Symbolic Computation of Some New Nonlinear Partial Differential Equations of Nanobiosciences Using Modified Extended Tanh-function Method

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Abstract – By means of computerized symbolic computation and a modified extended tanh-function method the multiple travelling wave solutions of nonlinear partial differential equations is presented and implemented in a computer algebraic system. Applying this method, we consider some of nonlinear partial differential equations of special interest in nanobiosciences and biophysics namely, the transmission line models of microtubules for nano-ionic currents. The nonlinear equations elaborated here are quite original and first proposed in the context of important nanosciences problems related with cell signaling. It could be even of basic importance for explanation of cognitive processes in neurons. As results, we can successfully recover the previously known solitary wave solutions that had been found by other sophisticated methods. The method is straightforward and concise, and it can also be applied to other nonlinear equations in physics.

Key Words: METF method; Soliton; Symbolic Computation; Microtubules; Nonlinear RLC Transmission Line; Travelling Wave Solution.

1. Introduction

Nonlinear phenomena play the crucial roles in applied mathematics and physics. Calculating exact and numerical solutions, in particular, travelling wave solutions of nonlinear equations in mathematical physics plays an important role in soliton theory [1]. Recently, it has become more interesting to obtain exact solutions of nonlinear partial differential equations by using symbolical computer programs such as Matlab, Maple, and Mathematica that facilitate complicated and tedious algebraical computations. It is also very important to find exact solutions of nonlinear partial differential equations. These equations are mathematical models of complex physical phenomena that arise in engineering, chemistry, biology, mechanics and applied physics. Various effective methods have been developed to understand the mechanisms of these physical models, to help engineers and physicists to ensure knowledge of very essence of underlying problems and facilitate their applications.

Several powerful methods for obtaining explicit travelling wave solitary solutions of nonlinear equations have been proposed such as: Hirota’s bilinear method [2], Painlevé expansions [3], the inverse scattering transform [2], homogeneous balance method [4, 5], F-expansion method [6] and Jacobi elliptic function method [7]. Recently, tanh-function method [8–12] has been proposed to find the exact solutions of nonlinear differential equations.
Further, Fan [13, 14] has proposed an extended tanh-function method yielding the new travelling wave solutions that cannot be obtained by tanh-function method. Most recently, El-Wakil [15, 16] and Soliman [17] modified extended tanh-function (METF) method and obtained some new exact solutions.

The rest of the Paper is organized as follows: In Section 2, a METF method is presented. In Section 3, we choose some new nonlinear differential equations of special interest in nanobiosciences which describe models of microtubules (MTs) as nonlinear RLC transmission lines to illustrate the validity and advantage of this method. Finally conclusion and discussion are given in Section 4.

2. Modified extended tanh-function (METF) method

To illustrate the basic concepts of the METF method [15, 16], we consider a given partial differential equation (PDE) with two variables given by the following expression

\[ F(u, u_x, u_t, u_{xx}, \ldots) = 0. \]  

When we look for its travelling wave solutions, the first step is to introduce the wave transformation \( u(x,t) = u(\xi), \xi = x \pm ct, \) (\( c \) is the wave velocity) and transform Eq. (1) to an ordinary differential equation (ODE)

\[ F(u, u', u'', u''', \ldots) = 0. \]  

The next crucial step is to introduce a new variable \( \phi = \phi(\xi) \) which is a solution of the Riccati equation

\[ \phi' = b + \phi^2, \]  

where \( b \) represents a parameter to be determined, \( \phi' = \frac{d\phi}{d\xi}. \)

Then we could use the following series expansion as a solution of Eq. (2):

\[ u(x,t) = u(\xi) = a_0 + \sum_{i=1}^{m} \left( a_i \phi^i + b_i \phi^{-i} \right), \]  

Where the positive integer \( m \) can be determined by balancing the highest-order derivative term with nonlinear terms in Eq. (2). Substituting Eqs. (3) and (4) into Eq. (2) and then equating with zero all coefficients of \( \phi' \), we can obtain a system of algebraic equations, from which the constants \( a_0, a_i, b_i, b \) and \( c \) could be obtained explicitly. Fortunately, the Riccati equation admits several types of solutions:

If \( b < 0 \)

\[ \phi = -\sqrt{-b} \tanh(\sqrt{-b} \xi), \]  

or \( \phi = -\sqrt{-b} \coth(\sqrt{-b} \xi), \) which depends on the initial conditions.

