Generalized Steiner Systems
$\text{GS}_d(2, 4, v, g)$ for $g = 2, 3, 6$

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Received 24 August 2000; accepted 25 January 2001

Abstract: Generalized Steiner systems $\text{GS}_d(t, k, v, g)$ were first introduced by Etzion and used to construct optimal constant-weight codes over an alphabet of size $g + 1$ with minimum Hamming distance $d$, in which each codeword has length $v$ and weight $k$. Much work has been done for the existence of generalized Steiner triple systems $\text{GS}_d(2, 3, v, g)$. However, for block size four there is not much known on $\text{GS}_d(2, 4, v, g)$. In this paper, the necessary conditions for the existence of a $\text{GS}_d(t, k, v, g)$ are given, which answers an open problem of Etzion. Some singular indirect product constructions for $\text{GS}_d(2, k, v, g)$ are also presented. By using both recursive and direct constructions, it is proved that the necessary conditions for the existence of a $\text{GS}_d(2, 4, v, g)$ are also sufficient for $g = 2, 3, 6$. © 2001 John Wiley & Sons, Inc. J Combin Designs 9: 401–423, 2001

Keywords: generalized Steiner systems; constant-weight codes; skew starter; singular indirect product construction

1. INTRODUCTION

The concept of an $H$-design was first introduced by Hanani [8] as a generalization of Steiner systems (the notion of $H$-design is due to Mills [11]). An $H(v, g, k, t)$ design is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$, where $\mathcal{X}$ is a set of points whose cardinality is $vg$, and $\mathcal{G} = \{G_1, \ldots, G_v\}$ is a partition of $\mathcal{X}$ into $v$ sets of cardinality $g$; the members of $\mathcal{G}$ are called groups. A transverse of $\mathcal{G}$ is a subset of $\mathcal{X}$ that meets each group in at most one point. The set $\mathcal{B}$ contains $k$-element transverse of $\mathcal{G}$, called blocks, with the property that each $t$-element transverse of $\mathcal{G}$ is contained in precisely one block. When $t = 2$, an $H(v, g, k, 2)$ is just a group-divisible design of group type $g^v$ and denoted by $k$-GDD($g^v$).
As stated in Etzion [7] and Yin et al. [17], an optimal \((g + 1)\)-ary \((v, k, d)\) constant-weight code (CWC) over \(Z_{g+1}\) can be constructed from a given \(H(v, g, k, t)\) \((I_v \times I_g, \{\{i\} \times I_g \mid i \in I_v\}, \mathcal{B})\), where \(I_m = \{1, 2, \ldots, m\}\) and \(d\) is the minimum Hamming distance of the resulting code. For each block \(\{(i_1, a_1), (i_2, a_2), \ldots, (i_k, a_k)\} \in \mathcal{B}\), we form a codeword of length \(v\) by putting \(a_j\) in position \(i_j\), \(1 \leq j \leq k\), and zeros elsewhere. For convenience, when two codewords obtained from blocks \(B_1\) and \(B_2\) have distance \(d\), we simply say that \(B_1\) and \(B_2\) have distance \(d\).

In the code which is related to an \(H(v, g, k, t)\), it is not difficult to see that the minimum Hamming distance \(d\) is bounded as

\[ k - t + 1 \leq d \leq 2(k - t) + 1. \]

An \(H(v, g, k, t)\) which forms a code with minimum Hamming distance \(d\) is denoted by \(GS_d(t, k, v, g)\) and called a generalized Steiner system. If \(d = 2(k - t) + 1\), it is simply denoted by \(GS(t, k, v, g)\).

Much work has been done for the existence of \(GS(t, k, v, g)\) when \(t = 2\) and \(k = 3\) (see [7, 2, 13, 12, 3, 4, 15]). However, not much is known for other cases. In particular, for the case of \(t = 2\) and \(k = 4\), there are only partial results on \(GS(2, 4, v, 2)\) (see [14]) and some product constructions stated in [7].

**Lemma 1.1** ([14]). For any prime power \(v \equiv 7 \pmod{12}\), there exists a \(GS(2, 4, v, 2)\) except \(v = 7\), for which there does not exist a \(GS(2, 4, 7, 2)\).

