One-Phase Inverse Stefan Problem Solved by Adomian Decomposition Method

R. GRZYMIAWSKI AND D. SLOT\textsuperscript{*}

Institute of Mathematics
Silesian University of Technology
Kaszubska 23, 44-100 Gliwice, Poland

d.slot@pois1.pl

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Abstract—In this paper, the solution of one-phase inverse Stefan problem is presented. The problem consists of the reconstruction of the function which describes the distribution of temperature on the boundary, when the position of the moving interface is well-known. The proposed solution is based on the Adomian decomposition method and the least square method. © 2006 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The inverse problems for differential equations consist of stating the boundary conditions, thermophysical properties of the body or initial conditions. The insufficiency of input information is compensated by some additional information on the effects of the input conditions. For the inverse Stefan problem, this additional information is the position of the freezing front, its velocity in normal direction, or temperature in selected points of the domain.

Most published materials involve the reconstruction of temperature or heat flux on the boundary of a domain. In papers [1,2], the problem is reduced to a system of integral equations. In paper [3], the solution is found in terms of an infinite series of one-dimensional integrals. Jochum considers the inverse Stefan problem as a problem of nonlinear approximation theory (see [4,5]). In paper [6], for solutions of one-phase two-dimensional problems, authors used a complete family of solutions to the heat equation to minimize the maximal defect in the initial-boundary data. Similar method was used in [7,8] for two- and multi-phase problems. The solution in this method is found in a linear combination form of the functions satisfying the equation of heat conduction. The coefficients of this combination are determined by the least square method for the boundary of a domain. In papers [9–11], authors used dynamic programming or minimization techniques in

*Author to whom all correspondence should be addressed.
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finite- and infinite-dimensional space. The inverse Stefan problems, where the thermal properties of materials (e.g., thermal conductivity, thermal diffusivity, coefficient of convective heat-transfer etc.) are reconstructed, are discussed in papers [12–16].

The Adomian decomposition method was developed by Adomian [17,18]. This method is useful for solving a wide class of problems. In papers [18,19], the review of applications of the Adomian decomposition method for differential and integral equations is presented. The Adomian decomposition method is used for obtaining solutions of linear or nonlinear heat equation in papers [20–24]. The application of this method for wave equation is studied in papers [25–27]. In papers [28,29], authors consider solving the inverse problems for the differential equations by the Adomian decomposition method. This method is also used for an approximate solution of the Stefan problem [30].

Using this method, we are able to solve a nonlinear operator equation,

\[ F(u) = f, \]

where \( F : H \rightarrow G \) is nonlinear operator, \( f \) is a known element from Hilbert space \( G \) and \( u \) is a sought element from Hilbert space \( H \). Operator \( F(u) \) can be written as

\[ F(u) = L(u) + R(u) + N(u), \]

where \( L \) is an invertible linear operator, \( R \) is a linear operator and \( N \) is a nonlinear operator. The solution of equation (1.1) is sought in the form of functional series,

\[ u = \sum_{i=0}^{\infty} g_i. \]

The nonlinear operator \( N \) is decomposed in series,

\[ N(u) = \sum_{i=0}^{\infty} A_i, \]

where \( A_i \) are Adomian polynomials, which can be generated according to algorithms set by Adomian [18,19] and Wazwaz [31,32],

\[ A_0 = N(g_0), \]
\[ A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{n} \lambda^i g_i \right) \right]_{\lambda=0}, \quad n \geq 1. \]

After the substitution of equations (1.2)–(1.4) into operator equations (1.1) and using inverse operator \( L^{-1} \), we get the following recurrent formula,

\[ g_0 = g^* + L^{-1}(f), \]
\[ g_n = -L^{-1} R(g_{n-1}) - L^{-1}(A_{n-1}), \quad n \geq 1, \]

where \( g^* \) is the function dependent on the initial and boundary conditions.

In this paper, we tried solving the one-phase inverse Stefan problem, which consists of the reconstruction of the function which describes the distribution of temperature on the boundary, when the position of the moving interfaces (freezing front) is well known. This kind of problem becomes an inverse design problem. The conditions for the existence and uniqueness of the solution to this problem can be found in [33]. The solution is based on the Adomian decomposition and the least square methods.
2. INVERSE STEFAN PROBLEM

Let $D = \{(x,t); \, t \in [0,t^*), \, x \in [\alpha,\xi(t)]\}$ be a domain in $\mathbb{R}^2$ (Figure 1). On the boundary of this domain three components are distributed,

\[
\Gamma_0 = \{(x,0); \, x \in [\alpha,\xi(0)]\},
\]
\[
\Gamma_1 = \{(\alpha,t); \, t \in [0,t^*)\},
\]
\[
\Gamma_g = \{(x,t); \, t \in [0,t^*), \, x = \xi(t)\},
\]

where the initial and boundary conditions are given. Assuming linear heat conduction with constant coefficients, after appropriate changes in the variables, without loss of generality, we will look for an approximate solution of the following dimensionless problem (see \[34,35\]).

