is, roughly speaking, a linear ordering. As will be explained in more detail later, the term layout is due to the early application to optimal layouts of circuits. We present here some background that motivates research on layout problems, as well as some of their applications. For a more detailed survey of graph layout problems, see [5]. We start with a historical overview.

MINCUT was first used in the seventies as a theoretical model for the number of channels in an optimal layout of a circuit [2]; see also the Introduction in [15]. More recent applications of this problem include network reliability [10], automatic graph drawing [19], and information retrieval [4].

Many layout problems are originally motivated as simplified mathematical models of VLSI layout. Given a set of modules, the VLSI layout problem consists of placing the modules on a board in a non-overlapping manner and wiring together the terminals on the different modules according to a given wiring specification in such a way that the wires do not interfere with each other. There are two stages in a VLSI layout: placement and routing. The placement problem consists of placing the modules on a board; the routing problem consists of wiring together the terminals on different modules that should be connected. A VLSI circuit can be modelled by means of a graph, whose vertices represent modules and the edges represent the wires. Of course, this graph is an over-simplified model of the circuit, but understanding and solving problems in this simple model may assist in obtaining better solutions for the real-world problem.

MINCUT gives a measure of the area needed to represent the graph in a VLSI layout when vertices are laid out in a row [14]. In fact, in [22] a new relationship is found between the value of MINCUT and the area of the VLSI layout of a graph: the minimal area of a VLSI layout of a graph is not less than the square of its MINCUT.

1.2. Known results

As mentioned above, MINCUT is NP-hard in general. It is known, though, to be efficiently solvable in certain special cases. In Table 1 we list these cases, indicate their known complexity, and mark those which admit an exact formula for the MINCUT value.

Many popular interconnection network topologies, such as hypercubes and toroidal meshes (products of simple cycles), are based on Cayley graphs of abelian groups. The symmetry and algebraic structure of these graphs result in many nice physical properties of the network concerning layout, routing algorithms, and load balancing. As was mentioned above, in this paper we find the exact values of MINCUT for abelian Cayley graphs with up to 4 generators.

1.3. Layout problems as embedding problems

Linear arrangements are a special case of embedding graphs in d-dimensional grids or other graphs. In its most general form, the embedding of a graph G into a host graph H consists of defining an injective function mapping the
vertices of $G$ to the vertices of $H$ and associating a path in $H$ with each edge of $G$. Three parameters are fundamental to assess the quality of an embedding: the dilation, the congestion, and the load. The dilation of an embedding is the length of the largest associated path. The congestion of an embedding is the maximal number of paths sharing an edge of $H$. The load of an embedding is the maximal number of vertices of $G$ mapped to the same vertex of $H$. Making use of good embeddings is essential in certain contexts, such as parallel computing, where embeddings can be used to simulate an algorithm designed for one type of network on a parallel machine with a different type of network; see [17] for a nice survey. The case in which a graph with $n$ vertices has to be embedded into a path graph $P_n$ of $n$ vertices with load 1 is perhaps the simplest non-trivial embedding problem and has been intensively studied in the literature [12,1,9,25,15,24,4,11,13,27]. In this particular case, some layout and embedding problems are closely related. There exist other interesting embeddings into graphs other than paths. For instance, [23] presents a survey on cyclic MINCUT, that is, when the graph is embedded into a cycle rather than a path. Few results are known for other cyclic width parameters. Rolim et al. [26] solved the Cyclic MINCUT problem for two-dimensional toroidal meshes.

1.4. Definitions and notations

Given a finite graph $G = (V, E)$ with $n$ vertices, the MINCUT problem seeks a vertex enumeration function $F : V → \{1, \ldots, n\}$ such that

$$
\max_{1 \leq t < n} |\{(u, v) ∈ E : F(u) ≤ t < F(v)\}|
$$

is minimal over all such enumerations:

$$
γ(G) = \min_{F : V → \{1, \ldots, n\}} \max_{1 \leq t < n} |\{(u, v) ∈ E : F(u) ≤ t < F(v)\}|.
$$

There are interesting variations in the linear arrangement theme. For example, rather than minimizing \(\max_{1 \leq t < n} |\{(u, v) ∈ E : F(u) ≤ t < F(v)\}|\), the objective function may be

$$
\left( \sum_{(u, v) ∈ E} (|F(u) − F(v)|)^p \right)^{1/p}
$$

where $p ∈ (0, ∞]$. For $p = 1$, this problem is known as the optimal linear arrangement (OLA) problem, and for $p = ∞$ as the bandwidth (BANDWIDTH) problem. These problems seek an enumeration that minimizes the sum of all edge lengths and the length of the largest edge, respectively. Restricted to trees, BANDWIDTH remains NP-complete [6], but OLA can be solved in polynomial time [8]. There are other extensions and modifications of the basic problem. For example, weights can be added to the edges, changing the objective function to

$$
\max_{1 \leq t < n} \sum_{(u, v) ∈ E} w(u, v) I(F(u) ≤ t < F(v)).
$$

The weighted MINCUT (and, in addition, the optimal cuts of each order) turns out to be NP-complete even for trees [16].

As mentioned above, we can view the ordering of the graph vertices as a special case of the congestion, i.e., define MINCUT and Cyclic MINCUT via the congestion.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $|V_1| = |V_2|$. An embedding of $G_1$ in $G_2$ consists of a pair of mappings $φ$ and $ψ$, where $φ$ is a bijection from $V_1$ to $V_2$ and $ψ$ a function from $E_1$ to the set of simple paths in $G_2$, such that, if $(u, v) ∈ E_1$, then $ψ((u, v))$ is a path between $φ(u)$ and $φ(v)$. The congestion of the edge $e_2 ∈ E_2$ under such an embedding of $G_1$ in $G_2$ is

$$
cg(G_1, G_2, φ, ψ, e_2) = |\{e_1 ∈ E_1 : e_2 ∈ ψ(e_1)\}|.
$$

Thereby we obtain the following alternative forms for the objective functions of MINCUT and Cyclic MINCUT:

$$
γ(G) = \min_{φ, ψ} \max_{e_2 ∈ E(P_n)} cg(G, P_n, φ, ψ, e_2)
$$

and

$$
θ(G) = \min_{φ, ψ} \max_{e_2 ∈ E(C_n)} cg(G, C_n, φ, ψ, e_2),
$$

where $P_n$ and $C_n$ denote an $n$-vertex path and an $n$-vertex cycle, respectively.
The following definitions play an important role in the sequel. Let \( G = (V, E) \) be a graph of order \( n \). A cut in \( G \) is a partition of \( V \) into two sets, say \((A, \bar{A})\). For \( A, B \subseteq V \), denote by \( e(A, B) \) the set of all \( A-B \) edges:
\[
e(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}.
\]
In the special case where \( B = \bar{A} \) we obtain the set of all cut edges:
\[
e(A, \bar{A}) = \{(u, v) \in E \mid u \in A, v \notin A\}.
\]
The size of the cut \((A, \bar{A})\), denoted by \( c(A, \bar{A}) \), is the number of edges having exactly one vertex in \( A \) and the other in \( \bar{A} \), namely \(|e(A, \bar{A})|\). Employing this notation, we may view \( \gamma(G) \) as the minimum, taken over all orderings of \( V \), of \( \max_{1 \leq i \leq n-1} c(D_i, \bar{D}_i) \), where \( D_i \subseteq V \) consists of the first \( i \) vertices in the ordering.

A minimum cut of order \( i \) is a cut \((A, \bar{A})\) minimizing \( c(A, \bar{A}) \) over all sets with \( A \) of size \( i \). Denote the size of this cut by \( \delta_i(G) \):
\[
\delta_i(G) = \min\{c(A, \bar{A}) : A \subseteq V, |A| = i\}, \quad i = 1, 2, \ldots, n-1.
\]
For arbitrary graphs \( G \), the problem of determining the \( \delta_i(G) \)’s is NP-hard [7].

The bisection width of a graph is the size of a minimum cut of order \( \lfloor n/2 \rfloor \), namely \( \delta_{\lfloor n/2 \rfloor}(G) \).

2. The main results

Recall the definition of a Cayley graph. Let \( H \) be a finite group and \( S \) a subset thereof, closed with respect to inversion and not containing the identity. The Cayley graph \( \text{Cay}(H, S) \) of \( H \) with respect to \( S \) is the graph \((H, E)\), where \((x, y) \in E \) if \( y = xs \) for some \( s \in S \). The elements of \( S \) are the generators of \( \text{Cay}(H, S) \).

Throughout this paper \( H \) is a finite group of order \( h \) and \( G = \text{Cay}(H, S) \). Usually, \( S \) will be the set \({a^{\pm 1}, b^{\pm 1}}\), where \( a \) and \( b \) are of orders \( \alpha \) and \( \beta \), respectively, with \( \alpha, \beta \geq 3 \).

For abelian \( H \) and \( a, b \in H \), denote
\[
\psi_2^+(a, b) = \min\{|m| + |n| : a^m b^n = 1, m \neq 0 \land n \neq 0\}
\]
and
\[
\psi_2(a, b) = \min\{x, \beta, \psi_2^+(a, b)\} = \min\{|m| + |n| : a^m b^n = 1, m \neq 0 \lor n \neq 0\}.
\]
Also denote, for positive integers \( x, y \):
\[
\eta(x, y) = \begin{cases} 0 & y|x, \\ 1 & \text{otherwise.} \end{cases}
\]

**Theorem 2.1.** If \( G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\}) \) with \( H \) abelian and \( h > 4 \), and \( a^2 \neq 1, b^2 \neq 1, a \neq b^{\pm 1} \), then:

(i) \( \gamma(G) = 2 \min\{|x+1, \beta+1, \psi_2^+(a, b)| \} \leq 2\sqrt{2h} + 2 \). Moreover, the ordering constructed in Algorithm 2 is optimal.

(ii) For \( 1 \leq i \leq h-1 \):
\[
\delta_i(G) = 2 \min\{x + \eta(i, x), \beta + \eta(i, \beta), \psi_2^+(a, b), [2\sqrt{i}], [2\sqrt{h-i}]\}.
\]
Moreover, the set obtained in Algorithm 3 is optimal.