If \( b = 0 \)

\[ \phi = -\frac{1}{\xi}. \]

If \( b > 0 \)

\[ \phi = \sqrt{b} \tan(\sqrt{b} \xi), \]  

or \( \phi = -\sqrt{b} \coth(\sqrt{b} \xi), \) depending on the initial conditions.
3. Numerical applications

In this section, we will demonstrate this adopted approach on some new nonlinear partial differential equations of special interest in nanobiosciences, namely, the transmission line models for nano-ionic currents along MTs playing the important roles in cellular signalling.

MTs are cytoskeleton biopolymers shaped as nanotubes that are essential for cell motility, cell division, intracellular trafficking and information processing within neuronal processes. MTs have also been implicated in higher neuronal functions, including memory and the emergence of consciousness. How MTs handle and process electrical information, however, is heretofore still unknown. We established a new model for ionic waves along MTs based on polyelectrolyte features of cylindrical biopolymers. Each tubulin dimmer protein is an electric element with a capacitive, resistive and negative incrementally resistive property \[18\]. The particular attention was paid in \([19, 20]\) to the role of nano-pores (NPs) existing between neighbouring dimmers within a MT wall which exhibit properties like ionic channels. These NPs could be used to explain the behaviour of MTs as biomolecular transistors capable of amplifying electrical information in neurons.

3.1. Equation of nano-ionic currents along MTs

Let us first consider the nonlinear partial differential equation which describes a model of MT as nonlinear transmission line. In the contest of this model a MT is segmented into identical elementary rings (ERs). The physical details of derivation of following equation describing their ionic currents are elaborated in \([19]\):

\[
\frac{l^2}{3}u_{xxx} + \frac{Z^{3/2}}{l}(\chi G_0 - 2\delta C_0)uu_x + 2u_x + \frac{ZC_0}{l}u_x + \frac{1}{l}(RZ^{-1} - G_0 Z)u = 0, \\
\]

where \(R = 0.34 \times 10^9 \Omega\) is the resistance of the ER with length \(l = 8 \times 10^{-9}\) m, \(C_0 = 1.8 \times 10^{-15}\) F is the total maximal capacitance of the ER, \(G_0 = 1.1 \times 10^{-13}\) Si is conductance of pertaining NPs and \(Z = 5.56 \times 10^{10}\) Ω is the characteristic impedance of our system. Parameters \(\delta\) and \(\chi\) describe nonlinearity of ER capacitor and conductance of NPs in ER, respectively.

In order to solve Eq. (5) by the METF method, we use the travelling wave transformations \(u(x,t) = u(\xi)\) with dimensionless wave variable \(\xi = \frac{1}{l}x - \frac{t}{c}\), \(c\) is the dimensionless velocity of wave, and the characteristic time of charging ER capacitor is \(\tau = RC_0 = 0.6 \times 10^{-6}\) s. Thus Eq. (5) takes the form of ODEs

\[
u'' + Acuu' + (6 - Bc)u' + Cu = 0, \\
\]

with following dimensionless parameters:

\[
A = 3\frac{Z^{3/2}}{\tau}(2\delta C_0 - \chi G_0), \quad B = 3\frac{ZC_0}{\tau}, \quad C = 3(RZ^{-1} - G_0 Z).
\]

Now, we compare the first linear term of highest-order derivative with the second, highest-order nonlinear term. So, balancing the order of \(u''\) with the order of \(uu'\) in Eq. (6), we obtain

\[
m + 3 = m + m + 1 \quad \Rightarrow \quad m = 2.
\]

