Multiple Description Coding over Erasure Channels

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Abstract—In this paper, we study the optimal rate allocation of multiple description coding over multiple erasure channels in order to attain the minimum average distortion of the recovered message at the receiver. The results of this investigation are subsequently used to determine conditions under which the optimal rate allocation is characterized by transmission of one description over a single channel, i.e. single description coding. We formulate the optimal rate allocation problem by using rate-distortion theory under the constraint that the total coding rate is fixed and solve the problem by using numerical methods. Furthermore, we derive a closed-form analytical solution to the optimal rate allocation problem under four conditions and verify that the analytical solution is consistent with the numerical results. Also, we present a heuristic that captures the solution to the optimal rate allocation problem and can be used to determine the number of descriptions and relative rates required to achieve optimality.

I. INTRODUCTION

Multiple description coding (MDC) refers to source coding in which the encoder generates multiple descriptions of the source. The different descriptions are transmitted over various paths in an unreliable network. The decoder can rely on any subset of the descriptions to reconstruct the transmitted message. The distortion level of the reconstructed message depends on the number and quality of the descriptions available at the decoder.

MDC offers a potentially attractive framework for data transmission in scenarios where packet loss is inevitable. For instance, in real-time multimedia communications, packet retransmission is not possible due to severe latency constraints. In this situation, the receiver must rely on the packets available at the decoder to recover the transmitted signal. The quality of the recovered signal will depend on the level of packet loss on the network.

MDC provides a mechanism to recover signals with reduced but acceptable quality even when some of the descriptions have been corrupted. Specifically, when all descriptions arrive at the decoder, the distortion is lowest. However, when some of the descriptions have been lost, the decoder can still rely on the descriptions that have arrived at the receiver to recover the transmitted signal at lower quality. This approach is fundamentally different from the use of scalable coding, where a specific enhancement layer can be used to improve the quality of the decoded signal provided that the base layer and all lower-level enhancement layers have been recovered by the receiver. This feature has made MDC very attractive for multimedia communication over multi-path networks which exploit network diversity to improve the transmitted signal quality [1]–[3].

In practice, especially in multimedia communication, the service company can just provide a fixed data rate according to the fee that consumers are ready to pay. In this case, we can design a channel with a fixed capacity and divide the channel optimally into two channels whose capacities will match the optimal data rates. Thus we formulate the problem by determining the optimal allocation of rates among the channels in order to minimize the average distortion subject to the constraint that the total coding rate is fixed. We address the problem of determining the optimal rate allocation of multiple descriptions among multiple erasure channels. Furthermore, we derive conditions to determine when the average distortion of multiple descriptions is lower than single description coding (SDC) as a particular case of the rate allocation problem.

Batllo et al. [4] studied the optimal bit allocation problem in order to minimize the central distortion given constraints on the side distortion and the total coding rate. Bernardini et al. [5] considered the problem of bit allocation and quantization optimization for frame-based multiple description coding schemes obtained by means of an oversampled filter bank. Coward et al. [6] considered a symmetric bit allocation and studied the effect of channel codes parameters on the the erasure probabilities and on the average distortion. Interestingly, they found that MDC outperforms SDC only when the channel code correction capabilities are poor. We shall see that our results analytically quantify this observation and extend it to the more general case where the MDC encoder can adapt the descriptions’ rate according to the channel erasure probability.

In this paper, we provide a numerical solution (and a closed-form analytical solution under certain conditions) to the optimal rate allocation for MDC over multiple erasure channels with different error rates by minimizing the average distortion. Furthermore, we rely on the results of the optimal rate allocation to determine conditions when MDC or SDC yield lower distortion levels.

The remainder of the paper is organized as follows: In Section II, we describe the system model. The main results of the optimal rate allocation problem are presented in Section III. The results consist of both numerical and analytical solutions to the optimal rate allocation problem. Moreover, a heuristic is proposed to instantly determine the optimal rate allocation. Finally, we conclude with a brief summary and discussion of our results in Section IV.
The simplest model for the MDC problem is the case of two channels and three receivers illustrated in Fig. 1. The MDC encoder generates two descriptions. Each individual description provides an approximation to the original message, and multiple descriptions can refine each other, to produce a better approximation than that attainable by any single one alone. If both descriptions are received, then the decoder can reconstruct the source at some small distortion value $D_0$ (the central distortion), but if either one is lost, the decoder can still reconstruct the source at some higher distortion $D_1$ or $D_2$ (the side distortions). Practical MDC designs appear in [7], [8].

