Weight distribution of the crown-weight space

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Abstract

We derive a formula that gives the weight distribution of finite-dimensional vector spaces over a finite field equipped with crown-weight.
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1. Introduction

Brualdi et al. [1] were led to introduce the notions of poset-weight, poset-metric and a poset-code. That was due to an attempt at further generalizing Niederreiter’s generalization [3–5] for finding a linear code over the finite field $\mathbb{F}_q$ of given length and dimension having the maximum possible minimum distance.

The vector space $\mathbb{F}_q^n$ over $\mathbb{F}_q$ equipped with a $P$-weight $w_P$ (cf. Section 2, for details) will be called a $P$-weight space and is denoted by $(\mathbb{F}_q^n, w_P)$. In this paper, we will consider the case where $P$ is the crown $CR = CR_{2m}$ (cf. Fig. 1) and derive a formula giving the $CR$-weight distribution of the crown-weight space $(\mathbb{F}_q^{2m}, w_{CR})$. For each integer $j$ with $0 \leq j \leq 2m$, let $A_{CR,j}$ denote the number of vectors $u$ in $\mathbb{F}_q^{2m}$ with $w_{CR}(u) = j$, so that $\{A_{CR,j}\}_{j=0}^{2m}$ is the $CR$-weight distribution of $(\mathbb{F}_q^{2m}, w_{CR})$. Then our result (cf. Theorem 4.4)

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Fig. 1. Crown $CR = CR_{2m}$.

\[ A_{CR,j} = \frac{1}{(2m - j)!} \sum \binom{2m - j}{i_1 i_2 \ldots i_m} \text{Tr} M^{(i_1)} M^{(i_2)} \cdots M^{(i_m)}, \]

where the sum runs over all integers $i_1, i_2, \ldots, i_m$ satisfying $0 \leq i_1, \ldots, i_m \leq 2$, $i_1 + \cdots + i_m = 2m - j$, $\binom{2m - j}{i_1 i_2 \ldots i_m}$ are the usual multinomial coefficients, $\text{Tr}$ is the usual trace of a square matrix, and

\[ M^{(0)} = \begin{bmatrix} 0 & 0 \\ (q - 1)q & (q - 1)q \end{bmatrix}, \quad M^{(1)} = \begin{bmatrix} q - 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \]

Using this formula, one easily computes, for example, that the $CR$-weight distribution of $(F_2^6, w_{CR})$ (i.e., $q = 2, m = 3$) is $(w_{CR,0}, w_{CR,1}, \ldots, w_{CR,6}) = (1, 3, 3, 13, 12, 24, 8)$ (cf. the Example in Section 4). In fact, we will derive a more general formula that expresses the $CR$-weight distribution of the dual code $C^\perp$ of a linear code $C$ in terms of $C$ (cf. Theorem 4.3).

One technical point in our derivation of the formula is computing the Fourier transform of the $CR$-weight function $g(u) = x^{2m - w_{CR}(u)} y^{w_{CR}(u)}$. It turns out that $\hat{g}(u)$ can be expressed as the trace of a product of $m$ $2 \times 2$ matrices with polynomial entries depending on $u$. Then our formula will follow immediately by using a discrete Poisson summation formula and elementary calculus.

2. Preliminaries

Let $F_q$ be the finite field with $q$ elements. Then, for $u = (u_1, \ldots, u_n) \in F_q^n$, the support of $u$ and the Hamming weight of $u$ are, respectively, given by

\[ \text{Supp}(u) = \{i|1 \leq i \leq n, u_i \neq 0\}, \]

\[ w(u) = |\text{Supp}(u)|. \]

Let $P$ be a poset on the underlying set

\[ [n] = \{1, 2, \ldots, n\} \]

of coordinate positions of vectors in $F_q^n$. Then, for any such a poset $P$, the $P$-weight $w_P(u)$ of $u \in F_q^n$ is defined in [1] to be
\[ w_P(u) = |\langle \text{Supp}(u) \rangle|, \]

where \( \langle \text{Supp}(u) \rangle \) denotes the smallest ideal containing \( \text{Supp}(u) \) (recall that a subset \( I \) of \([n]\) is an ideal if \( a \in I \) and \( b < a \Rightarrow b \in I \)). Then one shows that \( w_P(u) \geq 0 \) with equality if and only if \( u = 0 \) and \( w_P(u + v) = w_P(u) + w_P(v) \), for every \( u, v \in \mathbb{F}_q^n \), i.e., \( d_P(u, v) = w_P(u - v) \) is a metric, called the \( P \)-metric. The space \( \mathbb{F}_q^n \) equipped with \( w_P \), denoted by \( (\mathbb{F}_q^n, w_P) \), is called the \( P \)-weight space. If \( P \) is an antichain, \( w_P = w \) is the Hamming weight and \( (\mathbb{F}_q^n, w) \) is the Hamming (weight) space.