For a given position of freezing front $\Gamma_g$, the distribution of dimensionless temperature $u$ in domain $D$ is calculated as well as function $\vartheta(t)$ on boundary $\Gamma_1$, which satisfies the following equations,

\[
\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t}(x,t), \quad \text{in} \ D, \tag{2.4}
\]
\[
u(x,0) = \varphi(x), \quad \text{on} \ \Gamma_0, \tag{2.5}
\]
\[
u(\alpha,t) = \vartheta(t), \quad \text{on} \ \Gamma_1, \tag{2.6}
\]
\[
u(\xi(t),t) = u^*, \quad \text{on} \ \Gamma_g, \tag{2.7}
\]
\[
-\frac{\partial u(x,t)}{\partial x} = \frac{d\xi(t)}{dt}, \quad \text{on} \ \Gamma_g, \tag{2.8}
\]

where $u^*$ is the dimensionless phase-change temperature, $x = \xi(t)$ is the function describing the position of the freezing front $\Gamma_g$, and $u$, $t$, and $x$ refer to dimensionless temperature, dimensionless time, and dimensionless spatial location, respectively.

3. SOLUTION OF THE PROBLEM

In the considered problem, we have operator equations (1.1), where

\[
L(u) = \frac{\partial^2 u}{\partial x^2}, \tag{3.1}
\]
\[
R(u) = -\frac{\partial u}{\partial t}, \tag{3.2}
\]
\[
N(u) = 0, \tag{3.3}
\]
\[
f = 0. \tag{3.4}
\]

The inverse operator $L^{-1}$ is given by

\[
L^{-1}(u) = \int_{\alpha}^{x} \int_{\xi(t)}^{x} u(x,t) \, dx \, dx. \tag{3.5}
\]
Using the Stefan condition (2.8) and the boundary condition (2.6), we obtain
\[
L^{-1}L(u) = \int_{\alpha}^{\xi(t)} \int_{\alpha}^{\xi(t)} \frac{\partial^2 u(x, t)}{\partial x^2} \, dx \, dx \\
= \int_{\alpha}^{\xi(t)} \left( \frac{\partial u(x, t)}{\partial x} + \xi'(t) \right) \, dx \\
= u(x, t) - \vartheta(t) + \xi'(t)(x - \alpha),
\]
where \(\xi'(t) = \frac{d\xi(t)}{dt}\), hence,
\[
g^* = \vartheta(t) - \xi'(t)(x - \alpha).
\]
In this problem, we have
\[
g_0 = g^*, \\
g_n = -L^{-1}R(g_{n-1}), \quad n \geq 1.
\]
i.e.,
\[
g_0 = \vartheta(t) - \xi'(t)(x - \alpha), \\
g_n = \int_{\alpha}^{\xi(t)} \int_{\alpha}^{\xi(t)} \frac{\partial g_{n-1}}{\partial t} \, dx \, dx, \quad n \geq 1.
\]
We seek an approximation solution in the form,
\[
u_n = \sum_{i=0}^{n} g_i, \quad n \in \mathbb{N}.
\]
Because functions \(g_i\) (3.9) are dependent on an unknown function \(\vartheta(t)\), we derived this function in the form,
\[
\vartheta(t) = \sum_{i=1}^{m} p_i \psi_i(t),
\]
where \(p_i \in \mathbb{R}\) and the basis functions \(\psi_i(t)\) are linear independence.

To construct the measure of error the least square method is applied [36]. We want that solution (3.10) to satisfy the initial condition (2.5) and the boundary condition (2.7). Thus, we are looking for the minimum of the following functional,
\[
J(p_1, \ldots, p_m) = \int_{\alpha}^{\xi(t)} \int_{\alpha}^{\xi(t)} (u_n(x, 0) - \varphi(x))^2 \, dx \, dx + \int_{0}^{t^*} (u_n(\xi(t), t) - u^*)^2 \, dt.
\]
Substituting equations (3.10), (3.9), and (3.11) to functional \(J\) and differentiating it with respect to the coefficients \(p_i\) \((i = 1, \ldots, m)\) and equating the obtained derivatives to zero,
\[
\frac{\partial J}{\partial p_i}(p_1, \ldots, p_m) = 0, \quad i = 1, \ldots, m,
\]
a system of linear algebraic equations is obtained. In the course of solving this system, coefficients \(p_i\) are determined, and thereby the approximated distributions of temperature in domain \(D\) and on boundary \(\Gamma_1\) are obtained.