The characterization of the rate-distortion region for generic source and generic distortion measure is an open problem [7], [9]. The achievable rate-distortion region is completely known for a memoryless unit-variance Gaussian source with mean-squared error distortion [10]. In this case, the set of achievable rates is

$$D_1 \geq 2^{-2R_1}, \quad D_2 \geq 2^{-2R_2}, \quad D_0 \geq 2^{-2(R_1+R_2)}\gamma(D_1, D_2, R_1, R_2)$$

where

$$1/\gamma = 1 - \left(\sqrt{(1-D_1)(1-D_2)} - \sqrt{D_1D_2 - 2^{-2(R_1+R_2)}}\right)^2$$

for $D_1 + D_2 < 1 + 2^{-2(R_1+R_2)}$ and $\gamma = 1$ otherwise. In general, it is not possible to achieve equality simultaneously in the three equations since two individually good descriptions tend to be similar to each other. Thus, the second description will contribute very little to improving the quality of the first one. On the other hand, two descriptions which are complementary cannot be individually good [7].

Suppose we transmit the MDC coded i.i.d. unit-variance Gaussian source over two erasure channels with packet error probability $p_1$ and $p_2$ respectively. The erasure rates are assumed to be known and constant. Extensions to the case of unknown and/or time varying erasure rates are of great practical relevance and would require designing variable-rate and error-robust MDC coders, both problems are topics of current investigation but outside the scope of this work.

Extensions to non-Gaussian sources are desirable in practice. We remark that i.i.d. Gaussian sources have the highest differential entropy among sources with the same variance, thus they can be considered as the “most difficult to compress”. As such, our conclusions can be regarded as a “toughest source” scenario. The present analysis can also be directly extended to other source models and/or for the case of more than two channels. In those cases one can use achievable rate-distortion regions and/or outer-bound region instead of the actual capacity region. However, the results would not be conclusive and should be regarded as bounds on attainable performances.

Another important property assumed in our model is that errors introduced by the physical channel can always be detected and that packets in error are always erased. It is well known that error-correcting codes of finite length cannot have probability of undetected errors equal to zero. The sensitivity of our current analysis to channel error propagation is an interesting and challenging open question left for future work.

In a network of erasure channels, if a packet in one description is lost (i.e., erased), the appropriate side decoder output is used. If both descriptions are lost, the mean value of the source is output, which results in a distortion equal to the source variance. Let $p_{11} = p_1p_2$, $p_{01} = (1-p_1)p_2$, $p_{10} = p_1(1-p_2)$, and $p_{00} = (1-p_1)(1-p_2)$. Thus, the average distortion [6] is

$$D_{\text{ave}} = p_{11}D_1 + p_{10}D_2 + p_{00}D_0$$

In order to allow sensible comparisons between MDC and single description code (SDC), we further require that the total source coding rate satisfies $R_1 + R_2 = C$, where $C$ is a positive constant representing the total number of bits per source sample. The total rate constraint is introduced to ensure that we do not give any advantage to MDC over SDC. Furthermore, this restriction has practical significance when the costs of network services depend on transmission rates as well as in the design of multiplexing systems for transmission of multiple streams.

Our goal is to find the optimal rate allocation policy $\beta \in [0,1]$ such that $R_1 = \beta C, \ R_2 = (1-\beta)C$ minimize the average distortion in (3) within the rate-distortion region specified by (1)-(2). We denote with $D_{\text{MDC}}$ the optimal value of $D_{\text{ave}}$. With the introduction of the parameter $\beta$, the sum of the side distortions always satisfy

$$e^{-\beta c} + e^{-(1-\beta)c} \leq D_1 + D_2 < 1 + e^{-c}$$

for all $0 < \beta < 1$, where $c = 2 \ln(2)C$. In other words, the case $\gamma = 1$ corresponds to SDC, i.e., $\beta$ either 0 or 1. Hence, the domain of our minimization can be reformulated as

$$0 \leq \beta \leq 1, \quad e^{-\beta c} \leq D_1, \quad e^{-(1-\beta)c} \leq D_2$$

$$e^{-c} \leq \frac{1 - \left(\sqrt{(1-D_1)(1-D_2)} - \sqrt{D_1D_2 - e^{-c}}\right)^2}{1 - \left(\sqrt{(1-D_1)(1-D_2)} - \sqrt{D_1D_2 - e^{-c}}\right)^2} = D_0$$

since clearly the minimum distortion is attained when $D_0$ is minimum.