From now on, \( CR = CR_{2m} \) will denote the crown in Fig. 1. Explicitly, \( CR = CR_{2m} \) is the poset whose underlying set and order relations are given by

\[ [2m] = \{1, 2, \ldots, 2m - 1, 2m\}, \]
\[ 1 < m + 1, 1 < 2m, \text{ and } i < m + i, i < m + i - 1, \text{ for each } i = 2, 3, \ldots, m. \quad (2.1) \]

Let \( \delta \) be the Kronecker delta function on \( \mathbb{F}_q \), so that, for \( x \in \mathbb{F}_q \),

\[ \delta(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \]

Then, by inspection, one notes that the \( CR \)-weight \( w_{CR}(u) \) of \( u \in \mathbb{F}_q^{2m} \) is given by the following formulas.

**Proposition 2.1.** Let \( CR = CR_{2m} \) be the crown with \( 2m \) elements (cf. Fig. 1). Then, for \( u = (u_1, \ldots, u_{2m}) \in \mathbb{F}_q^{2m} \), its \( CR \)-weight \( w_{CR}(u) \) is given by each of the following two formulas:

(a)

\[ w_{CR}(u) = w(u) + \delta(u_1)(1 - \delta(u_{m+1})\delta(u_{2m})) + \sum_{j=2}^{m} \delta(u_j)(1 - \delta(u_{m+j-1})\delta(u_{m+j})), \]

(b)

\[ w_{CR}(u) = \sum_{j=1}^{m} (1 - \delta(u_{m+j})) + (1 - \delta(u_1)\delta(u_{m+1})\delta(u_{2m})) + \sum_{j=2}^{m} (1 - \delta(u_j)\delta(u_{m+j-1})\delta(u_{m+j})). \]

Here \( w(u) \) denotes the Hamming weight of \( u \).

**Proof.** See [2], Proposition 2.1. \( \square \)

Let \( \lambda \) be a fixed nontrivial additive character of \( \mathbb{F}_q \). Then, for a function \( f \) from \( \mathbb{F}_q^n \) to an algebra over \( \mathbb{C} \), its Fourier transform is defined to be

\[ \hat{f}(u) = \sum_{v \in \mathbb{F}_q^n} f(v)\lambda(u \cdot v), \quad (2.2) \]

(cf. (2.4)).
For a linear code $C$ of length $n$ over $\mathbb{F}_q$ and such a function $f$ on $\mathbb{F}_q^n$, the discrete Poisson summation formula says

$$\sum_{u \in C^\perp} f(u) = \frac{1}{|C|} \sum_{u \in C} \hat{f}(u),$$

(2.3)

where the orthogonal complement $C^\perp$ of $C$ is with respect to the usual inner product on $\mathbb{F}_q^n$ given by

$$u \cdot v = \sum_{i=1}^n u_i v_i,$$  
(2.4)

for $u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$.

3. The Fourier transform of the CR-weight function

This section will be devoted to the derivation of the Fourier transform of the CR-weight function $g$ defined by

$$g(v) = x^{2m - w_{CR}(v)} y^{w_{CR}(v)}.$$

For $u = (u_1, \ldots, u_{2m}), v = (v_1, \ldots, v_{2m}) \in \mathbb{F}_q^{2m}$, and from the expression for $w_{CR}(v)$ in (b) of Proposition 2.1, the Fourier transform $\hat{g}$ of $g$ (cf. (2.2)) can be written as

$$\hat{g}(u) = \prod_{j=1}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j})} y^{1-\delta(v_{m+j})} \right)$$

$$\times \sum_{v_1 \in \mathbb{F}_q} \lambda(u_1 v_1) x^{\delta(v_1)\delta(v_{m+1})\delta(v_{2m})} y^{1-\delta(v_1)\delta(v_{m+1})\delta(v_{2m})}$$

(3.1)

$$\times \prod_{j=2}^{m} \left( \sum_{v_j \in \mathbb{F}_q} \lambda(u_j v_j) x^{\delta(v_j)\delta(v_{m+j-1})\delta(v_{m+j})} y^{1-\delta(v_j)\delta(v_{m+j-1})\delta(v_{m+j})} \right).$$  
(3.2)

Define the function $\rho : \mathbb{F}_q \rightarrow \mathbb{C}$ by

$$\rho(u) = \begin{cases} -1, & \text{if } u \neq 0, \\ q - 1, & \text{if } u = 0. \end{cases}$$

Then, dividing it into subsums with $v_1 = 0$ and all $v_1 \neq 0$, the sum in (3.1) is

$$y\rho(u_1) + x^{\delta(v_{m+1})\delta(v_{2m})} y^{1-\delta(v_{m+1})\delta(v_{2m})}.$$