4. NUMERICAL EXAMPLES

EXAMPLE 1. The theoretical considerations introduced in the previous sections will be illustrated with an example, in which \(\alpha = 0, \varphi(x) = e^{-x}, \xi(t) = t, u^* = 1, t^* = 1/2\). Then, an exact solution of the inverse Stefan problem will be found by means of the following functions,
\[
u(x, t) = e^{t-x}, \quad (x, t) \in D, \\
\vartheta(t) = e^t, \quad t \in [0, t^*].
\]
As basis functions, we take
\[ \psi_i(t) = t^{i-1}, \quad i = 1, \ldots, m. \] (4.3)

For the calculations, we assume \( m \in \{2, 4, 6, 8\} \) and \( n = 1 \). The approximate solution is compared with the exact solution. The values of the dimensionless absolute errors are calculated from formulas,
\[ \delta_\vartheta = \left( \frac{1}{t^*} \int_0^{t^*} (\vartheta_e(t) - \vartheta_r(t))^2 \, dt \right)^{1/2}, \] (4.4)
\[ \delta_u = \left( \frac{1}{|D|} \int_D (u_e(x,t) - u_n(x,t))^2 \, dx \, dt \right)^{1/2}, \] (4.5)

where \( \vartheta_e(t) \) is an exact distribution of dimensionless temperature on boundary \( \Gamma_1 \), \( \vartheta_r(t) \) is a reconstructed distribution of dimensionless temperature on this boundary (see (3.11)), \( u_e(x,t) \) is an exact distribution of dimensionless temperature in domain \( D \), \( u_n(x,t) \) is a reconstructed distribution of dimensionless temperature in this domain (see (3.10)), and
\[ |D| = \int_D 1 \, dx \, dt. \] (4.6)

However, percentage relative errors are calculated from formulas,
\[ \Delta_\vartheta = \frac{\delta_\vartheta}{\sqrt{\frac{1}{t^*} \int_0^{t^*} (\vartheta_e(t))^2 \, dt}} \cdot 100\%, \] (4.7)
\[ \Delta_u = \frac{\delta_u}{\sqrt{\frac{1}{|D|} \int_D (u_e(x,t))^2 \, dx \, dt}} \cdot 100\%. \] (4.8)

Figure 2. Distribution of dimensionless temperature on boundary \( \Gamma_1 \) (solid line, exact value; dash line, reconstructed value).
Table 1. Values of error in the reconstruction of the dimensionless temperature distribution.

<table>
<thead>
<tr>
<th></th>
<th>m = 2</th>
<th>m = 4</th>
<th>m = 6</th>
<th>m = 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>0.0149879</td>
<td>0.0160398</td>
<td>0.0120165</td>
<td>0.0112571</td>
</tr>
<tr>
<td>( \Delta \delta ) [%]</td>
<td>1.1433923</td>
<td>1.2236371</td>
<td>0.9167045</td>
<td>0.8587736</td>
</tr>
<tr>
<td>( \delta u )</td>
<td>0.0126190</td>
<td>0.0095735</td>
<td>0.0070887</td>
<td>0.0061800</td>
</tr>
<tr>
<td>( \Delta u ) [%]</td>
<td>1.0528404</td>
<td>0.7987475</td>
<td>0.5914351</td>
<td>0.5156187</td>
</tr>
</tbody>
</table>

Figure 3. Distribution of the dimensionless temperature on boundary \( \Gamma_1 \) (solid line, exact value; reconstructed value: \( \diamond \) for \( m = 2 \), \( \blacksquare \) for \( m = 3 \)).

In Figure 2, we presented the exact and reconstructed distribution of the dimensionless temperature on boundary \( \Gamma_1 \) for different number of basis functions \( \psi_i(t) \), \( i = 1, \ldots, m \), \( m \in \{2, 4, 6, 8\} \). The values of the errors are presented in Table 1. The obtained results show that functions \( \varphi(t) \) and \( u(x, t) \) are reconstructed very well.

**Example 2.** We now present a next example in which \( \alpha = 0 \), \( u^* = 0 \), \( t^* = 3/2 \), and

\[
\begin{align*}
\varphi(x) &= \exp \left( 1 - 2^{-1/2} (1 + x) \right) - 1, \\
\xi(t) &= 2^{-1/2} (t + 2 - \sqrt{2}).
\end{align*}
\]

(4.9) \hspace{2cm} (4.10)

Then, an exact solution of the inverse Stefan problem is given by

\[
\begin{align*}
u(x, t) &= \exp \left( 1 - 2^{-1/2} (1 + x) + t/2 \right) - 1, \quad (x, t) \in D, \\
\vartheta(t) &= \exp \left( 1 - 2^{-1/2} + t/2 \right) - 1, \quad t \in [0, t^*].
\end{align*}
\]

(4.11) \hspace{2cm} (4.12)

In Figure 3, we presented results for \( \psi_i(t) = e^{-it} \), \( n \in \{1, 2\} \), and \( m \in \{2, 3\} \). In case of longer time intervals, the error can be reduced by selecting numbers \( n \) and \( m \) or the kind of the basis functions.

5. CONCLUSION

In this paper, we presented a method for solving the problem which consists of the reconstruction of the function describing the distribution of temperature on the boundary, when the position of the moving interface is well known. The discussed method makes use of the Adomian decomposition method and the least square method. The derived calculations show that this method is effective for solving one-phase inverse Stefan problem.

REFERENCES