The following algorithms perform the required tasks. Algorithm 1 finds the optimal ordering for MINCUT for connected graphs. Algorithm 2 does the same in the general case. Algorithm 3 finds optimal cuts of any order.

**Remark 2.2.** Taking the first \( i \) elements in the ordering provided by Algorithm 2, we do not necessarily obtain an optimal cut of order \( i \).
Algorithm 1. MINCUT (FOR CONNECTED GRAPHS)

Require: $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$, where $H = \langle a, b \rangle$ is abelian.

1: $h \leftarrow |H|$
2: $\alpha \leftarrow \text{order of } a$
3: $\beta \leftarrow \text{order of } b$
4: $\psi_2^+(a, b) \leftarrow \min\{|m| + |n| : a^m b^n = 1, m \neq 0 \wedge n \neq 0\}$
5: choose $w_1, w_2$ such that: \{by Proposition 4.8\}
6: (a) $a^{w_2} b^{w_1} = 1$
7: (b) $\psi^+_2(a, b) = |w_1| + |w_2|$
8: if $\min\{\alpha + 1, \beta + 1, \psi^+_2(a, b)\} = \psi^+_2(a, b)$ then \{by Proposition 4.10\}
9: (a) $g$ is a generator of $H$
10: (b) $g^{w_2} = a$ and $g^{w_1} = b$
11: return $1, g, g^2, \ldots, g^{h-1}$

14: else \{by Case 2 of Proposition 4.10\}
15: $t \leftarrow \gcd(w_1, w_2)$
16: $F \leftarrow \langle a^{w_1/t} b^{w_2/t} \rangle$
17: choose $Fg$ such that: \{by Proposition 4.8\}
18: (a) $Fg$ is a generator of $H/F$
19: (b) $(Fg)^{w_2/t} = Fa$ and $(Fg)^{w_1/t} = Fb$
20: return $1, f, \ldots, f^{t-1}, g, fg, \ldots, f^{t-1}g, \ldots, g^{h/t-1}$, $f g^{h/t-1}$, $\ldots, f^{t-1}g^{h/t-1}$

21: end if
22: else \{by Proposition 4.11\}
23: if $\alpha > \beta$ then
24: $a \leftarrow b$
25: end if
26: $\alpha \leftarrow \min\{\alpha, \beta\}$
27: $A \leftarrow \langle a \rangle$
28: $h' \leftarrow h/\alpha$
29: return $1, a, \ldots, a^{\alpha-1}$, $b, ab, \ldots, a^{\alpha-1}b$, $b^{h'-1}$, $ab^{h'-1}$, $\ldots, a^{\alpha-1}b^{h'-1}$

30: end if

Algorithm 2. MINCUT

Require: $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$, where $H$ is abelian.

1: $G_c \leftarrow \text{Cay}(\langle a, b \rangle, \{a^{\pm 1}, b^{\pm 1}\})$
2: $h_c \leftarrow |\langle a, b \rangle|$
3: $t \leftarrow h/h_c$
4: Apply Algorithm 1 to obtain an optimal ordering $v_1, v_2, \ldots, v_{h_c}$ of $G_c$
5: $H_1, H_2, \ldots, H_t \leftarrow \text{cosets of } H/\langle a, b \rangle$
6: choose $u_i \in H_i$, $1 \leq i \leq t$
7: return $u_1 v_1, u_1 v_2, \ldots, u_1 v_{h_c}, u_2 v_1, u_2 v_2, \ldots, u_2 v_{h_c}, \ldots, u_t v_1, u_t v_2, \ldots, u_t v_{h_c}$
Algorithm 3. MINIMAL CUT OF A GIVEN ORDER

Require $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$, where $H$ is abelian and $1 \leq i \leq h - 1$.

1: $h \leftarrow |H|$
2: $z \leftarrow \text{order of } a$
3: $\beta \leftarrow \text{order of } b$
4: if $\min\{x + \eta(i, x), x + \eta(i, \beta), \psi_2^+(a, b)\} > 2 \sqrt{\min\{i, h - i\}}$ then (by Proposition 4.14)
5: \[i_0 \leftarrow \min\{i, h - i\}\]
6: $x \leftarrow \lceil \sqrt{i_0} \rceil$
7: $y \leftarrow \lceil i_0/x \rceil$
8: $z \leftarrow xy - i_0$
9: $A \leftarrow \{a^y b^{y'} : 0 \leq x' \leq x - 1, \ 0 \leq y' \leq y - 1\}$
10: $B \leftarrow A\setminus\{1, a, a^2, \ldots, a^{z-1}\}$
11: if $i < h/2$ then
12: \[\text{return } B\]
13: else \[\text{return } \bar{B}\]
14: end if
15: end if

Theorem 2.1 deals with the only non-trivial case of abelian Cayley graphs with up to 4 generators. In fact:

1. If $|S| \leq 2$, then $G = \text{Cay}(H, \{a^{\pm 1}\})$, so that $G$ is a union of simple cycles if $a$ is not of order 2 and a union of disjoint 2-paths otherwise. Arranging these cycles or paths in the “natural” order, we see that $\gamma(G) = 2$ or $\gamma(G) = 1$, respectively.
2. If $|S| = 3$, then either $G = \text{Cay}(H, \{a, b, c\})$, where $a^2 = b^2 = c^2 = 1$, or $G = \text{Cay}(H, \{a^{\pm 1}, b\})$, where $a^2 \neq 1$ and $b^2 = 1$. In the first case the connected component of the identity is of order 4 or 8, which is trivial to deal with. In the second case, if $h > 4$ it is not hard to show that $\gamma(G) = 5$ and $\delta_i(G) = \min\{4 + \eta(i, 2), 3i, 3(h - i)\}$ (and find an optimal ordering or cut, respectively).
3. If $|S| = 4$, then there are three possibilities (in addition to the case of Theorem 2.1).
   (i) $G = \text{Cay}(H, \{a, b, c, d\})$, where $a^2 = b^2 = c^2 = d^2 = 1$. Similarly to Case 2, the connected component of the identity is of order 8 or 16, which is trivial to deal with.
   (ii) $G = \text{Cay}(H, \{a^{\pm 1}, b, c\})$, where $a^2 \neq 1$ and $b^2 = c^2 = 1$, $b = a^{x/2}$, $c \notin \langle a \rangle$. Similar to the case $|S| = 3$, we can prove that (for $h > 8$) $\gamma(G) = 10$ and $\delta_i(G) = \min\{10, 2\sqrt{i}, 2\sqrt{h - i}\}$ for $i = 1, 2, \ldots, h - 1$.
   (iii) $G = \text{Cay}(H, \{a^{\pm 1}, b, c\})$, where $a^2 \neq 1$, $b^2 = c^2 = 1$, and $b, c \notin \langle a \rangle$.

In this case it is easy to see that the connected component of $G$ is isomorphic to $G' = \text{Cay}(\langle a \rangle \times C_4, \{(a, 0)^{\pm 1}, (0, 1)^{\pm 1}\})$ (where $C_k$ is the cyclic group of order $k$), which is dealt with by Theorem 2.1.

Throughout the rest of this paper, unless specified otherwise, we assume (as in Theorem 2.1) that $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$, where $H$ is abelian and $a^2 \neq 1$, $b^2 \neq 1$, $a \neq b^{\pm 1}$.

3. General properties

In this section we discuss some properties of MINCUT which will be useful for Cayley graphs as well.

3.1. Connected graphs

We start with the following trivial
Lemma 3.1. For every graph $G$

$$\gamma(G) = \max_{1 \leq i \leq t} \gamma(G(A_i)),$$

where $A_1, A_2, \ldots, A_t$ are the connected components of $G$.

In fact, for any subgraph $G'$ of $G$ we obviously have $\gamma(G) \geq \gamma(G')$ and therefore

$$\gamma(G) \geq \gamma(G(A_i)), \quad 1 \leq i \leq t.$$ 

On the other hand, suppose we order the vertices of $G$ by putting first the vertices of $A_1$, and so forth, where each $A_i$ is ordered optimally. The value of our objective function for this ordering is exactly $\max_{1 \leq i \leq t} \gamma(G(A_i))$, so that

$$\gamma(G) \leq \gamma(G(A_i)), \quad 1 \leq i \leq t.$$ 

Therefore,

$$\gamma(G) = \max_{1 \leq i \leq t} \gamma(G(A_i)).$$

3.2. Minimum cut of a given order

The main reason for being interested in minimum cuts of a given order is the following obvious property.

Lemma 3.2. For every graph $G = (V, E)$:

$$\gamma(G) \geq \max_{1 \leq i \leq |V|-1} \delta_i(G).$$

Example 3.3. The inequality in the lemma may well be strict. Let $G$ be a disconnected graph of order $n = 2^t + 1 - 1$ with $t + 1$ connected components $K_1, K_2, K_4, \ldots, K_{2^t}$, where $K_j$ denotes a clique of order $j$. Given $i \in \{1, \ldots, n\}$, write

$$i = x_0 + 2x_1 + 4x_2 + \cdots + 2^t x_t, \quad (x_j \in \{0, 1\}, 0 \leq j \leq t).$$

Let

$$A = \bigcup_{x_j=1} V(K_{2^j}).$$

Since $|A| = i$ and $c(A, \bar{A}) = 0$ we obtain

$$\delta_i(G) = 0, \quad 1 \leq i \leq n.$$ 

On the other hand, Lemma 3.1 gives

$$\gamma(G) = \max_{1 \leq j \leq t} \gamma(K_{2^j}) = \gamma(K_{2^t}) = 2^{2t-2}.$$ 

3.3. Splitting lemma

Lemma 3.4. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_n = (V_n, E_n)$ be $n$ graphs of order $m$ with disjoint vertex sets. Let $G' = (V', E')$ be a graph of order $n$, say $V' = \{v'_1, v'_2, \ldots, v'_n\}$. Let $G = (V, E)$ be a graph having the following properties:

1. $V = V_1 \cup V_2 \cup \cdots \cup V_n$.
2. $E \supseteq E_1 \cup E_2 \cup \cdots \cup E_n$.
3. There exists an $l \leq m$ such that, if $(v'_i, v'_j) \in E'$, then $|\{v^j \in V_j: (v^i, v^j) \in E\}| = l$ for every $v^i \in V_i$ (see Fig. 1).
(4) If \((v^i_j, v^j_i) \notin E\), then \((v^i_j, v^j_i) \notin E\) for every \(v^i \in V_i\) and \(v^j \in V_j\). Then

\[
\gamma(G) \leq ml \cdot \gamma(G') + \max_{1 \leq i \leq n} \gamma(G_i).
\]

**Proof.** Write

\[
V_i = \{v^i_1, v^i_2, \ldots, v^i_m\}, \quad 1 \leq i \leq n.
\]

Suppose without loss of generality that

(a) for each \(1 \leq i \leq n\) the ordering

\[
v^i_1, v^i_2, \ldots, v^i_m,
\]

is optimal for \(G_i\),

(b) the ordering \(v^i_1, v^i_2, \ldots, v^i_m\) is optimal for \(G'\).