Thus, the solution of Eq. (6) takes the form

\[
u(\xi) = a_0 + a_1\phi(\xi) + b_1\phi(\xi)^{-1} + a_2\phi(\xi)^2 + b_2\phi(\xi)^{-2},
\]
where \( a_0, a_1, a_2, b_1, b_2 \) are to be determined and \( \phi(\xi) \) satisfies Eq. (3). Inserting Eq. (8) into Eq. (6) with the aid of Eq. (3), we get a system of algebraic equations for \( a_0, a_1, a_2, b_1, b_2, b \) and \( c \):

\[
\phi^0: \quad 2a_0 b^2 - 2a_0 b - 6a_0 b - 6b_1 + Aa_0 a_1 c - Aa_0 b_1 c - Aa_2 b_1 c + Aa_2 b_2 c + Aa_2 b_2 c + B a_1 + B a_2 b + C a_1 = 0, \\
\phi^1: \quad 16a_2 b^2 + 12a_2 b + 2A a_0 a_2 b + A a_0 a_2 b + A a_1 c - 2Ba_2 c + C a_1 = 0, \\
\phi^{-1}: \quad -16a_2 b - 12a_2 - 2A a_0 b_2 c - Ab_1^2 c + 2B b_2 c + C b_1 = 0, \\
\phi^2: \quad 8a_2 b + 6a_1 + A a_0 a_1 c + 3A a_0 a_2 b + A a_2 b c - Ba_2 c + C a_2 = 0, \\
\phi^{-2}: \quad -8a_2 b^2 - 6a_2 b - A a_0 b_1 c - A a_2 b_1 c + 3A b_1 c + B b_2 c + C b_2 = 0, \\
\phi^3: \quad 40a_2 b + 12a_2 + 2A a_0 a_2 c + A a_1 c + 2A a_2 c - 2B a_2 c = 0, \\
\phi^{-3}: \quad -40a_2 b^2 - 2A a_0 b_2 c - Ab_1^2 c - 2A b_2 c + 2B b_2 c = 0, \\
\phi^4: \quad 6a_1 + 3A a_1 a_2 c = 0, \\
\phi^{-4}: \quad -6a_2 b^3 - 3A b_1 b_2 c = 0, . \\
\phi^5: \quad 24a_2 + 2A a_2^2 c = 0, . \\
\phi^{-5}: \quad -24a_2 b^3 - 2A b_2^2 b c = 0. (9)
\]

Solving the above set of equations with the aid of Matlab, we can distinguish different cases, as follows:

Case I:
\[ a_0 = a_0, a_1 = 0, a_2 = a_2, b_1 = 0, b_2 = 0, c = \frac{12}{A a_2}, b = \frac{3}{4a_2} \left( 2a_0 - a_2 - \frac{B}{A} \right) \]
with \( a_0 \) and \( a_2 \) being arbitrary constants.

According to the value of \( b \) (\( b < 0 \)), we obtain the exact travelling wave solution in the following form

\[
u(x,t) = a_0 - \frac{3}{4} \left( 2a_0 - a_2 - \frac{2B}{A} \right) \tanh^2 \left( \frac{3}{4a_2} \left( 2a_0 - a_2 - \frac{2B}{A} \right) \left( \frac{1}{t} x - \left( \frac{12}{A a_2} \right) \frac{1}{\tau} \right) \right). \]

The solitary wave and behavior of the solution \( \nu(x,t) \) is shown in Figure 1 for some fixed values of the parameters \( a_0 \) and \( a_2 \). In the case b) we have slowly moving localized pulse with constant amplitude and with plausible physical interpretation, while in the case a) the corresponding pulse is spreading and getting very asymmetric.

For the special choice of constants \( a_0 = 1 \), and \( a_2 = 2(2a_0 - a_2 = 0) \), we could discuss the properties of solution Eq. (10). The solitonic amplitude, \( \frac{3}{2} \frac{B}{A} = \frac{3}{2} \frac{C_0}{Z^{1/2}} \left( 2\delta C_0 - \chi G_0 \right) \), is proportional to ER capacitance \( C_0 \). It also increases when the capacitive nonlinearity approaches to nonlinear conductance. At the same time, the inverse width of soliton is proportional to \( C_0 \). The solitonic speed increases with decreasing difference between nonlinear capacitance and conductance.