Similarly, each sum in (3.2) is

$$y\rho(u_j) + x^{\delta(v_{m+j-1})\delta(v_{m+j})} y^{1-\delta(v_{m+j-1})\delta(v_{m+j})},$$

for $j$ with $2 \leq j \leq m$. Hence we have

$$\hat{g}(u) = \prod_{j=1}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j})} y^{1-\delta(v_{m+j})} \right)$$

$$\times \prod_{j=2}^{m} (y\rho(u_j) + x^{\delta(v_{m+j-1})\delta(v_{m+j})} y^{1-\delta(v_{m+j-1})\delta(v_{m+j})})$$
Now, we put
\[ \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j}) y^{1-\delta(v_{m+j})}} \]
\[ \lambda(u_{m+1} v_{m+1}) x^{\delta(v_{m+1}) y^{1-\delta(v_{m+1})}} \]
\[ \lambda(u_{m+2} v_{m+2}) x^{\delta(v_{m+2}) y^{1-\delta(v_{m+2})}} \]
\[ \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j}) y^{1-\delta(v_{m+j})}} \]
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\[ \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j}) y^{1-\delta(v_{m+j})}} \]
Again, dividing it into subsums with \( v_{m+1} = 0 \) and all \( v_{m+1} \neq 0 \), the sum in (3.4) is seen
to be equal to
\[ x(y \rho(u_1) + y^{\delta(v_{m+2})} y^{1-\delta(v_{m+2})}) + y^{2}(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}). \]

Now, we put
\[ x(y \rho(u_1) + y^{\delta(v_{m+2})} y^{1-\delta(v_{m+2})}) = A_1, \]
\[ y^{3}(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) = B_1. \]
Combining (3.3) and (3.5), using the notation in (3.6), and extracting one factor from each
product in (3.3), we obtain the following expression for \( g(u) \):
\[ g(u) = A_1 \prod_{j=3}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j}) y^{1-\delta(v_{m+j})}} \right) \]
\[ \times \prod_{j=4}^{m} (y \rho(u_j) + y^{\delta(v_{m+j-1})} y^{1-\delta(v_{m+j-1})} y^{1-\delta(v_{m+j})}) \]
\[ \times \left( \sum_{v_{m+2} \in \mathbb{F}_q} \lambda(u_{m+2} v_{m+2}) x^{\delta(v_{m+2}) y^{1-\delta(v_{m+2})}} \right) \]
\[ \times (y \rho(u_3) + y^{\delta(v_{m+2})} y^{1-\delta(v_{m+2})}) \]
\[ \times (y \rho(u_2) + y^{\delta(v_{m+2})} y^{1-\delta(v_{m+2})}) \]
\[ + B_1 \prod_{j=3}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j}) y^{1-\delta(v_{m+j})}} \right) \]
\[ \times \prod_{j=4}^{m} (y \rho(u_j) + y^{\delta(v_{m+j-1})} y^{1-\delta(v_{m+j-1})} y^{1-\delta(v_{m+j})}) \]
Proceeding just as before, we see that the sums in (3.7) and (3.8) are respectively given by

\[
\begin{align*}
\sum_{j=4}^{m} (y \rho(u_j) + x \delta(v_{m+j-1}) y^{1-\delta(v_{m+j-1})} \\
&\times \left\{ (y \rho(u_3) + x) (y \rho(u_3) + x \delta(v_{m+3}) y^{1-\delta(v_{m+3})}) \\
&\quad + y^3 (\rho(u_2) + 1) (\rho(u_3) + 1) \rho(u_{m+2}) \right\} \\
&+ B_1 \prod_{j=3}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j})} y^{1-\delta(v_{m+j})} \right) \\
&\times \left\{ x (y \rho(u_2) + x) (y \rho(u_3) + x \delta(v_{m+3}) y^{1-\delta(v_{m+3})}) \\
&\quad + y^3 (\rho(u_2) + 1) (\rho(u_3) + 1) \rho(u_{m+2}) \right\}.
\end{align*}
\]

We extract one factor from each product in (3.11), rearrange the terms and sum over \(v_{m+3} \in \mathbb{F}_q\). Then we have

\[
\hat{g}(u) = A_2 \prod_{j=4}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j})} y^{1-\delta(v_{m+j})} \right) \\
\times \left\{ x (y \rho(u_2) + x) (y \rho(u_3) + x \delta(v_{m+4}) y^{1-\delta(v_{m+4})}) \\
&\quad + y^3 (\rho(u_2) + 1) (\rho(u_4) + 1) \rho(u_{m+3}) \right\} \\
&+ B_2 \prod_{j=4}^{m} \left( \sum_{v_{m+j} \in \mathbb{F}_q} \lambda(u_{m+j} v_{m+j}) x^{\delta(v_{m+j})} y^{1-\delta(v_{m+j})} \right)
\]
Rearranging the terms in (3.12), we obtain

\[
\prod_{j=5}^{m} (y\rho(u_j) + x^\delta(v_{m+j-1})^\delta(v_{m+j})^1 - \delta(v_{m+j})^1 - \delta(v_{m+j})),
\]

\[
\times \{x(y\rho(u_4) + x^\delta(v_{m+4})^1 - \delta(v_{m+4})) + y^2(\rho(u_4) + 1)\rho(u_{m+3})\},
\]

where we put

\[
A_1x(y\rho(u_2) + x) + B_1x = A_2,
A_1y^3(\rho(u_2) + 1)(\rho(u_3) + 1)\rho(u_{m+2}) + B_1y^2(\rho(u_3) + 1)\rho(u_{m+2}) = B_2.
\]