We claim that the ordering

\[
v_1^1, v_2^1, \ldots, v_m^1, v_1^2, v_2^2, \ldots, v_m^2, \ldots, v_1^n, v_2^n, \ldots, v_m^n
\]

satisfies the following property: for any \(1 \leq j \leq nm - 1\)

\[
c(D_j, \overline{D}_j) \leq ml \cdot \gamma(G') + \max_{1 \leq i \leq n} \gamma(G_i),
\]

where \(D_j\) denotes the first \(j\) vertices of the above arrangement of \(V\). Denote \(E_+ = E_1 \cup E_2 \cup \cdots \cup E_n\). Obviously

\[
c(D_j, \overline{D}_j) = |e(D_j, \overline{D}_j)|
\]

\[
= |e(D_j, \overline{D}_j) \setminus E^+| + |e(D_j, \overline{D}_j) \cap E^+|.
\]

Since each \(V_i\) is separately ordered optimally, we have

\[
|e(D_j, \overline{D}_j) \cap E^+| \leq \max_{1 \leq i \leq n} \gamma(G_i).
\]

It remains to show that \(|e(D_j, \overline{D}_j) \setminus E^+| \leq ml \cdot \gamma(G')\) for each \(j\). Let

\[
F(j) = |e(D_j, \overline{D}_j) \setminus E^+|, \quad 1 \leq j \leq nm - 1.
\]
We distinguish between two cases:

Case 1: \( j = tm \) for some \( 1 \leq t \leq n - 1 \).

In this case \( D_j = V_1 \cup V_2 \cup \cdots \cup V_t \). Denote \( D'_j = \{ v'_1, v'_2, \ldots, v'_t \} \). From the third property of \( G \) in the lemma it follows that to any edge \((u', v') \in e(D'_j, \bar{D}_j)\) in \( G' \) there correspond exactly \( lm \) edges in \( e(D_j, \bar{D}_j) \cdot E^+ \) in \( G \). Therefore,

\[
F(tm) = |e(D_j, \bar{D}_j) \cdot E^+| \leq ml \cdot \gamma(G').
\]

Case 2: \( j = tm + s \) for some \( 1 \leq t \leq n - 1, 1 \leq s \leq m - 1 \).

In this case \( D_j = V_1 \cup V_2 \cup \cdots \cup V_t \cup \{ v'_t+1, v'_t+2, \ldots, v'_{t+s} \} \). By the third property of \( G \), for any \( i \neq qt + 1 \), each of the vertices in \( V_{t+i} \) has the same number of neighbors in \( V_t \). In particular, each of the vertices of \( V_{t+i} \) has the same number \( N_1 \) of neighbors in \( \bigcup_{i=1}^{n} V_t \) and the same number \( N_2 \) of neighbors in \( \bigcup_{i=1}^{n} V_t \). Consequently

\[
F(r) = F(tm) + (r - tm)(N_1 - N_t), \quad tm \leq r \leq (t + 1)m,
\]

so that \( F(j) \leq \max\{F(tm), F(tm + m)\} \). By Case 1 we have \( F(j) \leq ml \cdot \gamma(G') \). This completes the proof. \( \square \)

4. Explicit constructions attaining the minimal cut

We now start a detailed discussion of the construction of an enumeration attaining the minimal cut for the family of Cayley graphs.

4.1. Solution generation

Denote by \( T \) the circle group \( \mathbb{R}/\mathbb{Z} \). This group will be identified with the interval \([0, 1)\) when convenient. Let

\[
\|x\| = \min\{x, 1 - x\}, \quad x \in \mathbb{T}.
\]

The function \( \| \cdot \| \) gives rise to a metric \( d \) on \( \mathbb{T} \), defined by

\[
d(u, v) = \|u - v\|, \quad u, v \in \mathbb{T}.
\]

Note that, when identifying \( \mathbb{T} \) with \([0, 1)\), the open ball of radius \( r \) around \( 1 \) is the set \((0, r) \cup (1 - r, 1) \). The sum metric on \( \mathbb{T}^k \) will also be denoted by \( d \), that is

\[
d(u, v) = \|u_1 - v_1\| + \|u_2 - v_2\| + \cdots + \|u_k - v_k\|,
\]

for \( u = (u_1, u_2, \ldots, u_k), v = (v_1, v_2, \ldots, v_k) \in \mathbb{T}^k \). Also note that, due to the identification of \( \mathbb{T}^2 \) with \([0, 1)^2 \), we may refer to the area of a subset of \( \mathbb{T}^2 \).

Proposition 4.1. Let \( F \subseteq \mathbb{T}^2 \) with \( 2 \leq |F| < \infty \). Then there exist two distinct points \( u, v \in F \) such that \( d(u, v) \leq \sqrt{2/|F|} \).

Proof. Let \( f = |F| \). For any point \( v \in \mathbb{T}^2 \), denote by \( B_{\varepsilon}(v) \) the closed ball of radius \( \varepsilon \) around \( v \):

\[
B_{\varepsilon}(v) = \{ w \in \mathbb{T}^2 : d(v, w) \leq \varepsilon \}.
\]

It is easy to see that, for \( \varepsilon \leq \frac{1}{\sqrt{2f}} \), the area of \( B_{\varepsilon}(v) \) is \( 2\varepsilon^2 \). Choose \( \varepsilon = \sqrt{1/2f} \). The area of \( B_{\sqrt{1/2f}}(v) \) is \( 1/f \) for all \( v \in \mathbb{T}^2 \). Since the area of \( \mathbb{T}^2 \) is \( 1 \), there exist \( u, v \in F \) such that \( B_{\sqrt{1/2f}}(u) \cap B_{\sqrt{1/2f}}(v) \neq \emptyset \) (see Fig. 2). Let \( w \) belong to this intersection. Then \( d(u, w) \leq \sqrt{1/2f} \) and \( d(v, w) \leq \sqrt{1/2f} \).

Consequently,

\[
d(u, v) \leq d(u, w) + d(w, v) \leq \sqrt{1/2f} + \sqrt{1/2f} = \sqrt{2f} = \sqrt{2/|F|}.
\]

\( \square \)

Corollary 4.2. For any abelian group \( H \) and \( a, b \in H \),

\[
\psi_2(a, b) \leq \sqrt{2h}.
\]
Proof. Let \( H = \{z_0, z_1, \ldots, z_{h-1}\} \). Denote

\[
F_i = \{(v_1/h, v_2/h) : 0 \leq v_1, v_2 \leq h - 1, a^{v_1}b^{v_2} = z_i\} \subseteq T^2, \quad i = 0, 1, \ldots, h - 1.
\]

Let

\[
F = F_0 \cup F_1 \cup \cdots \cup F_{h-1}.
\]

Then \( |F| = h^2 \), and hence there exists \( j \in \{0, 1, \ldots, h - 1\} \) such that \( |F_j| \geq h \). By the previous lemma there exist distinct points \( x = (u_1/h, u_2/h) \) and \( y = (v_1/h, v_2/h) \) in \( F_j \) such that \( d(x, y) \leq \sqrt{2/h} \). Consequently,

\[
a^{u_1-v_1}b^{u_2-v_2} = a^{u_1}b^{u_2}/a^{v_1}b^{v_2} = z_j/z_j = 1,
\]

so that

\[
\psi_2(a, b) \leq h \cdot \|u_1/h - v_1/h\| + h \cdot \|u_2/h - v_2/h\|
\]

\[
\leq h \cdot d(x, y) \leq h \sqrt{2/h} = \sqrt{2h}.
\]

\( \square \)

Remark 4.3. If the set \( \{a, b\} \) generates \( H \), then all sets \( F_j \) in the last proof are of the same size. Denote by \( f : F \rightarrow H \) the homomorphism defined by \( f((v_1/h, v_2/h)) = a^{v_1}b^{v_2} \). Then \( F_0 = \ker(f) \), and by Lagrange’s Theorem we obtain \( |F_j| = |\ker(f)| \) for \( j = 0, 1, \ldots, h - 1 \).