It means that soliton with greater amplitude moves faster, which is typical solitonic property. The values of constants \( a_0 \) and \( a_2 \) are expected to be determined by initial conditions, eg. the input ionic influx.
Figure 1. The soliton solution $u(x, t)$, where a) $a_0=1$, $a_2=1.17$ and b) $a_0=1$, $a_2=2$.

Due to case I, with $b > 0$ we get

$$u(x, t) = a_0 + \frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \tan^2 \left( \frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \left( \frac{1}{l} x - \left( -12 \frac{c}{Aa_2} \right) t \right) \right).$$  (11)

The plots for the solution $u(x, t)$ are given in Figure 2, for some fixed values of the parameters $a_0$ and $a_2$. Here we see the static kinks meaning that strong nonlinearity term ($A$ much greater than $B$) leads to localization of ionic pulse. This solution does not have proper physical interpretation since the values of voltage (current) tend to infinity.

Case II:

$$a_0 = a_0, a_1 = 0, a_2 = 0, b_1 = 0, b_2 = -24 \frac{b^2(4b+3)}{B - Aa_0}, c = \frac{8b + 6}{B - Aa_0}, b = b, \text{ with } a_0 \text{ and } b \text{ being arbitrary constants.}$$

Pertaining to the value of $b$ ($b < 0$), we the exact travelling wave solution has the following form

$$u(x, t) = a_0 + 24 \frac{b(4b+3)}{B - Aa_0} \coth^2 \left( \sqrt{-b} \left( \frac{1}{l} x - \frac{8b + 6}{B - Aa_0} \frac{t}{\tau} \right) \right).$$  (12)

Due to case II, with $b > 0$ we get

$$u(x, t) = a_0 - 24 \frac{b(4b+3)}{B - Aa_0} \cot^2 \left( \sqrt{b} \left( \frac{1}{l} x - \frac{8b + 6}{B - Aa_0} \frac{t}{\tau} \right) \right).$$  (13)
Case III:

\[ a_0 = a_0, a_1 = 0, a_2 = a_2, b_1 = 0, b_2 = \frac{9}{16a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right)^2, \]
\[ c = -\frac{12}{Aa_2}, b = \frac{3}{4a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right), \]  
with \( a_0 \) and \( a_2 \) being arbitrary constants.

The exact travelling wave solution for the condition \( b < 0 \) now reads

\[ u(x, t) = a_0 - \frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \tanh^2 \left( -\frac{3}{4a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \left( \frac{1}{l} x - \left( -\frac{12}{Aa_2} \right) \frac{t}{\tau} \right) \right) - \]
\[ -\frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \coth^2 \left( -\frac{3}{4a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \left( \frac{1}{l} x - \left( -\frac{12}{Aa_2} \right) \frac{t}{\tau} \right) \right) \]  
(14)

For the case III, with \( b > 0 \) we get new solution possessing the singularity

\[ u(x, t) = a_0 + \frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \tan^2 \left( -\frac{3}{4a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \left( \frac{1}{l} x - \left( -\frac{12}{Aa_2} \right) \frac{t}{\tau} \right) \right) + \]
\[ +\frac{3}{4} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \cot^2 \left( -\frac{3}{4a_2} \left( 2a_0 - a_2 - 2 \frac{B}{A} \right) \left( \frac{1}{l} x - \left( -\frac{12}{Aa_2} \right) \frac{t}{\tau} \right) \right) \]  
(15)

The solutions of cases II and III also exhibit nonphysical features since ionic currents could tend to infinity.

### 3.2. Equation of MTs as nonlinear RLC transmission line

A second instructive example is also connected with modelling of MTs as nonlinear RLC transmission line.

\[ R_2 C_0 l^2 u_{xx} + l^2 u_{xx} + 2R_1 C_0 \delta u_t - R_1 C_0 u_t = 0. \]  
(16)
The details of this model are elucidated in [18]. Here \( l = 8 \times 10^{-9} \) m and \( C_0 = 1.32 \times 10^{15} \) F have the same meanings as in Eq. (5), while \( R_1 = 10^9 \) \( \Omega \) and \( R_2 = 7 \times 10^6 \) \( \Omega \) stand for longitudinal and transversal component of resistance of an ER. Again, parameter \( \delta (\delta < 1) \) describes nonlinearity of ER capacitor in MT.