Proceeding in this fashion, we arrive at the following expression for \(\hat{g}(u)\):

\[
\hat{g}(u) = A_{m-2} \sum_{v_{2m} \in \mathbb{F}_q^2} \lambda(u_{2m}v_{2m})x^\delta(v_{2m})^1 - \delta(v_{2m})
\]

\[
\times \{x(y\rho(u_{m-1}) + x)(y\rho(u_m) + x^\delta(v_{2m})^1 - \delta(v_{2m})) + y^3(\rho(u_{m-1}) + 1)(\rho(u_m) + 1)\rho(u_{2m-1})\}
\]

\[
+ B_{m-2} \sum_{v_{2m} \in \mathbb{F}_q^2} \lambda(u_{2m}v_{2m})x^\delta(v_{2m})^1 - \delta(v_{2m})
\]

\[
\times \{x(y\rho(u_m) + x^\delta(v_{2m})^1 - \delta(v_{2m})) + y^2(\rho(u_m) + 1)\rho(u_{2m-1})\},
\]

where

\[
A_{m-3}x(y\rho(u_{m-2}) + x) + B_{m-3}x = A_{m-2},
A_{m-3}y^3(\rho(u_{m-2}) + 1)(\rho(u_{m-1}) + 1)\rho(u_{2m-2})
\]

\[
+ B_{m-3}y^2(\rho(u_{m-1}) + 1)\rho(u_{2m-2}) = B_{m-2}.
\]

Rearranging the terms in (3.12), we obtain

\[
\hat{g}(u) = \sum_{v_{2m} \in \mathbb{F}_q^2} \lambda(u_{2m}v_{2m})x^\delta(v_{2m})^1 - \delta(v_{2m}) (y\rho(u_m) + x^\delta(v_{2m})^1 - \delta(v_{2m})) A_{m-1}
\]

\[+(\sum_{v_{2m} \in \mathbb{F}_q^2} \lambda(u_{2m}v_{2m})x^\delta(v_{2m})^1 - \delta(v_{2m}) B_{m-1},
\]

where

\[
A_{m-2}x(y\rho(u_{m-1}) + x) + B_{m-2}x = A_{m-1},
A_{m-2}y^3(\rho(u_{m-1}) + 1)(\rho(u_m) + 1)\rho(u_{2m-1})
\]

\[
+ B_{m-2}y^2(\rho(u_m) + 1)\rho(u_{2m-1}) = B_{m-1}.
\]

Now, for \(u = (u_1, \ldots, u_{2m}) \in \mathbb{F}_q^{2m}\), we define the \(2 \times 2\) matrix \(M_j(x, y : u)\) with polynomial entries by

\[
M_j(x, y : u) = \begin{bmatrix}
    x(y\rho(u_j) + x) & x
    
y^3(\rho(u_j) + 1)(\rho(u_{j+1}) + 1)\rho(u_{m+j}) & y^2(\rho(u_{j+1}) + 1)\rho(u_{m+j})
\end{bmatrix},
\]

(3.14)
for \( j = 1, 2, \ldots, m - 1 \). Then, with \( M_j = M_j(x, y : u) \),

\[
\begin{bmatrix}
A_{m-1} \\
B_{m-1}
\end{bmatrix} = M_{m-1} \begin{bmatrix}
A_{m-2} \\
B_{m-2}
\end{bmatrix}, \\
\ldots, \\
\begin{bmatrix}
A_3 \\
B_3
\end{bmatrix} = M_3 \begin{bmatrix}
A_2 \\
B_2
\end{bmatrix}, \\
\begin{bmatrix}
A_2 \\
B_2
\end{bmatrix} = M_2 \begin{bmatrix}
A_1 \\
B_1
\end{bmatrix},
\]

where

\[
\begin{bmatrix}
A_1 \\
B_1
\end{bmatrix} = \begin{bmatrix}
x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1})
\end{bmatrix}
\]

(cf. (3.6)). So, with \( M = M_{m-2} \cdots M_3 M_2 \), \( A_{m-1} \) and \( B_{m-1} \) are given by

\[
A_{m-1} = \left[ x(y\rho(u_{m-1}) + x) \right] M \left[ x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right],
\]

\[
B_{m-1} = \left[ y^3(\rho(u_{m-1}) + 1)(\rho(u_m) + 1)\rho(u_{2m-1}) \right] M \left[ x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right].
\]

With these expressions for \( A_{m-1} \) and \( B_{m-1} \), and from (3.13),

\[
\hat{g}(u) = \sum_{v_{2m} \in \mathbb{F}_q} \lambda(u_2v_{2m})x^\delta(v_{2m})y^{1-\delta(v_{2m})}(y\rho(u_m) + x^\delta(v_{2m})y^{1-\delta(v_{2m})})
\]

\[
\times \left[ x(y\rho(u_{m-1}) + x) \right] M \left[ x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right] + \sum_{v_{2m} \in \mathbb{F}_q} \lambda(u_2v_{2m})x^\delta(v_{2m})y^{1-\delta(v_{2m})}
\]

\[
\times \left[ y^3(\rho(u_{m-1}) + 1)(\rho(u_m) + 1)\rho(u_{2m-1}) \right] M \left[ x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right].
\]