Corollary 4.4. For any abelian group \( H \) with \( \sqrt{2h} \in \mathbb{Z} \) and \( a, b \in H \), we have

\[
\min\{\alpha + 1, \beta + 1, \psi_2^+(a, b)\} \leq \sqrt{2h}.
\]

Proof. If \( \psi_2^+(a, b) = \psi_2(a, b) \), then by Corollary 4.2 we are done. Thus, without loss of generality we may assume that

\[
\psi_2^+(a, b) > \psi_2(a, b) = \alpha > \sqrt{2h} - 1.
\]

By Corollary 4.2 we have \( \alpha \leq \sqrt{2h} \). Since \( \sqrt{2h} \) is an integer, this implies \( \alpha = \sqrt{2h} \). Hence, \( h = \alpha^2/2 \). Let \( A \) be the subgroup generated by \( a \). The quotient group \( H/A \) is of order \( \alpha/2 \). Therefore, there exists an \( l \in \{1, \ldots, \alpha/2\} \) such that \( b^l \in A \), say \( b^l = a^m \) with \( -\alpha/2 < m \leq \alpha/2 \). Hence

\[
\psi_2^+(a, b) \leq |l| + |m| \leq \alpha/2 + \alpha/2 = \alpha.
\]

This contradicts (2) and thereby proves the corollary. \( \square \)

Corollary 4.5. For any abelian group \( H \) with \( h \) odd and \( a, b \in H \), we have

\[
\min\{\alpha + 1, \beta + 1, \psi_2^+(a, b)\} \leq \sqrt{2h}.
\]
**Proof.** As in the proof of the previous corollary we may assume that (2) holds. By Corollary 4.2 we have $\alpha \leq \sqrt{2h}$. Since $\alpha$ divides $h$, we may write $h = k\alpha$, where $k$ is odd. Therefore, $\sqrt{2k\alpha} - 1 < \alpha \leq \sqrt{2k\alpha}$. This easily implies $2k = \alpha + 1$. Hence $h = \alpha(\alpha + 1)/2$. Let $A$ be the subgroup generated by $a$. The quotient group $H/A$ is of order $k = (\alpha + 1)/2$. Therefore, there exists an $l \in \{1, \ldots, (\alpha + 1)/2\}$ such that $b^l \in A$, say $b^l = a^m$ with $-(\alpha - 1)/2 \leq m \leq (\alpha - 1)/2$. Hence

$$\psi_2^+(a, b) \leq |l| + |m| \leq (\alpha + 1)/2 + (\alpha - 1)/2 = \alpha,$$

which leads to a contradiction. □

**Remark 4.6.** We may write (1) equivalently as $\min\{\alpha, \beta, \psi_2^+(a, b)\} \leq \sqrt{2h}$. Thus, Corollaries 4.4 and 4.5 improve Corollary 4.2 in that they allow the replacement of $\alpha$ and $\beta$ by $\alpha + 1$ and $\beta + 1$, respectively.

The following example shows that the condition $\sqrt{2h} \in \mathbb{Z}$ or $h \in 2\mathbb{Z} + 1$ in Corollaries 4.4 and 4.5 is not redundant.

**Example 4.7.** Let $H = C_{12}$ and $a = 3$, $b = 2$. It is easy to see that

1. The orders are $\alpha = 4$ and $\beta = 6$.
2. $a^2 = b^3$, so that $\psi_2^+(a, b) = 5$.
3. $\sqrt{2h} = \sqrt{24} \notin \mathbb{Z}$ and $h$ is even.

Hence,

$$\min\{\alpha + 1, \beta + 1, \psi_2^+(a, b)\} = \min\{5, 7, 5\} = 5 > \sqrt{24} = \sqrt{2h}.$$

**Proposition 4.8.** Let $H$ be an abelian group and $a, b \in H$ such that

1. The set $\{a, b\}$ generates $H$.
2. $a^n = b^n$, where $\gcd(m, n) = 1$ and $m, n \neq 0$.

Then $H$ is cyclic. Moreover, there exists a generator $g$ of $H$ such that $a = g^n$ and $b = g^m$.

**Proof.** Since $\gcd(m, n) = 1$ there exist $r_1, r_2 \in \mathbb{Z}$ such that $r_1m + r_2n = 1$. Let $g = a^{r_2}b^{r_1}$. Then

$$g^n = (a^{r_2}b^{r_1})^n = a^{nr_2}b^{nr_1} = a^{nr_2}a^m = a^{nr_2+mr_1} = a^1 = a$$

and similarly $g^m = b$. □

The following proposition (unlike most of our other results) applies to any Cayley graph, i.e., $H$ may be non-abelian and the set of generators $S$ may be of arbitrary size.

**Proposition 4.9.** For every $G = \text{Cay}(H, S)$

$$\gamma(G) = \gamma(G_c),$$

where $G_c = \text{Cay}(\langle S \rangle, S)$.

**Proof.** Since all connected components of $G$ are isomorphic to $G_c = \text{Cay}(\langle a, b \rangle, \langle a^{\pm 1}, b^{\pm 1} \rangle)$, by Lemma 3.1 we have $\gamma(G) = \gamma(G_c)$. □

**Proposition 4.10.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$, then

$$\gamma(G) \leq 2\psi_2^+(a, b).$$

**Proof.** By Proposition 4.9 we may assume that $G = G_c$ is connected, i.e., $\langle a, b \rangle = H$.

Let $\psi_2^+(a, b) = |w_1| + |w_2|$, where $a^{w_1}b^{w_2} = 1$. 
Fig. 3. The ordering of $G$ in Case 1, where $|w_1| = 3$ and $|w_2| = 2$.

Case 1: $\gcd(w_1, w_2) = 1$.

By Proposition 4.8, $H$ is cyclic and there exists a generator $g$ of $H$ such that $a = g^{w_2}$ and $b = g^{w_1}$. We claim that the ordering

$$1, g, g^2, \ldots, g^{h-1},$$

satisfies the requirement, i.e.,

$$c(D_k, \overline{D_k}) \leq 2\psi_2(a, b), \quad 1 \leq k \leq h - 1,$$

where $D_k = \{1, g, g^2, \ldots, g^k\}$ (see Fig. 3).

Note that there is an $a \pm 1$-edge between $g^i$ and $g^j$ if and only if $i - j \equiv \pm w_2 \pmod{h}$ and a $b \pm 1$-edge if and only if $i - j \equiv \pm w_1 \pmod{h}$. Therefore, if $w_2 > 0$, only the last $|w_2|$ vertices $\{g^{k-|w_2|}, g^{k-|w_2|+1}, \ldots, g^{k-1}\}$ of $D_k$ have an $a$-cut edge in $e(D_k, \overline{D_k})$. Similarly, if $w_2 < 0$, only the first $|w_2|$ vertices $\{1, g, g^2, \ldots, g^{w_2-1}\}$ of $D_k$ have an $a$-cut edge in $e(D_k, \overline{D_k})$. Thus,

$$|e_a(D_k, \overline{D_k})| \leq |w_2|$$

and analogously

$$|e_b(D_k, \overline{D_k})| \leq |w_1|.$$

By Lemmas 5.4 and 5.5,

$$c(D_k, \overline{D_k}) = 2|e_a(D_k, \overline{D_k})| + 2|e_b(D_k, \overline{D_k})|$$

$$\leq 2(|w_1| + |w_2|) = 2\psi_2(a, b).$$

Case 2: $\gcd(w_1, w_2) > 1$.

Denote $t = \gcd(w_1, w_2)$ and $w_i' = w_i/t$, $i = 1, 2$. Then,

$$(a^{w_i'}b^{w_2'})^t = a^{tw_i'}b^{tw_2'} = a^{w_1}b^{w_2} = 1.$$

Let $f = a^{w_i'}b^{w_2'}$. The minimality property of $|w_1| + |w_2|$ ensures that $f$ is of order $t$. Denote by $F$ the subgroup generated by $f$. Let $H' = H/F$. Denote by $h' = h/t$ the order of $H'$.

First suppose that $Fa \neq Fb^{\pm 1}$. Let $G' = \text{Cay}(H', \{Fa^{\pm 1}, Fb^{\pm 1}\})$. Since $\{a, b\}$ generates $H$, the set $\{Fa, Fb\}$ generates $H'$. Now in $H'$

$$(Fa)^{w_i'}(Fb)^{w_2'} = F a^{w_i'}b^{w_2'} = Ff = 1$$

and we easily obtain $\psi_2^t(Fa, Fb) = |w_i'| + |w_2'|$. The graph $G'$ satisfies the condition in Case 1, so that we can find a generator $Fg$ of $H'$ such that for the ordering

$$F, Fg, Fg^2, \ldots, Fg^{h'-1}$$
we have $c(D'_k, \bar{D}'_k) \leq 2(|w'_1| + |w'_2|)$ for $1 \leq k \leq h' - 1$, where $D'_k = \{F, F_g, F_{g^2}, \ldots, F_{g^{k-1}}\}$ (see Fig. 4). Denote by $G_1, G_2, \ldots, G_{h'}$ the subgraphs of $G$ induced by $F, F_g, \ldots, F_{g^{h'-1}}$, respectively. The graphs $G, G', G_1, G_2, \ldots, G_{h'}$ satisfy the conditions of Lemma 3.4 with $n = h', m = t$, and $l = 1$, and therefore,

\[ \gamma(G) \leq kl \cdot \gamma(G') + \max_{1 \leq i \leq h'} \gamma(G_i). \]

Consequently,

\[ \gamma(G) \leq t \cdot \gamma(G') + 0 \leq 2t \psi_2^+(Fa, Fb) \]

\[ = 2t(|w'_1| + |w'_2|) = 2(|w_1| + |w_2|) = 2\psi_2^+(a, b). \]

Now suppose $Fa = Fb$ (or $Fa = Fb^{-1}$). Then $G' = \text{Cay}(H', \{Fa^{\pm 1}\})$ is a simple cycle. We easily obtain that the ordering $F, Fa, Fa^2, \ldots, Fa^{h'-1}$ satisfies $c(D'_k, \bar{D}'_k) \leq 2$ for $0 \leq k \leq h' - 2$, where $D'_k = \{F, Fa, Fa^2, \ldots, Fa^k\}$. Similarly to the case $Fa \neq Fb^{\pm 1}$, denote by $G_1, G_2, \ldots, G_{h'}$ the subgraphs of $G$ induced by $F, Fa, \ldots, Fa^{h'-1}$, respectively. The graphs $G, G', G_1, G_2, \ldots, G_{h'}$ satisfy the conditions of Lemma 3.4 with $n = h', m = t$, and $l = 2$, and therefore:

\[ \gamma(G) \leq kl \cdot \gamma(G') + \max_{1 \leq i \leq h'} \gamma(G_i). \]

Consequently,

\[ \gamma(G) \leq 2t \cdot \gamma(G') + 0 = 4t \]

\[ \leq 2t(|w'_1| + |w'_2|) = 2(|w_1| + |w_2|) \]

\[ = 2\psi_2^+(a, b). \]
Proposition 4.11. If \( G = \text{Cay}(H, \{a^{\pm1}, b^{\pm1}\}) \), then \( \gamma(G) \leq 2x + 2 \).