$$D_{\text{MDC}} \leq D_{\text{SDC}} = \min\{p_1, p_2\} + (1 - \min\{p_1, p_2\}) e^{-c}$$

III. MAIN RESULTS

We can immediately derive simple bounds on $D_{\text{MDC}}$. By considering a SDC source over the channel with the smallest erasure rate we obtain

$$D_{\text{MDC}} \leq D_{\text{SDC}} = \Delta = \min\{p_1, p_2\} + (1 - \min\{p_1, p_2\}) e^{-c}$$

Fig. 1. MDC with two channels and three receivers
where $e^{-c}$ is the distortion for SDC. Similarly, by considering a genie-aided scheme in which the encoder is informed about the description that will be received and assigns all the $C$ bits to that description using a SDC we obtain

$$D_{\text{MDC}} \geq D_G \Delta p_1 p_2 + (1 - p_1 p_2)e^{-c} \quad (5)$$

since whenever a description is received the distortion is $e^{-c}$.

**A. Optimal numerical solution**

It is not possible to find a closed form solution for the optimal average distortion. We have used the numerical sequential quadratic programming (SQP) and Quasi-Newton optimization (i.e., standard Matlab functions in the Optimization Toolbox) to derive the results showed in Fig. 2 and Fig. 3.

Fig. 2(a) shows a typical curve for the optimal $\beta$. The solid curve is the optimal solution while the dotted line refers to the case where $D_1 = e^{-\beta c}$ and $D_2 = e^{-(1-\beta)c}$, referred to as “with three lower bounds”. In general, we observe that the larger fraction of bits is allocated to the better channel, that is, $R_1 > R_2$, i.e., $\beta > 0.5$, when $p_1 < p_2$, and $R_1 < R_2$ when $p_1 > p_2$. In particular, we notice that MDC ($\beta$ strictly in $(0,1)$) outperforms SDC ($\beta$ either 0 or 1) only when the erasure probabilities of the two channels are comparable.

Fig. 2(b) compares the optimal average distortion with the bounds $D_{\text{SDC}}$ in (4), $D_G$ in (5), and the case “with three lower bounds”. We notice that the latter is very close to the optimal $D_{\text{MDC}}$.

Fig. 2(c) shows when MDC (inside the ellipsoid region) outperforms SDC, as function of $(p_1, p_2)$ for different values of $C$. We notice that MDC is better when $C$ is large. On the other hand, the SDC region enlarges as $C$ decreases. Intuitively, when very few bits per source samples are available to describe the Gaussian source, it is better to use them all on the channel with the lower erasure rate. For small $C$ we have in fact that $D_{\text{MDC}} \approx D_{\text{SDC}}$ as shown in Fig. 3(b) and Fig. 3(c). Only when the erasure probabilities are comparable in magnitude and relatively high, MDC achieves lower distortion than SDC. Notice also from Fig. 3(b) that $D_{\text{MDC}} \approx D_G$ for large $C$. That is, when we can use many bits to describe the Gaussian source, then almost perfect reconstruction is possible no matter which description is received since $D_1$, $D_2$ and $D_3$ all tend to zero. The only event affecting the average distortion is when both descriptions are erased, which happens with probability $p_1 p_2$ and coincides with the distortion floor for large $C$. Finally, Fig. 3(a) and Fig. 2(a) are consistent with Fig. 2(c).

To gain further insights into the behavior of the optimal rate allocation and of the average distortion, we reason as follows. The minimization of the average distortion is equivalent to the
minimization of
\[
\frac{D_{\text{ave}}}{p_1 p_2} = 1 - \frac{p_1}{p_1} D_1 + \frac{1 - p_2}{p_2} D_2 + \frac{1 - p_1}{p_1} 1 - p_2 D_0 \tag{6}
\]
Notice that the function \(\frac{1 - p}{p}\), \(p \in [0, 1]\), is monotonically decreasing in \(p\) and approaches 0 when \(p\) approaches 1 (we shall refer to this case as "large \(p\)") and approaches \(\infty\) when \(p\) approaches 0 (we shall refer to this case as "small \(p\)"). We utilize this feature to analyze approximations of \(D_{\text{MDC}}\) in the four corners of the region \((p_1, p_2) \in [0, 1]^2\). Next we will analyze these four cases respectively.