Here the sums in (3.15) and (3.16) respectively equal

\[
x(y\rho(u_m) + x) [x(y\rho(u_{m-1}) + x) \times \left[ x(y\rho(u_1) + x) \right] M \left[ x(y\rho(u_1) + x^\delta(v_{2m})y^{1-\delta(v_{2m})}) \\
y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right] + y^2(\rho(u_m) + 1)\rho(u_{2m}) [x(y\rho(u_{m-1}) + x) \times \left[ x(y\rho(u_1) + y) \right] M \left[ y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right],
\]
and
\[ x \left[ y^3(\rho(u_{m-1}) + 1)\rho(u_m) + 1)\rho(u_{2m-1}) \ y^2(\rho(u_m) + 1)\rho(u_{2m-1}) \right] \]
\[ \times M \left[ x(y\rho(u_1) + x) \right. \]
\[ + y\rho(u_{2m}) \left[ y^3(\rho(u_{m-1}) + 1)(\rho(u_m) + 1)\rho(u_{2m-1}) \ y^2(\rho(u_m) + 1)\rho(u_{2m-1}) \right] \]
\[ \times M \left[ y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right]. \tag{3.19} \]

On the one hand, the sum of (3.17) and (3.19) is
\[ \left[ x(y\rho(u_m) + x) \right] M_{m-1} M \left[ x(y\rho(u_1) + x) \right. \]
\[ \left. y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right], \tag{3.21} \]
and, on the other hand, that of (3.18) and (3.20) is
\[ y^2(\rho(u_m) + 1)\rho(u_{2m}) \ y\rho(u_{2m}) \]
\[ \times M_{m-1} M \left[ xy(\rho(u_1) + 1) \right. \]
\[ \left. y^3(\rho(u_1) + 1)(\rho(u_2) + 1)\rho(u_{m+1}) \right] \]
\[ = \left[ y^3(\rho(u_1) + 1)(\rho(u_m) + 1)\rho(u_{2m}) \ y^2(\rho(u_1) + 1)\rho(u_{2m}) \right] \]
\[ \times M_{m-1} M \left[ y^2(\rho(u_2) + 1)\rho(u_{m+1}) \right]. \tag{3.22} \]

With \( M_1 \) as in (3.14),
\[ M_m = \left[ x(y\rho(u_m) + x) \right. \]
\[ \left. y^3(\rho(u_m) + 1)(\rho(u_1) + 1)\rho(u_{2m}) \ y^2(\rho(u_1) + 1)\rho(u_{2m}) \right], \]
and, by adding up (3.21) and (3.22), we finally get a nice expression for \( \hat{g}(u) \):
\[ \hat{g}(u) = [1 \ 0] M_m M_{m-1} M M_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + [0 \ 1] M_m M_{m-1} M M_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ = \text{Tr} M_m M_{m-1} M M_1. \]

Here \( \text{Tr} \) denotes the usual trace of a square matrix.

Now, we summarize our result in the following theorem.

**Theorem 3.1.** Let \( g \) be the CR-weight function on \( \mathbb{F}_q^{2m} \) defined by
\[ g(u) = x^{2m\text{-wcr}(u)} y^{w_{\text{cr}}(u)} \]
(cf. Proposition 2.1(b)). Then its Fourier transform \( \hat{g}(u) \) is given by
\[ \hat{g}(u) = \text{Tr} \prod_{j=0}^{m-1} M_{m-j}(x, y : u), \tag{3.23} \]
where \( \text{Tr} \) denotes the usual trace of a square matrix,
\[ M_j(x, y : u) = \begin{bmatrix} x(y\rho(u_j) + x) \\ y^3(\rho(u_j) + 1)(\rho(u_{j+1}) + 1)\rho(u_{m+j}) \\ y^2(\rho(u_{j+1}) + 1)\rho(u_{m+j}) \end{bmatrix}, \]

for \( j = 1, 2, \ldots, m - 1, \)

\[ M_m(x, y : u) = \begin{bmatrix} x(y\rho(u_m) + x) \\ y^3(\rho(u_m) + 1)(\rho(u_1) + 1)\rho(u_{2m}) \\ y^2(\rho(u_1) + 1)\rho(u_{2m}) \end{bmatrix}, \]

and \( \rho(u_j) = q - 1 \) if \( u_j = 0 \), and \( \rho(u_j) = -1 \) otherwise. Here we understand that

\[ \prod_{j=0}^{m-1} M_{m-j}(x, y : u) = M_m M_{m-1} \cdots M_1, \]

with \( M_{m-j} = M_{m-j}(x, y : u) \).

4. Weight distribution of the crown-weight space

Here we will derive a formula giving CR-weight distribution of the crown-weight space \( (\mathbb{F}_q^{2m}, w_{CR}) \), as a corollary of Theorem 4.3 below. That theorem expresses the CR-weight distribution of the dual code \( C^\perp \) of a linear code \( C \) in terms of \( C \). However, it requires too much computation to be useful for a ‘general’ linear code \( C \).