**Proof.** Similarly to the proof of the previous proposition, we may assume that \( G \) is connected.

Denote by \( A \) the subgroup of \( H \) generated by \( a \). Then \(|A| = x \). Let \( H' = H/A, h' = h/x \), and \( G' = \text{Cay}(H', \{Ab^{\pm1}\}) \). Since \( Ab \) generates \( H' \) and the arrangement \( A, Ab, Ab^2, \ldots, Ab^{h'-1} \) is optimal for \( G' \), we have \( \gamma(G') = 2 \) (see Fig. 5). Denote by \( G_1, G_2, \ldots, G_{h'} \) the subgraphs of \( G \) induced by \( A, Ab, \ldots, Ab^{h'-1} \), respectively. The graphs \( G, G', G_1, G_2, \ldots, G_{h'} \) satisfy the conditions of Lemma 3.4 with \( n = h' \), \( m = x \), and \( l = 1 \) and therefore,

\[
\gamma(G) \leq ml \cdot \gamma(G') + \max_{1 \leq i \leq h'} \gamma(G_i) = x \cdot \gamma(G') + 2.
\]

Since the \( G_i \)'s are simple cycles,

\[
\gamma(G) \leq 2x + 2. \quad \square
\]

Propositions 4.10 and 4.11 jointly yield:

**Corollary 4.12.** If \( G = \text{Cay}(H, \{a^{\pm1}, b^{\pm1}\}) \) with \( H \) abelian, where \( a^2 \neq 1 \) and \( b^2 \neq 1 \), then

\[
\gamma(G) \leq 2 \min\{x + 1, \beta + 1, \psi_2^+(a, b)\}.
\]

**Remark 4.13.** For any \( G = \text{Cay}(H, \{a^{\pm1}, b^{\pm1}\}) \) and for any \( 1 \leq i \leq h - 1 \) we have:

\[
\delta_i(G) \leq 2(x + \eta(i, x)).
\]

If \( x \) does not divide \( i \) we obtain this result applying Proposition 4.11 and Lemma 3.2. Hence assume that \( x \) divides \( i \). Let \( B = \bigcup_{j=0}^{i/x} Ab^j \), where \( A \) is the subgroup of \( H \) generated by \( a \). It is easy to see that \(|B| = i \) and \( c(B, B') \leq 2x \), and consequently \( \delta_i(G) \leq 2x \).

**Proposition 4.14.** For any \( G = \text{Cay}(H, \{a^{\pm1}, b^{\pm1}\}) \) we have

\[
\delta_i(G) \leq 2[2\sqrt{i}], \quad 1 \leq i \leq h - 1.
\]

**Proof.** Let \( x = \lceil \sqrt{i} \rceil, y = \lfloor i/x \rfloor, z = xy - i \). It is easy to see that \( x \geq y \geq z \) and \( x + y = 2\sqrt{i} \).

If \( x \geq z \), then by Lemma 3.2 and Proposition 4.11 we obtain:

\[
\delta_i(G) \leq \gamma(G) \leq 2x + 2 \leq 2\sqrt{i} + 2 \leq 2[2\sqrt{i}]\]

Fig. 5. The ordering of \( G \).
Let $B = b^j P$, $0 \leq j \leq y - 1$.

Case 1: The sets $P_0, P_1, \ldots, P_{y-1}$ are disjoint.

Let $A = \bigcup_{j=0}^{y-1} P_j$. Obviously, $|A| = xy$. If $xy > i$ let $z = xy - i$ and let $B = A \setminus \{1, a, a^2, \ldots, a^{z-1}\}$, so that $|B| = i$.

The number of $a$-paths in $B$ does not exceed $y$, and hence $e_a(B, \bar{B}) \leq y$. Similarly $e_a(B, \bar{B}) \leq x$. By Lemmas 5.4 and 5.5,

$$
\delta_i(G) = \delta_i|B|(G) \leq c(B, \bar{B}) \leq 2|e_a(B, \bar{B})| + 2|e_a(B, \bar{B})| \leq 2(x + y) \leq 2[2\sqrt{i}].
$$

Case 2: The sets $P_0, P_1, \ldots, P_{y-1}$ are not disjoint.

In this case there exist $0 \leq k \neq l \leq y - 1$ such that $P_k \cap P_l \neq \emptyset$. Choose $c \in P_k \cap P_l$. Since $c \in P_k$ we have $c = b^k a^{m_1}$ where $0 \leq m_1 < x$, and similarly $c = b^l a^{m_2}$, where $0 \leq m_2 < x$. It follows that $a^{m_1-m_2} b^{k-l} = 1$. Since $x < \beta$ and $y < \beta$, $\psi_2(a, b) \leq x + y - 2$ and by Lemma 3.2 and Proposition 4.10 we obtain

$$
\delta_i(G) \leq \gamma(G) \leq 2\psi_2(a, b) \leq 2(x + y) - 2 < 2[2\sqrt{i}]. \quad \Box
$$

Lemma 3.2, Corollary 4.12 with Remark 4.13 and Proposition 4.14 jointly yield:

**Corollary 4.15.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ with $H$ abelian, where $a^2 \neq 1$ and $b^2 \neq 1$, then

$$
\delta_i(G) \leq 2 \min\{x + \eta(i, x), \beta + \eta(i, \beta), \psi_2(a, b), [2\sqrt{i}], [2\sqrt{h-i}]\}.
$$

5. Proof of optimality

5.1. Connectivity of the optimal solution

The following proposition (unlike most of our other results) applies to Cayley graphs with a set of generators $S$ of arbitrary size.

**Proposition 5.1.** Let $G = \text{Cay}(H, S)$ with $H$ abelian and $A \subset H$ with $\delta_{|A|}(G) = c(A, \bar{A})$. If $G$ is connected, then the subgraph of $G$ induced by $A$ is connected as well.

**Proof.** Suppose $A$ is disconnected. Write $A = A' \cup A''$, where $A', \ A'' \neq \emptyset$, $A' \cap A'' = \emptyset$ and no vertex in $A'$ is adjacent to a vertex in $A''$. Since $G$ is connected, there is at least one path between $A'$ and $A''$. Denote the length of the shortest of these paths by $k$. By the choice of $A'$ and $A''$ we have $k \geq 2$. Take $v' \in A'$, $v'' \in A''$, and $s_1, \ldots, s_k \in S$ such that $v's_1 \ldots s_k = v''$. Let

$$
A' = \{v'_1, v'_2, \ldots, v'_{|A'|}\},
$$

$$
A'' = \{v''_1, v''_2, \ldots, v''_{|A''|}\}.
$$

Let $B' = \{v'_1 s', v'_2 s', \ldots, v'_{|A'|} s'\}$ and $B'' = A''$, where $s' = s_1 \ldots s_k$. Finally, let $B = B' \cup B''$. Since the distance of any vertex in $B'$ from $A'$ is at most $k - 1$, the sets $B'$ and $B''$ are disjoint. In particular

$$
|B| = |B'| + |B''| = |A'| + |A''| = |A|,
$$

so that $\delta_{|A|}(G) = \delta_{|B|}(G)$.

Now we wish to show that $\delta_{|B|}(G) < c(A, \bar{A})$, thereby arriving at a contradiction. In fact, since the vertices $v's_1 \ldots s_{k-1} \in B'$ and $v'' = v's_1 \ldots s_k \in B''$ are adjacent, there is at least one edge between $B'$ and $B''$. Also note
that the graphs induced by $A'$ and $B'$ are isomorphic, and in particular $|e(G(A'))| = |e(G(B'))|$, where $e(G(A'))$ and $e(G(B'))$ are the sets of edges of these graphs. Consequently,

$$\delta_{|B|}(G) \leq c(B, \tilde{B})$$

$$= |B| \cdot |S| - 2|e(G(B))|$$

$$= |A| \cdot |S| - 2|e(G(A'))| - 2c(B', B'')$$

$$= c(A, \tilde{A}) - 2c(B', B'')$$

$$< c(A, \tilde{A}).$$

The contradiction proves the proposition. □

**Example 5.2.** The following example shows that the condition that $G$ is a Cayley graph is not redundant. Let $G' = (V', E')$ be a clique of order $n \geq 3$. Choose $v'_1, v'_2 \in V'$ and pick $v_1, v_2 / \in V$. Let

$$G = (V' \cup \{v_1, v_2\}, E' \cup \{(v_1, v'_1), (v_2, v'_2)\}).$$

It is easy to see that $c(\{v_1, v_2\}, V') = \delta_2(G) = 2$, but the subgraph induced by $\{v_1, v_2\}$ is disconnected.

**Definitions 5.3.** For $A, B \subseteq H$ and $s \in S$, the set $e_s(A, B)$ of $s$-cut edges between $A$ and $B$ induced by $s$ is given by

$$e_s(A, B) = \{(u, v) \in E | u \in A, v \in B, us = v\}.$$ 

Note that here we view the edges of $G$ as directed. For example, if $A = \{u\}$ and $B = \{us\}$, then $e_s(A, B) = \{(u, v)\}$ while $e_s(B, A) = \emptyset$.

The following two lemmas are trivial.

**Lemma 5.4.** For $s \in S$ and $A \subseteq H$,

$$|e_s(A, \tilde{A})| = |e_{s^{-1}}(A, \tilde{A})|.$$ 

**Lemma 5.5.** For $A, B \subseteq H$,

$$e(A, B) = \bigcup_{s \in S} e_s(A, B).$$

The following binary relation on the set of subsets of $H$ will play an important role in the sequel.

**Definitions 5.6.** Let $G = \text{Cay}(H, S)$, $A', A'' \subseteq H$ (not necessarily disjoint), and $s \in H$. $A''$ is an $s$-successor of $A'$ (and $A'$ is an $s$-predecessor of $A''$), and we denote $A' \equiv_s A''$, if there exist $u \in A'$ and $v \in A''$ such that $us = v$.