In order to solve Eq. (16) by the METF method, we use the travelling wave transformations \( u(x,t) = u(\xi) \) with dimensionless velocity \( \xi = l x - c t \), \( c \) is the dimensionless velocity of wave, and the characteristic time of charging ER capacitor is \( \tau = R_1 C_0 = 1.32 \times 10^{-6} \) s. Thus Eq. (16) takes the form of ODEs

\[
\frac{d^2 u}{d\xi^2} - \frac{\alpha}{c} \frac{du}{d\xi} + \beta uu' - \gamma u' = 0, \tag{17}
\]

with dimensionless parameters:

\[
\alpha = \frac{\tau}{R_1 C_0}, \quad \beta = \frac{2\delta R_1}{R_2}, \quad \gamma = \frac{R_1}{R_2}.
\]

Integrating Eq (17) once we get

\[
\frac{d^2 u}{d\xi^2} - \frac{\alpha}{c} \frac{du}{d\xi} - \gamma u + \beta u^2 = 0. \tag{18}
\]

Now, we compare the derivative term of highest-order with the highest-order nonlinear term. So, balancing the order of \( \frac{d^2 u}{d\xi^2} \) with the order of \( u^2 \) in Eq. (18), we obtain

\[
m + 2 = 2m \quad \Rightarrow \quad m = 2.
\]

So the solution takes the form

\[
u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \phi(\xi)^{-1} + a_2 \phi(\xi)^2 + b_2 \phi(\xi)^{-2}, \tag{20}
\]

where \( a_0, a_1, a_2, b_1, b_2 \) are to be determined and \( \phi(\xi) \) satisfies Eq. (3). Inserting Eq. (20) into Eq. (18) with the aid of Eq. (3), we get a system of algebraic equations for \( a_0, a_1, a_2, b_1, b_2, b \) and \( c \):

\[
\phi^0 : \quad 2a_2b^2 + 2b_2 - \frac{\alpha}{c} a_1 b + \frac{\alpha}{c} b_1 + \beta a_0^2 + 2\beta a_1 b_1 + 2\beta a_2 b_2 - \gamma a_0 = 0,
\]

\[
\phi^1 : \quad 2a_1 b - 2\frac{\alpha}{c} a_2 b + 2\beta a_0 a_1 + 2\beta a_1 b_1 - \gamma a_1 = 0,
\]

\[
\phi^{-1} : \quad 2b_1 b + 2\frac{\alpha}{c} b_2 + 2\beta a_0 b_1 + 2\beta a_1 b_2 - \gamma a_1 = 0,
\]

\[
\phi^2 : \quad 8a_2 b - \frac{\alpha}{c} a_1 + \beta a_1^2 + 2\beta a_0 a_2 - \gamma a_2 = 0,
\]

\[
\phi^{-2} : \quad 8b_1 b + \frac{\alpha}{c} b_2 + \beta a_1^2 + 2\beta a_1 b_2 - \gamma b_2 = 0,
\]

\[
\phi^3 : \quad 2a_1 - 2\frac{\alpha}{c} a_2 + 2\beta a_1 a_2 = 0,
\]

\[
\phi^{-3} : \quad 2b_1 b^2 + 2\frac{\alpha}{c} b_2 + 2\beta b_1 b_2 = 0,
\]

\[
\phi^4 : \quad 6a_2 + \beta a_1^2 = 0,
\]

\[
\phi^{-4} : \quad 6b_1 b^2 + \beta b_2^2 = 0.
\]
Solving the above set of equations with the aid of Matlab, we can distinguish several different cases, as follows:

Case I:
\[ a_0 = \frac{3}{4} \gamma \beta, \quad a_1 = a, \quad a_2 = -\frac{6}{\beta}, \quad b_1 = 0, \quad b_2 = 0, \quad c = \frac{6}{5} \beta a_1, \quad b = -\frac{1}{24} \gamma, \quad \text{with } a \text{ being arbitrary constant.} \]