We will need the following facts from elementary calculus.

**Lemma 4.1.** Let \( f(x, y) \) be the homogeneous polynomial of degree \( n \) given by

\[ f(x, y) = \sum_{i=0}^{n} f_i x^{n-i} y^i. \]

Then, for \( i = 0, 1, \ldots, n, \)

\[ f_i = \frac{1}{(n-i)!} \left( \frac{\partial}{\partial x^{n-i}} \right) f(0, 1) = \frac{1}{i!} \left( \frac{\partial}{\partial y^i} \right) f(1, 0). \]

Lemma 4.1 will be applied below to the CR-weight enumerator \( W_{C,CR}(x, y) \) of a linear code \( C \), defined by

\[ W_{C,CR}(x, y) = \sum_{u \in C} x^{2m-w_{CR}(u)} y^{w_{CR}(u)} = \sum_{i=0}^{2m} A_{CR,i} x^{2m-i} y^i. \]

Let \( M(x, y) \) be an \( n \times n \) square matrix whose entries are smooth functions \( f_{ij}(x, y) \) in \( x, y: \)

\[ M(x, y) = \begin{bmatrix} f_{11}(x, y) & f_{12}(x, y) & \cdots & f_{1n}(x, y) \\ f_{21}(x, y) & f_{22}(x, y) & \cdots & f_{2n}(x, y) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(x, y) & f_{n2}(x, y) & \cdots & f_{nn}(x, y) \end{bmatrix}. \]
Then we agree that $\frac{\partial^m}{\partial x^m} M(x, y)$ (resp. $\frac{\partial^m}{\partial y^m} M(x, y)$) means the $n \times n$ matrix whose entries consist of $m$-th order partial derivatives $\frac{\partial^m}{\partial x^m} f_{ij}(x, y)$ (resp. $\frac{\partial^m}{\partial y^m} f_{ij}(x, y)$) of $f_{ij}(x, y)$.

**Lemma 4.2.** Let $M(x, y), M_1(x, y), \ldots, M_s(x, y)$ be $n \times n$ matrices whose entries consist of smooth functions in $x, y$. Then

(a) $$\frac{\partial^m}{\partial x^m} \text{Tr} M(x, y) = \text{Tr} \frac{\partial^m}{\partial x^m} M(x, y),$$ $$\frac{\partial^m}{\partial y^m} \text{Tr} M(x, y) = \text{Tr} \frac{\partial^m}{\partial y^m} M(x, y).$$

(b) $$\frac{\partial^m}{\partial x^m} (M_1 \cdots M_s) = \sum \left( \begin{array}{c} m \\ i_1 i_2 \cdots i_s \end{array} \right) \frac{\partial^{i_1}}{\partial x^{i_1}} M_1 \cdots \frac{\partial^{i_s}}{\partial x^{i_s}} M_s,$$

$$\frac{\partial^m}{\partial y^m} (M_1 \cdots M_s) = \sum \left( \begin{array}{c} m \\ i_1 i_2 \cdots i_s \end{array} \right) \frac{\partial^{i_1}}{\partial y^{i_1}} M_1 \cdots \frac{\partial^{i_s}}{\partial y^{i_s}} M_s,$$

where the unspecified sums run over all nonnegative integers $i_1, \ldots, i_s$ satisfying $i_1 + \cdots + i_s = m$, and

$$\left( \begin{array}{c} m \\ i_1 i_2 \cdots i_n \end{array} \right) = \left( \begin{array}{c} m \\ i_1 \end{array} \right) \left( \begin{array}{c} m - i_1 \\ i_2 \end{array} \right) \cdots \left( \begin{array}{c} m - i_1 - i_2 - \cdots - i_{n-1} \\ i_n \end{array} \right)$$

are multinomial coefficients.

**Theorem 4.3.** Let $C$ be a linear code of length $2m$ over the finite field $\mathbb{F}_q$, and let $\{B_{\text{CR}, j}\}_{j=0}^{2m}$ be the CR-weight distribution for the dual code $C^\perp$ of $C$. Then, for $j = 0, 1, \ldots, 2m$,

$$B_{\text{CR}, j} = \frac{1}{(2m - j)! |C|} \sum_{u \in C} \sum_{i_1, i_2, \ldots, i_m} \left( \begin{array}{c} 2m - j \\ i_1 i_2 \cdots i_m \end{array} \right),$$

$$= \text{Tr} \prod_{k=0}^{m-1} \frac{\partial^{i_k}}{\partial x^{i_k}} M_{m-k}(0, 1 : u), \quad (4.2)$$

where the unspecified sum runs over all nonnegative integers $i_1, i_2, \ldots, i_m$ less than or equal to 2 satisfying $i_1 + i_2 + \cdots + i_m = 2m - j$.