**Definitions 5.7.** Let $s \in S$ and $A \subseteq H$. An $s$-component $P^s(A)$ of $A$ is a connected component of $A$ in the graph $G^s = \text{Cay}(H, \{s, s^{-1}\})$. An $s$-component of $A$ is an $s$-cycle if it is a coset of the subgroup generated by $s$, and an $s$-path otherwise.

Note that any $s$-component $P^s(A)$ of $A$ is of the form:

$$P^s(A) = \{x, xs, \ldots, xs^k\}.$$ 

$P^s(A)$ is an $s$-cycle if $k$ is the order of $s$, and an $s$-path if it is smaller.
Obviously, in general, the edges between vertices of $G^s$ in an $s$-component $P^s_i(A)$ are between $xs^j$ and $xs^{j+1}$. There is an edge between $xs^i(P^s(A))^{-1}$ and $x$ if and only if $P^s(A)$ is a $s$-cycle. Therefore, if $s$ is a generator of $H$, then the number of $s$-cut edges between $A$ and $\bar{A}$ is equal to the number of $s$-paths of $A$.

Denoting the $s$-components of $A$ by $P^s_1(A)$, $P^s_2(A)$, $\ldots$, $P^s_k(A)$, we have

$$A = \bigcup_{i=1}^{k} P^s_i(A),$$

where the union is disjoint.

**Lemma 5.8.** Let $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ and $A \subset H$. If there exist two distinct $a$-components $P^a_1(A)$ and $P^a_2(A)$ of $A$ such that $P^a_1(A) \supseteq bP^a_2(A)$ and $P^a_1(A)b \neq P^a_2(A)$, then there exists a vertex $x \in \bar{A}$ such that

$$c(A \cup \{x\}, \bar{A}\setminus\{x\}) \leq c(A, \bar{A}).$$

**Proof.** Choose $P^a_1(A)$ and $P^a_2(A)$ to satisfy the requirements. Denote by $l_1$ and $l_2$ the lengths (i.e., the number of vertices) of $P^a_1(A)$ and $P^a_2(A)$, respectively. Write $P^a_1(A) = \{x_1, x_1a, \ldots, x_1a^{l_1-1}\}$ and $P^a_2(A) = \{x_2, x_2a, \ldots, x_2a^{l_2-1}\}$. We may assume without loss of generality (by interchanging $P^a_1(A)$ and $P^a_2(A)$ or $a$ and $a^{-1}$) that an $i \in [0, \ldots, l_1 - 1]$ exists, such that $x_1a^ib \notin P^a_2(A)$ and $x_1a^{i+1}b \in P^a_2(A)$. Choose $x = x_1a^ib$. Then $x$ has at least two adjacent vertices in $A$, namely $x_1a^i \in P^a_2(A)$ and $x_1a^{i+1}b \in P^a_2(A)$ (see Fig. 6). Consequently,

$$c(A \cup \{x\}, \bar{A}\setminus\{x\}) = c(A, \bar{A}) - |e(A, \{x\})| + |e(\{x\}, \bar{A}\setminus\{x\})|$$

$$\leq c(A, \bar{A}) - 2 + 2$$

$$= c(A, \bar{A}).$$

□

**Corollary 5.9.** Let $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ and $B \subset H$. Let $A$ be any connected component of $B$. If

$$c(B, \bar{B}) < c(B \cup \{x\}, \bar{B}\setminus\{x\}), \quad x \in \bar{B},$$

then all $a$-components of $A$ have the same length and all $b$-components of $A$ have the same length.

**Proof.** Suppose, say, that there exist two $a$-components $P^a_1(A)$ and $P^a_2(A)$ with different lengths. Since $A$ is connected we can find a chain of $a$-components leading from $P^a_1(A)$ to $P^a_2(A)$

$$P^a_1(A) = P^a_{10}(A) \supseteq bP^a_{11}(A) \supseteq b \cdot \supseteq \cdots \supseteq bP^a_{1k}(A) = P^a_2(A)$$

(or a chain leading from $P^a_2(A)$ to $P^a_1(A)$). Since $|P^a_2(A)| \neq |P^a_1(A)|$, we can find two consecutive $a$-components of distinct length in the chain. Hence, we may assume that there exists some $0 \leq i \leq k - 1$ such that $P^a_{1i}(A)b \neq P^a_{1,i+1}(A)$. Now every $a$-path of $A$ is also an $a$-path of $B$. Applying Lemma 5.8 we obtain an $x \in \bar{B}$ for which

$$c(B, \bar{B}) \leq c(B \cup \{x\}, \bar{B}\setminus\{x\}).$$

The contradiction proves the corollary. □
Remark 5.10. Under the conditions of Corollary 5.9 it is actually easy to show, moreover, that \( A \) is a “rectangle”-like set, namely
\[
A = \{ x^{a_i}b^j : 0 \leq i < k, 0 \leq j < l \}
\]
for some \( x \in H \) and \( k \leq \alpha, l \leq \beta \), where \( x^{a_i}b^j = x^{a_{i+1}}b^{j+1} \) if and only if \( i_1 = i_2 \) and \( j_1 = j_2 \) (see Fig. 7).

Proposition 5.11. Let \( G = \text{Cay}(H, \{a^\pm 1, b^\pm 1\}) \) and \( A \) be a nonempty connected subset of \( H \), which does not contain \( a \)- and \( b \)-cycles. If
\[
c(A, \bar{A}) < c(A \cup \{x\}, \bar{A}\setminus\{x\}), \quad x \in \bar{A},
\]
then
\[
c(A, \bar{A}) \geq 4\sqrt{|A|}.
\]
Moreover, all \( a \)-components of \( A \) have the same length, say \( k \), and \( c(A, \bar{A}) \geq 2(k + L) \), where \( L \) is the number of \( a \)-components of \( A \).

Proof. Since \( A \) does not contain \( a \)- and \( b \)-cycles, all \( a \)-components of \( A \) are \( a \)-paths and the same holds for \( b \)-components. Applying the previous corollary with \( B = A \) we obtain that all \( a \)-paths of \( A \) have the same length, and the same holds for \( b \)-paths. Denote those lengths by \( k \) and \( l \), respectively. Let \( P^a(A) = \{y, ya, \ldots, y^{k-1}\} \) be any \( a \)-path of \( A \). Note that \( y^{a^k} \in \bar{A} \).

Let \( L \) be the number of \( a \)-paths of \( A \). In view of the above, \( e_a(A, \bar{A}) \) consists of \( L \) edges. Analogously, there are \( K \) edges in \( e_b(A, \bar{A}) \), where \( K \) is the number of \( b \)-paths of \( A \). Obviously, \( kL = Kl = |A| \). Applying Lemmas 5.5 and 5.4 we obtain
\[
|e(A, \bar{A})| = \left| \bigcup_{s \in S} e_s(A, \bar{A}) \right|
= |e_a(A, \bar{A})| + |e_{a^{-1}}(A, \bar{A})| + |e_b(A, \bar{A})| + |e_{b^{-1}}(A, \bar{A})|
= 2|e_a(A, \bar{A})| + 2|e_b(A, \bar{A})|
= 2(K + L).
\]

Let us show now that \( K \geq k \) (and similarly \( L \geq l \)). In fact, suppose \( K < k \). Since there are exactly \( k \) vertices in \( P^a(A) \), there exists a \( b \)-path \( P^b(A) \) of \( A \) containing at least two vertices of \( P^a(A) \). Write \( P^b(A) = \{z, zb, \ldots, z^{b^{l-1}}\} \). Let \( zb^j \) and \( zb^j \) be two vertices belonging to both \( P^a(A) \) and \( P^b(A) \), where \( 0 \leq i < j \leq l-1 \). If \( zb^{j-1} \) and \( zb^{j-1} \) do not belong to...
Proposition 5.12. Let \( G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\}) \) and \( B \) a subset of \( H \), which does not contain \( a \)- or \( b \)-cycles. If

\[
c(B, \bar{B}) < c(B \cup \{x\}, \bar{B} \setminus \{x\}), \quad x \in \bar{B},
\]

then

\[
c(B, \bar{B}) \geq 4\sqrt{|B|}.
\]

Proof. Denote by \( A_1, A_2, \ldots, A_t \) all connected components of \( B \). Obviously, \( c(B, \bar{B}) = \sum_{i=1}^t c(A_i, \bar{A}_i) \). In view of Proposition 5.11 we have \( c(A_i, \bar{A}_i) \geq 4\sqrt{|A_i|} \) for each \( i \). Therefore,

\[
c(B, \bar{B}) = \sum_{i=1}^t c(A_i, \bar{A}_i)
\]

\[
\geq 4 \sum_{i=1}^t \sqrt{|A_i|} \geq 4 \sqrt{\sum_{i=1}^t |A_i|} = 4\sqrt{|B|}. \quad \Box
\]

Proposition 5.13. Let \( G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\}) \) be connected. Let \( \emptyset \neq A \subset H \) be such that \( \delta_{|A|}(G) = c(A, \bar{A}) \). If there exists an \( a \)-path \( P^a(A) \) of \( A \) which has no \( b \)-successor in \( A \) (see Fig. 8), then

\[
c(A, \bar{A}) \geq 2 \min\{x + 1, \beta + 1, 2\sqrt{|A|}\}.
\]

Proof. We distinguish between two cases.
Let $B$ does not increase, we obtain a sequence of sets $B_1 \supseteq B_2 \supseteq \ldots \supseteq B_m$ such that $|B_m| = |B|$. Consequently,
\[
\begin{align*}
|B| &= |B_1| - |B_2| - \ldots - |B_m| \\
&\leq |B_1| - |B_2| - \ldots - |B_m| = |B_1| - |B_2| - \ldots - |B_m|.
\end{align*}
\]

Case 2: $|P^a(A)| < 2\sqrt{|A|} - 1$. Let $B = A \setminus P^a(A)$. Since there are no $b$-cut edges between $P^a(A)$ and $B$, we have $|e(P^a(A), B)| = |e_{b^{-1}}(P^a(A), B)| \leq |P^a(A)|$.