According to the value of \( b (b < 0) \), we obtain the exact travelling wave solution in the following form
\[ u(x, t) = \frac{3}{4} \gamma \beta \sqrt{\frac{24}{\gamma}} \tanh \left( \sqrt{\frac{1}{24} \gamma \left( \frac{1}{l} x - \frac{6}{5} \beta a_1 \sqrt{\gamma} \right)} \right) - \frac{4}{\gamma} \tanh^3 \left( \sqrt{\frac{1}{24} \gamma \left( \frac{1}{l} x - \frac{6}{5} \beta a_1 \sqrt{\gamma} \right)} \right). \quad (22) \]

The solitary wave time-space behavior of the solution \( u(x, t) \), is shown in Figure 3, for two fixed values of the parameter \( a_1 \). These solutions are physically very reasonable representing localized and very stable ionic pulses reaching long distances with constant velocity.

The constant \( a_1 \) affects the solitonic amplitude and its speed. It is plausible that this constant arises from initial physical conditions depending on the amount of ions injected along MT from corresponding ionic channels of cell membrane. The solitonic speed is determined by the ratio \( \frac{6}{5} \beta a_1 \tau = \frac{3}{5} \frac{1}{\delta a_1 R_1 C_0} \). It means that greater nonlinearity \( \delta \), and greater longitudinal resistance \( R_1 \), as well as, capacitance \( C_0 \) diminish the solitonic speed. The ratio \( \frac{\gamma}{\beta} = \frac{1}{2 \delta} \) impacts the solitonic amplitude through first and third term of Eq. (22). Eventually, the resistivity ratio \( \gamma = \frac{R_1}{R_2} \equiv 143 \) determines the width of soliton through \( \sqrt{\frac{\gamma}{24}} \equiv 2.45 \). This result shows that soliton, Eq. (22), is highly localized, practically on the single ER.

Figure 3. The soliton solution \( u(x, t) \), where a) \( a_1 = 0.8 \) and b) \( a_1 = 40 \).
Case II:

\[ a_0 = \frac{1}{4} \beta, a_1 = a_2 = -\frac{6}{5} \beta, b_1 = 0, b_2 = 0, \quad c = \frac{6}{5} \beta \alpha, b = \frac{1}{24} \gamma, \text{ with } a_i \text{ being arbitrary constant.} \]

Considering the value of \( b \) \( (b > 0) \), we obtain the exact travelling wave solution in the following form

\[ u(x, t) = \frac{1}{4} \beta + a_1 \sqrt{\frac{1}{24} \gamma} \tan \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{6}{5} \beta \alpha \gamma \right) \right) - \frac{1}{24} \gamma \tan^2 \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{6}{5} \beta \alpha \gamma \right) \right). \quad (23) \]

The plots for the solution \( u(x, t) \) are given in Figure 4, for some fixed values of the parameter \( a_i \). Similarly as in the case of Eq. 6 here static kink solutions arise. These solutions do not sustain the physical phenomena since the ionic current does not tend to infinity.

![Figure 4](image)

**Figure 4.** The solution \( u(x, t) \), where a) \( a_1 = 0.8 \) and b) \( a_1 = 40 \).

Case III:

\[ a_0 = \frac{3}{4} \beta, a_1 = 0, a_2 = 0, b_1 = b_2 = -\frac{1}{96} \beta^2, c = \frac{1}{20} \beta \alpha, b = -\frac{1}{24} \gamma, \text{ with } b_i \text{ being arbitrary constant.} \]

For the value of \( b \) \( (b < 0) \), the corresponding singular solution has the following form

\[ u(x, t) = \frac{3}{4} \beta - b_1 \sqrt{\frac{24}{\gamma}} \coth \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{1}{20} \beta \alpha \gamma \right) \right) + \frac{1}{24} \gamma \coth^2 \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{1}{20} \beta \alpha \gamma \right) \right). \quad (24) \]

Case IV:

\[ a_0 = \frac{1}{4} \beta, a_1 = 0, a_2 = 0, b_1 = b_2 = -\frac{1}{96} \beta^2, c = -\frac{1}{20} \beta \alpha, b = \frac{1}{24} \gamma, \text{ with } b_i \text{ being arbitrary constant.} \]