### B. Best single descriptions approximation

Consider the case when both \(p_1\) and \(p_2\) are large, both \(1 - \frac{p_1}{p_2}\) and \(1 - \frac{p_2}{p_1}\) are small, but larger than \(1 - \frac{p_1}{p_1} 1 - p_2\). This means that the side distortions \(D_1\) and \(D_2\) contribute more to the average distortion than \(D_0\). In this situation, we try to make \(D_1\) and \(D_2\) as small as possible, that is, we need best single descriptions [7]. Then our original optimization constraints can be simplified to

\[
D_1 = e^{-\beta c}, \quad D_2 = e^{-(1-\beta)c}, \quad D_0 = e^{-\beta c} + e^{-(1-\beta)c} - e^{-c}
\]

Let us consider the special case \(p_1 = p_2 = p\) first. In this case the average distortion depends on \(\beta\) only through \(y = D_1 + D_2 - e^{-c}\) and minimizing \(D_{\text{MDC}}\) is equivalent to minimizing

\[
(1 - p)y + (1 - p)^2 e^{-c} \left( \frac{1}{y} \right) \tag{7}
\]

The minimum in (7) is attained for \(y = \sqrt{\frac{1 - p}{p}} e^{-c}\). Since \(y \in [2e^{-c/2} - e^{-c}, 1]\) when \(\beta \in [0, 1]\), the optimal \(y\) is

\[
y^* = \begin{cases} 
2e^{-c/2} - e^{-c} & \text{if } \frac{1 - p}{p} \geq e^{c/2} \text{ (i.e., } \beta = 0 \text{ or } 1) \\
\sqrt{\frac{1 - p}{p}} e^{-c} & \text{elsewhere}
\end{cases}
\]

Finally, by solving \(y^*\) for \(\beta\) in the range \(y \in [2e^{-c/2} - e^{-c}, 1]\), we get

\[
\beta = \frac{1}{2} - \frac{1}{c} \cosh^{-1} \left( \frac{1}{2} e^{-c/2} + \frac{1}{2} \sqrt{\frac{1 - p}{p}} \right) \tag{8}
\]

We can verify result from Fig. 4 in which the optimal solution (solid line) and the above approximated closed-form solution (dotted line) are practically superimposed. Notice that the “best single descriptions case” is actually the same as “with three lower bounds case”. Notice that, for \(p_1 = p_2 = p\), the “best single descriptions” approximation coincides with SDC for

\[
p \leq p^* = \frac{1}{e^c + 1} \tag{9}
\]

For the general case \(p_1 \neq p_2\), consider again the objective function to be minimized under the “best single descriptions” approximation, i.e.,

\[
F(\beta) = \frac{1 - p_1}{p_1} e^{-\beta c} + \frac{1 - p_2}{p_2} e^{-(1-\beta)c} + \frac{1 - p_1}{p_1} 1 - p_2 e^{-c}
\]

The KKT conditions for the optimality of MDC (i.e., \(\beta\) strictly in \((0, 1)\)) guarantee that in the region

\[
\frac{dF(\beta)}{d\beta} |_{\beta=0} = - \frac{1 - p_1}{p_1} + \frac{1 - p_2}{p_2} e^{c} + \frac{1 - p_1}{p_1} 1 - p_2 e^{-c} (1 - e^{-c}) < 0 \tag{10}
\]

\[
\frac{dF(\beta)}{d\beta} |_{\beta=1} = - \frac{1 - p_2}{p_2} e^{c} + \frac{1 - p_1}{p_1} 1 - p_2 e^{-c} (1 - e^{-c}) > 0 \tag{11}
\]

MDC is superior to SDC. Fig. 3(c) shows three such regions for different values of \(C\). The region where MDC beats SDC intersects with the line \(p_1 = p_2 = p\) at \(p = 1\) and \(p = p^*\) given by (9), as we already found before. We can further approximate the ellipsoid region where MDC beats SDC by a polygon whose facets are

\[
p_2 - 1 = e^{\pm c} (p_1 - 1), \quad p_2 - p^* = e^{\pm c} (p_1 - p^*) \tag{12}
\]

where \(p^*\) is given in (9). Indeed, equations (10) (11) with strict equalities implicitly define one part of contour of the ellipsoid region. By invoking the differentiation theorem for implicit function, we get that

\[
dp_2 = - \frac{\partial}{\partial p_1} \left\{ \frac{dF(\beta)}{d\beta} |_{\beta=0} \right\} = e^c - 1 - p_2 e^{2c} \tag{13}
\]

\[
dp_1 = \frac{\partial}{\partial p_2} \left\{ \frac{dF(\beta)}{d\beta} |_{\beta=0} \right\} = e^c - 1 + p_1 \tag{14}
\]