Or equivalently,

$$B_{\text{CR}, j} = \frac{1}{j! |C|} \sum_{u \in C} \sum_{i_1, i_2, \ldots, i_m} \left( \begin{array}{c} j \\ i_1 i_2 \cdots i_m \end{array} \right) \text{Tr} \prod_{k=0}^{m-1} \frac{\partial^{i_k}}{\partial y^{i_k}} M_{m-k}(1, 0 : u), \quad (4.3)$$

where the unspecified sum runs over all nonnegative integers $i_1, i_2, \ldots, i_m$ less than or equal to 3 satisfying $i_1 + i_2 + \cdots + i_m = j$. Here the product of matrices is taken in the order of $k = 0, 1, \ldots, m - 1$, from the left, and
Applying the discrete Poisson summation formula (2.3) to the CR-weight function \( g(u) = x^{2m-a_{CR}(u)}y^{a_{CR}(u)} \) and using (3.23), we have

\[
M_m(x, y : u) = \left[ \frac{x^2 + \rho(u_m)xy}{(\rho(u_m) + 1)(\rho(u_{m+1}) + 1)(\rho(u_{2m}) + 1)(\rho(u_{2m+1}) + 1)} \right],
\]

for \( j = 1, 2, \ldots, m - 1, \)

\[
M_j(x, y : u) = \left[ \frac{x^2 + \rho(u_j)xy}{(\rho(u_j) + 1)(\rho(u_{j+1}) + 1)(\rho(u_{2m}) + 1)(\rho(u_{2m+1}) + 1)} \right],
\]

with

\[
\rho(u) = \begin{cases} -1 & \text{if } u \neq 0, \\ q - 1 & \text{if } u = 0. \end{cases}
\]

**Proof.** By Lemma 4.1 (cf. (4.1)),

\[
B_{CR, j} = \frac{1}{(2m - j)!} \frac{\partial^{2m-j}}{\partial x^{2m-j}} W_{C^\perp, CR}(0, 1). \tag{4.4}
\]

Applying the discrete Poisson summation formula (2.3) to the CR-weight function \( g(u) = x^{2m-a_{CR}(u)}y^{a_{CR}(u)} \) and using (3.23), we have

\[
W_{C^\perp, CR}(x, y) = \sum_{u \in C^\perp} g(u) = \frac{1}{|C|} \sum_{u \in C} \hat{g}(u) = \frac{1}{|C|} \sum_{u \in C} \text{Tr} \prod_{k=0}^{m-1} M_{m-k}(x, y : u).
\]

Thus, using (a) and (b) of Lemma 4.2, we see that

\[
\frac{\partial^{2m-j}}{\partial x^{2m-j}} W_{C^\perp, CR}(x, y) \tag{4.5}
\]

\[
= \frac{1}{|C|} \sum_{u \in C} \frac{\partial^{2m-j}}{\partial x^{2m-j}} \text{Tr} \prod_{k=0}^{m-1} M_{m-k}(x, y : u) = \frac{1}{|C|} \sum_{u \in C} \text{Tr} \prod_{k=0}^{m-1} M_{m-k}(x, y : u)
\]

\[
= \frac{1}{|C|} \sum_{u \in C} \text{Tr} \sum_{(i_{m-1} \cdots i_1) \leq 2} \prod_{k=0}^{m-1} \frac{\partial^{i_{m-k}}}{\partial x^{i_{m-k}}} M_{m-k}(x, y : u),
\]

where the unspecified sum runs over all nonnegative integers \( i_1, i_2, \ldots, i_m \) less than or equal to 2 satisfying \( i_1 + \cdots + i_m = 2m - j \). Here one must observe that the entries of \( M_i(x, y : u) \) have degree at most 2 in \( x \).
Now, (4.2) follows on combining (4.4) and (4.5). (4.3) can be derived in exactly the same manner as (4.2) by noting that the entries of \( M_i(x, y : u) \) have degree at most 3 in \( y \). □

The next theorem is actually a corollary to Theorem 4.3.

**Theorem 4.4.** The CR-weight distribution \( \{ACR, j\}_{j=0}^{2m} \) of the crown-weight space \((\mathbb{F}_q^{2m}, w_{CR})\) is, for \( j = 0, 1, \ldots, 2m \), given by

\[
ACR, j = \frac{1}{(2m - j)!} \sum_{0 \leq i_1, \ldots, i_m \leq 2} \binom{2m - j}{i_1 \ldots i_m} \text{Tr} M^{(i_1)} \cdots M^{(i_m)},
\]

where

\[
M^{(0)} = \begin{bmatrix} 0 & 0 \\ (q - 1)q^2 & (q - 1)q \end{bmatrix}, \quad M^{(1)} = \begin{bmatrix} q - 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
\]

**Proof.** As \( \mathbb{F}_q^{2m} = \{0\}^\perp \), this is just a special case of (4.2) with \( C = \{0\} \). Observe also that, for all \( j = 1, \ldots, m \),

\[
M_j(x, y : 0) = \begin{bmatrix} x^2 + (q - 1)xy & x \\ (q - 1)q^2y^3 & (q - 1)qy^2 \end{bmatrix}.
\]

So, if we denote (4.7) simply by \( M(x, y) \), then

\[
M(0, 1) = \begin{bmatrix} 0 & 0 \\ (q - 1)q^2 & (q - 1)q \end{bmatrix} = M^{(0)},
\]

\[
\frac{\partial}{\partial x} M(0, 1) = \begin{bmatrix} q - 1 & 1 \\ 0 & 0 \end{bmatrix} = M^{(1)},
\]

and

\[
\frac{\partial^2}{\partial x^2} M(0, 1) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = M^{(2)}. \quad \square
\]

Finally, we present an example that illustrates the formula (4.6).