Also
\[
|e(P^a(A), \tilde{A})| \geq |e_{a}(P^a(A), \tilde{A})| + |e_{a^{-1}}(P^a(A), \tilde{A})| + |e_{b}(P^a(A), \tilde{A})| \\
\geq 2 + |P^a(A)|.
\]

Consequently,
\[
c(B, \tilde{B}) = c(A, \tilde{A}) - |e(P^a(A), \tilde{A})| + |e(P^a(A), B)| \\
\leq c(A, \tilde{A}) - 2 - |P^a(A)| + |P^a(A)| = c(A, \tilde{A}) - 2.
\]

Suppose there exists a vertex $x \in \tilde{B}$ such that
\[
c(B \cup \{x\}, \tilde{B} \setminus \{x\}) < c(B, \tilde{B}).
\]

Let $B_1 = B \cup \{x\}$. Continuing this process of augmentation of $B$ as long as the size of the cut of the set and its complement does not increase, we obtain a sequence of sets $B = B_0, B_1, \ldots, B_m$ and points $x = x_0, x_1, \ldots, x_m$ with

\[
B_{i+1} = B_i \cup \{x_i\}, \quad 0 \leq i \leq m - 1,
\]

such that
\[
c(B_{i+1}, \tilde{B}_{i+1}) \leq c(B_i, \tilde{B}_i).
\]

At the end of the process we have:
\[
c(B_m, \tilde{B}_m) < c(B_m \cup \{x\}, \tilde{B}_m \setminus \{x\}), \quad x \in \tilde{B}_m.
\]

If $m \geq j = |P^a(A)|$, then by (3)
\[
c(A, \tilde{A}) \geq c(B_0, \tilde{B}_0) + 2 \geq c(B_j, \tilde{B}_j) + 2 \geq c(B_j, \tilde{B}_j),
\]

which, as $|B_j| = |B| + |P^a(A)| = |A|$, contradicts the minimality property of $c(A, \tilde{A})$. Hence $m < |P^a(A)|$.

First suppose that $B_m$ contains an $a$-cycle. Let $A_m$ be a connected component of $B_m$ which contains such a cycle. By Corollary 5.9 all $a$-components of $A_m$ have the same length, and in particular all of them are $a$-cycles. If all these cycles have a $b$-successor in $A_m$, then $A_m$ is a union of a chain of $a$-cycles $P^a_1(A_m), P^a_2(A_m), \ldots, P^a_k(A_m) = P^a_1(A_m)$, such that
\[
P^a_{i+1}(A_m) = bP^a_i(A_m), \quad 1 \leq i \leq k - 1.
\]

Since $G$ is connected, this means that $A_m = H$, which is a contradiction. Thus, $A_m$ contains at least one $a$-cycle with no $b$-successor in $A_m$, and consequently no $b$-successor in $B_m$. Therefore, $c(B_m, \tilde{B}_m) \geq 2|e_b(B, \tilde{B}_m)| \geq 2\alpha$. By (3)
\[
c(A, \tilde{A}) \geq c(B, \tilde{B}) + 2 \geq c(B_m, \tilde{B}_m) + 2 \\
\geq 2\alpha + 2 \geq 2\min\{\alpha + 1, \beta + 1, 2\sqrt{|A|}\}.
\]
Similarly, if \( B_m \) contains a \( b \)-cycle, then
\[
c(A, \bar{A}) \geq 2\beta + 2 \geq 2 \min\{x + 1, \beta + 1, 2\sqrt{|A|}\}.
\]
We may assume therefore that \( B_m \) does not contain an \( a \)- or \( b \)-cycle. Since \( \delta_{|A|}(G) = c(A, \bar{A}) \) and \( G \) is connected, Proposition 5.1 implies that \( A \) is connected. Distinguish between two cases:

Case 2.1: \( b^{-1} P^a(A) \subseteq B \).

Since \( A \) is connected and \( b^{-1} P^a(A) \subseteq B \), the set \( B \) is connected as well. As each \( B_{i+1} \) was obtained from \( B_i \) by adjoining a vertex adjacent to \( B_i \), the set \( B_m \) is also connected. By Corollary 5.9 all \( a \)-components of \( B_m \) are \( a \)-paths of the same length, say \( k \). Denote by \( L \) the number of \( a \)-paths of \( B_m \). Since \( |P^a(A)| \leq k \), Proposition 5.11 gives
\[
c(B_m, \bar{B}_m) \geq 2(k + L) = 2(k + L + 1) - 2
\]
\[
\geq 4\sqrt{k(L + 1) - 2} = 4\sqrt{kL + k - 2}
\]
\[
\geq 4\sqrt{|B_m| + |P^a(A)| - 2} \geq 4\sqrt{|A| - 2}.
\]

By (3)
\[
c(A, \bar{A}) \geq c(B, \bar{B}) + 2 \geq c(B_m, \bar{B}_m) + 2 \geq 4\sqrt{|A|}.
\]

Case 2.2: \( b^{-1} P^a(A) \not\subseteq B \).

In this case
\[
|e_{b^{-1}}(P^a(A), B)| \leq |P^a(A)| - 1
\]
and
\[
|e_{b^{-1}}(P^a(A), \bar{A})| \geq 1.
\]

Consequently,
\[
c(B, \bar{B}) = c(A, \bar{A}) - |e(P^a(A), \bar{A})| + |e(P^a(A), B)|
\]
\[
\leq c(A, \bar{A}) - 2 - |P^a(A)| - 1 + |P^a(A)| - 1
\]
\[
= c(A, \bar{A}) - 4. \tag{5}
\]

By (4) we can apply Proposition 5.12 to obtain
\[
c(B_m, \bar{B}_m) \geq 4\sqrt{|B_m|} \geq 4\sqrt{|A| - |P^a(A)|}
\]
\[
> 4\sqrt{|A| - 2\sqrt{|A|} + 1} = 4\sqrt{(\sqrt{|A|} - 1)^2} = 4\sqrt{|A|} - 4.
\]

By (5)
\[
c(A, \bar{A}) \geq c(B, \bar{B}) + 4 \geq c(B_m, \bar{B}_m) + 4 \geq 4\sqrt{|A|}.
\]

This completes the proof. □

**Lemma 5.14.** Let \( G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\}) \). Let \( \emptyset \neq A \subset H \) be such that

1. all \( a \)-components of \( A \) have a unique \( b \)-successor and a unique \( b \)-predecessor in \( A \), and all \( a \)-components of \( \bar{A} \) have the analogous property.
2. \( A \) and \( \bar{A} \) contain neither \( a \)- nor \( b \)-cycles.

Then,
\[
c(A, \bar{A}) \geq 2 \min\{\beta + 1, \psi_2^+(a, b)\}.
\]
Proof. Pick some $a$-component $P_1^a(A) = \{x_1, x_1a, \ldots, x_1a^{\lfloor P_1^a(A) \rfloor - 1}\}$. Let $P_2^a(A) = \{x_2, x_2a, \ldots, x_2a^{\lfloor P_2^a(A) \rfloor - 1}\}$ be its $b$-successor. Consider a simple path from $x_1$ to $x_2$ as follows:

- If $x_1 b \in P_2^a(A)$, the path is (see Fig. 9):
  \[
  x_1, x_1b = x_2a^i, x_2a^{i-1}, \ldots, x_2 \quad (i \in \{0, \ldots, \lfloor P_2^a(A) \rfloor - 1\}).
  \]

- If $x_1 b \notin P_2^a(A)$ (so that $x_2b^{-1} \in P_1^a(A)$), the path is (see Fig. 10):
  \[
  x_1, x_1a, \ldots, x_1a^i, x_1a^{i-1}b = x_2 \quad (i \in \{1, \ldots, \lfloor P_1^a(A) \rfloor - 1\}).
  \]

The path has the following properties:

1. For each edge of the form $(x, xa)$ along the path, $xb \notin A$.
2. For each edge of the form $(xa, x)$ along the path, $xb^{-1} \notin A$.

Continuing in the same manner, we can find a cyclic chain of distinct $a$-components

$$P_j^a(A) = \{x_j, x_ja, \ldots, x_ja^{\lfloor P_j^a(A) \rfloor - 1}\}, \quad 1 \leq j \leq k,$$

such that

$$P_1^a(A) \models b P_2^a(A) \models b \cdots \models b P_k^a(A) \models b P_1^a(A).$$

Combining the paths from $x_1$ to $x_2$, from $x_2$ to $x_3$, \ldots, from $x_k$ to $x_1$, we obtain a cycle containing all $x_j, 1 \leq j \leq k$, possessing the above-mentioned properties (1) and (2). (This cycle is not necessarily simple, but it traverses no edge more than once in the same direction. Denote by $t^+$ and $t^-$ the number of edges in the cycle of the form $(x, xa)$ and $(x, xa^{-1})$, respectively. Then $x_1 b^k a^{i^*} a^{-i^*} = x_1$, so that $b^k a^{i^*} a^{-i^*} a^{-1} = 1$. Since $A$ does not contain $a$- and $b$-cycles, $\max\{t^+, t^-\} > 0$, and consequently $k + \max\{t^+, t^-\} \geq \min(\beta + 1, \psi_2^+(a, b))$. By properties (1) and (2) we have $|e_b(A, \tilde{A})| \geq t^+$ and $|e_{b^{-1}}(A, \tilde{A})| \geq t^-$, respectively, and thus Lemmas 5.4 and 5.5 give

$$|e_b(A, \tilde{A})| = |e_{b^{-1}}(A, \tilde{A})| \geq \max\{t^+, t^-\}.$$
Note that the number of $a$-components of $A$ is at least $k$, so that $\lvert e_a(A, \tilde{A}) \rvert \geq k$. Therefore,
\[
c(A, \tilde{A}) = 2(\lvert e_a(A, \tilde{A}) \rvert + \lvert e_b(A, \tilde{A}) \rvert) \\
\geq 2(k + \max\{t^+, t^-\}) \\
\geq 2 \min\{\beta + 1, \psi_2^+(a, b)\}. \quad \square
\]

**Lemma 5.15.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ is connected, then for all $1 \leq i \leq h - 1$
\[
\delta_i(G) \geq 2 \min\{x + \eta(i, x), \beta + \eta(i, \beta), \psi_2^+(a, b), 2\sqrt{i}, 2\sqrt{h-i}\}.
\]

**Proof.** Let $i$ be arbitrary and fixed. Choose $A = \{x_1, x_2, \ldots, x_i\}$ such that $c(A, \tilde{A}) = \delta_i(G)$. If the condition of Lemma 5.14 holds, then we are done by the conclusion of that lemma. Hence we may assume without loss of generality that one of the following holds:

(a) There exists an $a$-component of $A$ with no $b$-successor in $A$.
(b) There exists an $a$-component of $A$ with at least two $b$-successors in $A$.
(c) $A$ does contain an $a$-cycle.