For \(p_1 = p_2 = 1\), \(dp_2/dp_1 = e^c\), as claimed. The approximation in (12) is illustrated in Fig. 5(a): when \(p_1 < p_2\) (above the main diagonal) and outside the polygon, we use SDC on channel 1 (\(\beta = 1\)); when \(p_1 > p_2\) (below the main diagonal) and outside the polygon, we use SDC on channel 2 (\(\beta = 0\)); when \((p_1, p_2)\) is inside the polygon, we approximate the optimal \(\beta\) with a linear function. Notice that, outside of the polygon, \(\beta\) is equal to its optimal value. Notice that the optimal \(\beta\) inside the ellipsoid region is in general non-linear. For example, equation (8) tells us that on the main diagonal \(p_1 = p_2 > p^*\) the optimal \(\beta\) follows a \(\cosh^{-1}\) type law. Fig. 5(b) and Fig. 5(c) show how the proposed approximation compare with the optimal solution and the “best single description” approximation. We observe that at one endpoint the approximate \(\beta\) is close to optimal one, but at the other endpoint it is not. However, the approximation error decreases as \(C\) increases. Nonetheless, the average distortion curves are almost superimposed. A better approximation can be obtained by using a linear function between the boundaries of the ellipsoid region (which can be easily numerically evaluated.)

### C. Best central description approximation

Now consider that both \(p_1\) and \(p_2\) are small, both \(1 - \frac{p_1}{p_1}\) and \(1 - \frac{p_2}{p_2}\) are large, but smaller than \(1 - \frac{p_1}{p_1} 1 - p_2\). Hence, the contribution of \(D_0\) may become dominant in \(D_{\text{MDC}}\). In this situation, we try to make \(D_0\) as small as possible, that is, we aim at the best possible central description, i.e., \(\gamma = 1\). Our original optimization can now be simplified into a linear program with constraints \(D_1 \geq e^{-\beta c}, D_2 \geq e^{-(1-\beta)c}\), and \(D_1 + D_2 \geq 1 + e^{-c}\). Since \(D_{\text{MDC}}\) is linear in \(D_1, D_2\), and \(D_0\), the optimization domain is defined by linear
equations in $D_1$, $D_2$ and $D_0$, the minimum for $D_{MDC}$ is therefore attained at one of the vertices of the “dominant” face $D_1 + D_2 = 1 + e^{-c}$. Hence, our linear program is solved by either $(D_1, D_2) = (1, e^{-c})$ or by $(D_1, D_2) = (e^{-c}, 1)$. Therefore, the optimal solution in this case coincides with our SDC inner bound $D_{SDC}$. Simulation results are illustrated in Fig. 3(b). We can see that in this region, the “best central description” approximation overlaps with the inner bound as expected.

**D. Best SDC approximation**

When $p_1$ is small and $p_2$ is large, or vice versa, then all the bits may be used for the description to be sent to the best channel. In this case we can achieve distortion $D_{SDC}$. Notice that, $D_{SDC}$ approaches $\min\{p_1, p_2\}$ for large $C$ while $D_{MDC}$ approaches $p_1p_2 < \min\{p_1, p_2\}$. This suggests that SDC performs close to optimal when $\max\{p_1, p_2\} \approx 1$. Indeed, Fig. 2(c) confirms that SDC is optimal when $\max\{p_1, p_2\} \approx 1$.

In conclusion, the “best single descriptions” approximation is very close to optimal in all parameter range. However, the other approximations are accurate for moderate $C$ only.

**IV. CONCLUSION AND FUTURE WORK**

In this paper, we studied the optimal rate allocation problem for MDC in order to attain the minimum average distortion over multiple independent erasure channels under the constraint that the total coding rate is fixed. We derived a numerical solution to the optimal rate allocation problem. Moreover, we showed that the optimal solution to the rate allocation problem can be approximated by assuming that the average distortion is computed using the lower bounds of the distortion of the side and central information. Furthermore, we presented an analytical solution to the optimal rate allocation problem under four conditions and showed that it matches our numerical results. Also, we proposed a heuristic to instantly determine the optimal rate allocation for MDC over multiple erasure channels. In the future, we will compare our results with other rate allocation methods for MDC and implement the practical video transmission experiments using the obtained analytical results in this paper. Also we plan to investigate the optimal rate allocation for MDC when we consider the relation between the data rate and the probability of erasure and when the erasure channels are correlated.

**REFERENCES**