**Example.** According to the above theorem, the CR-weight distribution \( \{ACR, j\}_{j=0}^{6} \) of \((\mathbb{F}_2^{6}, w_{CR})\) is given by

\[
ACR, j = \frac{1}{(6 - j)!} \sum_{0 \leq i_1, i_2, i_3 \leq 2} \binom{6 - j}{i_1 i_2 i_3} \text{Tr} M^{(i_1)} M^{(i_2)} M^{(i_3)},
\]

where

\[
M^{(0)} = \begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}, \quad M^{(1)} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad M^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.
\]
If \( j = 0 \), \((i_1, i_2, i_3)\) runs over the set \(\{(2, 2, 2)\}\). So
\[
A_{CR,0} = \frac{1}{6!} \left( \begin{array}{ccc} 6 \\ 2 & 2 & 2 \end{array} \right) \text{Tr} M^{(2)} M^{(2)} M^{(2)} = \frac{1}{6!} \cdot 90 \cdot 8 = 1.
\]

If \( j = 1 \), \((i_1, i_2, i_3)\) runs over the set \(\{(1, 2, 2), (2, 1, 2), (2, 2, 1)\}\). So
\[
A_{CR,1} = \frac{1}{5!} \left( \begin{array}{ccc} 5 \\ 1 & 2 & 2 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(1)} M^{(2)} M^{(2)} = \frac{1}{5!} \cdot 30 \cdot 3 \cdot 4 = 3.
\]

If \( j = 2 \), \((i_1, i_2, i_3)\) runs over the set
\(\{(2, 1, 1), (1, 2, 1), (1, 1, 2), (0, 2, 2), (2, 0, 2), (2, 2, 0)\}\).
So
\[
A_{CR,2} = \frac{1}{4!} \left\{ \left( \begin{array}{ccc} 4 \\ 2 & 1 & 1 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(2)} M^{(1)} M^{(1)} + \left( \begin{array}{ccc} 4 \\ 0 & 2 & 2 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(0)} M^{(2)} M^{(2)} \right\}
= \frac{1}{4!} (12 \cdot 3 \cdot 2 + 6 \cdot 3 \cdot 0) = 3.
\]

If \( j = 3 \), \((i_1, i_2, i_3)\) runs over the set
\(\{(1, 1, 1), (0, 1, 2), (2, 0, 1), (1, 2, 0), (0, 2, 1), (1, 0, 2), (2, 1, 0)\}\).
So
\[
A_{CR,3} = \frac{1}{3!} \left\{ \left( \begin{array}{ccc} 3 \\ 1 & 1 & 1 \end{array} \right) \text{Tr} M^{(1)} M^{(1)} M^{(1)} + \left( \begin{array}{ccc} 3 \\ 0 & 1 & 2 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(0)} M^{(1)} M^{(2)}
+ \left( \begin{array}{ccc} 3 \\ 0 & 2 & 1 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(0)} M^{(2)} M^{(1)} \right\}
= \frac{1}{3!} (6 \cdot 1 + 3 \cdot 3 \cdot 0 + 3 \cdot 3 \cdot 8) = 13.
\]

If \( j = 4 \), \((i_1, i_2, i_3)\) runs over the set
\(\{(0, 1, 1), (1, 0, 1), (1, 1, 0), (2, 0, 0), (0, 2, 0), (0, 0, 2)\}\).
So
\[
A_{CR,4} = \frac{1}{2!} \left\{ \left( \begin{array}{ccc} 2 \\ 0 & 1 & 1 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(0)} M^{(1)} M^{(1)} + \left( \begin{array}{ccc} 2 \\ 2 & 0 & 0 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(2)} M^{(0)} M^{(0)} \right\}
= \frac{1}{2!} (2 \cdot 3 \cdot 4 + 1 \cdot 3 \cdot 0) = 12.
\]

If \( j = 5 \), \((i_1, i_2, i_3)\) runs over the set \(\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\). So
\[
A_{CR,5} = \frac{1}{1!} \left( \begin{array}{ccc} 1 \\ 1 & 0 & 0 \end{array} \right) \cdot 3 \cdot \text{Tr} M^{(1)} M^{(0)} M^{(0)} = 3 \cdot 8 = 24.
\]

Finally, if \( j = 6 \), \((i_1, i_2, i_3)\) runs over the set \(\{(0, 0, 0)\}\). So
\[
A_{CR,6} = \text{Tr} M^{(0)} M^{(0)} M^{(0)} = 8.
\]
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