We first claim that each of the properties (a), (b), and (c) implies the following property:

(d) Either $A$ has an $a$-component with no $b$-successor in $A$ or $\tilde{A}$ has an $a$-component with no $b$-predecessor in $\tilde{A}$.

In fact
\[(a) \implies (d): \text{Trivial}.\]
\[(b) \implies (d): \text{Let } P_a^a(A) \text{ be an } a\text{-component of } A \text{ with at least two } b\text{-successors in } A. \text{ Let } P_a^a(\tilde{A}) \text{ be an } a\text{-component of } \tilde{A}, \text{ contained in } P_a^a(A)b. \text{ Obviously, } P_a^a(\tilde{A}) \text{ has no } b\text{-predecessor in } \tilde{A} \text{ (see Fig. 11).}\]
\[(c) \implies (d): \text{Suppose that every } a\text{-component of } A \text{ has a } b\text{-successor in } A. \text{ Let } P_1^a(A) \text{ be an } a\text{-cycle and } P_2^a(A) \text{ a } b\text{-successor of } P_1^a(A). \text{ First suppose that } P_2^a(\tilde{A}) \text{ is an } a\text{-path. Let } P_1^a(A) = \{w, wa, \ldots, wa^{x-1} = wa^{-1}\} \text{ and } P_2^a(A) = \{wb, wba, \ldots, wba^{l-1}\} \text{ for an appropriately chosen } w \in P_1^a(A), \text{ where } l < x \text{ is the length of } P_2^a(A). \text{ Then the set } P_1^a(A)b \setminus P_2^a(A) = \{wba^l, wba^{l+1}, \ldots, wba^{x-1}\} \text{ contains at least one } a\text{-component of } \tilde{A} \text{ with no } b\text{-predecessor in } \tilde{A}, \text{ so that } (d) \text{ holds. Assume therefore that } P_2^a(\tilde{A}) \text{ is an } a\text{-cycle. Continuing in the same manner, we obtain a chain of } a\text{-cycles } P_1^a(A), P_2^a(A), \ldots, P_k^a(A) = P_1^a(A) \text{ such that } P_1^a(A)b = \cdots = bP_k^a(A) = P_1^a(A). \text{ Since } G \text{ is connected, this means that } A = H, \text{ which is a contradiction. Thus } (d) \text{ holds.}\]

Suppose, say, that $A$ has an $a$-component with no $b$-successor in $A$. (The other possibility in (d), namely that $\tilde{A}$ has an $a$-component with no $b$-predecessor in $\tilde{A}$, is completely analogous.) Let $P_a^a(A)$ be such an $a$-component. Then $\lvert e_b(P_a^a(A), \tilde{A}) \rvert = \lvert P_a^a(A) \rvert$. Distinguish between two cases:

*Case 1: $P_a^a(A)$ is an $a$-cycle.*

By Lemmas 5.4 and 5.5 we have
\[
c(A, \tilde{A}) \geq 2\lvert e_b(P_a^a(A), \tilde{A}) \rvert = 2x.
\]
If $x$ divides $i$ then we are done. Otherwise, $A$ must contain at least one $a$-path, so that $|e_a(A, \bar{A})| \geq 1$. Hence
\[
c(A, \bar{A}) = |e(A, \bar{A})| = 2|e_a(A, \bar{A})| + 2|e_b(A, \bar{A})| \\
\geq 2 + 2x = 2(x + \eta(i, x)).
\]

Case 2: $P^a(A)$ is an $a$-path. Since $c(A, \bar{A}) = \delta_i(G)$, we can apply Proposition 5.13 to obtain
\[
c(A, \bar{A}) \geq 2 \min \{x + 1, \beta + 1, 2\sqrt{|A|}\} \\
\geq 2 \min \{x + 1, \beta + 1, 2\sqrt{i}, 2\sqrt{h-i}\}. \quad \square
\]

**Corollary 5.16.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ with $H$ abelian, where $h > 4$ and $a^2 \neq 1$ and $b^2 \neq 1$, then for any $1 \leq i \leq h$
\[
\delta_i(G) = 2 \min \{x + \eta(i, x), \beta + \eta(i, \beta), \psi_2^+(a, b), [2\sqrt{i}], [2\sqrt{h-i}]\}.
\]

**Proof.** Since $\delta_i(G)$ is even for all $i$, we can obtain this result immediately by applying Corollary 4.15 and Lemma 5.15. \square

**Proposition 5.17.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ is connected, then
\[
\gamma(G) \geq 2 \min \{x + 1, \beta + 1, \psi_2^+(a, b)\}.
\]

**Proof.** Let $i = \lceil h/2 \rceil$. By Lemma 3.2 we obtain
\[
\gamma(G) \geq \max \{\delta_i(G), \delta_{i+1}(G)\}.
\]

We distinguish between two cases:

**Case 1:** $h$ is even.

We have $i = h/2$ and $\eta(i - 1, x) = 1$, so that by Lemma 5.15 and Corollary 4.2
\[
\gamma(G) \geq 2 \min \{\min \{x, \beta, \psi_2^+(a, b), 2\sqrt{i}\}, \min \{x + 1, \beta + 1, \psi_2^+(a, b), 2\sqrt{i - 1}\}\} \\
= 2 \max \{\psi_2^+(a, b), \min \{x + 1, \beta + 1, \psi_2^+(a, b), 2\sqrt{i - 1}\}\}.
\]

Suppose without loss of generality that $x \leq \beta$. If $x \geq \psi_2^+(a, b)$, then $\gamma(G) \geq 2\psi_2^+(a, b) = 2\psi_2^+(a, b)$. Suppose therefore that
\[
x = \psi_2^+(a, b) < \psi_2^+(a, b). \quad (6)
\]

Then $\gamma(G) \geq 2 \max \{x, \min \{x + 1, 2\sqrt{i - 1}\}\} \geq 2x$. Since $\gamma(G)$ is even, if $\gamma(G) > 2x$, then we are done. Assume therefore that $\gamma(G) = 2x$. Hence $2\sqrt{i - 1} \leq x$. By Corollary 4.2 we have $x = \psi_2^+(a, b) \leq 2\sqrt{i - 1}$. Since $x$ divides $h$, and $h \geq 4$, we easily obtain $x \neq 2\sqrt{h/2 - 1} = 2\sqrt{i - 1}$. Altogether
\[
2\sqrt{h/2 - 1} < x \leq 2\sqrt{h/2}.
\]

(7)

Now, if $x$ is even this implies $x = 2\sqrt{h/2}$, so that $\gamma(G) = 2\sqrt{2h} \in \mathbb{Z}$. Corollary 4.4 gives
\[
\min \{x + 1, \beta + 1, \psi_2^+(a, b)\} \leq \sqrt{2h}
\]

and therefore
\[
\gamma(G) \geq 2 \min \{x + 1, \beta + 1, \psi_2^+(a, b)\}.
\]

In the second possibility, namely that $x$ is odd, (7) and the fact that $x$ divides $h$ easily give $x = 3$ and $h = 6$. A routine check yields that both $2\psi_2^+(a, b)$ and $2\psi_2^+(a, b)$ must be 3, which contradicts (6) and thereby settles this case.
Case 2: $h$ is odd.
In this case

$$\gamma(G) \geq 2 \max \{ \min \{ x + \eta(i, x), \beta + \eta(i, \beta), \psi_2^+ (a, b), 2 \sqrt{i} \},$$

$$\min \{ x + \eta(i - 1, x), \beta + \eta(i - 1, \beta), \psi_2^+ (a, b), 2 \sqrt{i} \} \}$$

$$= \min \{ x + 1, \beta + 1, \psi_2^+ (a, b), 2 \sqrt{i} \}.$$

If $\min \{ x + 1, \beta + 1, \psi_2^+ (a, b), 2 \sqrt{i} \} = \min \{ x + 1, \beta + 1, \psi_2^+ (a, b) \}$, then we are done. Thus, suppose that $\gamma(G) \geq 4 \sqrt{i}$. Since $h$ is odd, $\sqrt{2h} \notin \mathbb{Z}$, and consequently by Corollary 4.5

$$\min \{ x + 1, \beta + 1, \psi_2^+ (a, b) \} \leq \sqrt{2h} - 2 \leq \sqrt{4i} = 2 \sqrt{i}.$$

Hence

$$\gamma(G) \geq 4 \sqrt{i} \geq 2 \min \{ x + 1, \beta + 1, \psi_2^+ (a, b) \}.$$

This completes the proof. □

Lemma 3.1, Corollary 4.12 and Proposition 5.17 jointly yield:

**Corollary 5.18.** If $G = \text{Cay}(H, \{a^{\pm 1}, b^{\pm 1}\})$ with $H$ abelian, where $h > 4$ and $a^2 \neq 1$ and $b^2 \neq 1$, then

$$\gamma(G) = 2 \min \{ x + 1, \beta + 1, \psi_2^+ (a, b) \}.$$

Corollary 5.16 and 5.18 jointly prove Theorem 2.1 in case both $a$ and $b$ are of order at least 3.

**References**