For the positive values of constant \( b \) \( (b > 0) \), the exact travelling wave solution is expressed as follows

\[ u(x, t) = \frac{1}{4} \beta + b_1 \sqrt{\frac{24}{\gamma}} \cot \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{1}{20} \beta \alpha \gamma \right) \right) - \frac{1}{4} \gamma \cot^2 \left( \frac{1}{24} \gamma \left( \frac{1}{\tau} x - \frac{1}{20} \beta \alpha \gamma \right) \right). \quad (25) \]

Case V:
\[ a_0 = \frac{5}{8} \gamma, a_1 = a_2 = -\frac{6}{\beta}, b_1 = -\frac{\gamma}{96} a_1, b_2 = \frac{1}{1536} \gamma^2, c = \frac{6}{5 \beta a_1}, b = -\frac{1}{96} \gamma, \text{ with } a_i \text{ being arbitrary constant.} \]

Taking the value of \( b \) (\( b < 0 \)), we obtain the exact travelling wave solution in the form containing singularities

\[ u(x,t) = \frac{5}{8} \gamma - a_1 \sqrt{\frac{1}{96} \gamma} \tanh \left( \frac{1}{96} \gamma \left( \frac{1}{l} x - \frac{6 \alpha}{5 \beta a_1} \tau \right) \right) + a_1 \sqrt{\frac{1}{96} \gamma} \coth \left( \frac{1}{96} \gamma \left( \frac{1}{l} x - \frac{6 \alpha}{5 \beta a_1} \tau \right) \right) \]

Case VI:

\[ a_0 = \frac{5}{8} \gamma, a_1 = a_2 = -\frac{6}{\beta}, b_1 = -\frac{\gamma}{96} a_1, b_2 = -\frac{1}{1536} \gamma^2, c = \frac{6}{5 \beta a_1}, b = \frac{1}{96} \gamma, \text{ with } a_i \text{ being arbitrary constant.} \]

According to the value of \( b \) (\( b > 0 \)), we obtain the exact travelling wave solution in the following form

\[ u(x,t) = \frac{5}{8} \gamma + a_1 \sqrt{\frac{1}{96} \gamma} \tan \left( \frac{1}{96} \gamma \left( \frac{1}{l} x - \frac{6 \alpha}{5 \beta a_1} \tau \right) \right) - a_1 \sqrt{\frac{1}{96} \gamma} \cot \left( \frac{1}{96} \gamma \left( \frac{1}{l} x - \frac{6 \alpha}{5 \beta a_1} \tau \right) \right) \]

The solutions Eqs. (26) and (27) are mathematically the most general, but physically intractable, due to the presence of singularities.

4. Conclusions and discussion

In this Paper, the METF method has been successfully applied to find the solution of nonlinear partial differential equation such as the nonlinear RLC transmission line model of MTs and model of nano-ionic currents along MTs. By applying this method we have obtained new travelling wave solutions of two models of special interest in nanobiosciences and biophysics. The obtained solutions include solitons, kinks and plane periodic solutions. We found new exact solutions that are not obtained by an extended tanh-function method, hyperbolic function method. It is worth noting that the new solutions obtained by means of METF method confirm the correctness of those obtained by other methods. The results show that the METF method is a powerful mathematical tool, straightforward and concise, and it can also be applied to other nonlinear evolution equations in physics.

In this work, we presented a METF method based on the general ansatz (4) in which the exponent of tanh function may take both positive and negative values on the contrary to the solution ansatz of tanh function method where its exponent is only positive values. This, of course, leads to the conclusion that the method of extended tanh method with the ansatz (4) can be used to improve the tanh function method.

The physically acceptable solutions, Eq. (10) and E. (22), are interpreted in the context of relevant biophysical parameters already available. These equations contain the respective simpler solutions obtained by direct integration done in papers [18, 19].
Finally, it is worthwhile to mention that the proposed method is reliable and effective and gives more solutions. The applied method will be used in further works to establish more entirely new solutions for other kinds of nonlinear equations.

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Glossary

ER – elementary ring
METF – modified extended tanh-function
MT – microtubule
NP – nano-pore
ODE – ordinary differential equation
PDE – partial differential equation

References