**Lemma 1.2** ([7, Construction J]). If there exist a \(GS(2, k, m, g)\), a \(GS(2, k, n, g)\), and an \(OA(k, n)\), then there exists a \(GS(2, k, mn, g)\).

**Lemma 1.3** ([7, Construction K]). If there exist a \(GS(2, k, m, g)\), a \(GS(2, k, n + 1, g)\), and an \(OA(k, n)\), then there exists a \(GS(2, k, mn + 1, g)\).

Etzion mentioned several open problems for further research in [7, Section 9]. We restate below the problems 2 and 9 in [7].

**Problem 1** ([7, Problem 2]). One can easily verify that for a \(GS(t, k, v, g)\) with \(g > 1\), and \(t \neq k\), we must have \(v > k\). But a better lower bound on the length of the code would be very interesting.

**Problem 2** ([7, Problem 9]). Find more constructions for generalized designs \(GS_d(t, k, v, g)\), where \(d > k - t + 1\).

In this paper, we first give an answer for Problem 1 and show an implicit bound as follows.

**Theorem 1.4.** If there exists a \(GS_d(t, k, v, g)\), then

1. \(\left[ \frac{v - t}{k - l} \right] \cdots \left[ \frac{v - t}{k - t - s} \right] \geq g + \delta\), where \(s = 2(k - t) + 1 - d\), and
   \[
   \delta = \begin{cases} 
   1, & \text{if } s = 1, \quad v - t - 1 \equiv 0 \pmod{k - t - 1} \text{ and } \\
   0, & \text{otherwise.}
   \end{cases}
   \]
2. \(\binom{t - i}{l - i}\) divides \(\binom{v - i}{l - i}g^i\) for any \(0 \leq i \leq t - 1\).
This provides an explicit answer to Problem 1, as stated in Corollary 2.2, with a better lower bound: \( v \geq k + (g - 1)(k - t) \).

When \( t = 2 \) and \( k = 4 \), both of the cases for \( d = 4 \) and 5 are of interest from Problem 2. As stated in Lemmas 1.1–1.3, very little is known for the existence of a GS(2, 4, v, g). In fact, only \( g = 2 \) was discussed and nothing is known for \( g \geq 2 \). With the results in Lemmas 1.1–1.3 and the known GS(2, 4, 10, 2) from Etzion [7] one can use PBD-closure to get a GS(2, 4, v, 2) for all \( v \equiv 1 \pmod{3} \) and \( v \) is large enough. Unfortunately, the existence for small (perhaps most interesting) values \( v \) remains unknown. After some intensive efforts spent on the case \( d = 5 \) we realized that at the present state of authors knowledge the case \( d = 4 \) is a more realistic goal to work on.

The existence of a GS(2, 4, v, g) is still useful since the code related to it is capable of correcting all patterns of one error and detecting all patterns of two errors. For \( d = 4 \), we obtain from Theorem 1.4 the following.

**Lemma 1.5.** The necessary conditions for the existence of a GS(2, 4, v, g) are:

1. \( \binom{v-2}{2} \geq g \); and
2. \( v \equiv 1, 4 \pmod{12} \), if \( g \equiv 1, 5 \pmod{6} \),
   - \( v \equiv 1 \pmod{3} \), if \( g \equiv 2, 4 \pmod{6} \),
   - \( v \equiv 0, 1 \pmod{4} \), if \( g \equiv 3 \pmod{6} \).

In this paper, we show that the necessary conditions in Lemma 1.5 are also sufficient for \( g = 2, 3, 6 \).

**Theorem 1.6.** The necessary conditions \( v \equiv 1 \pmod{3} \) and \( v \geq 7 \) are also sufficient for the existence of a GS(2, 4, v, 2).

**Theorem 1.7.** The necessary conditions \( v \equiv 0, 1 \pmod{4} \) and \( v \geq 5 \) are also sufficient for the existence of a GS(2, 4, v, 3).

**Theorem 1.8.** The necessary condition \( v \geq 6 \) is also sufficient for the existence of a GS(2, 4, v, 6).

Theorem 1.4 will be proved in Section 2. To show the sufficiency, we need the singular indirect product (SIP) constructions, a generalization of Lemma 1.2 and Lemma 1.3, which will be stated in Section 3. Special constructions are stated in Sections 4 and 5. Theorems 1.6–1.8 will be proved in Sections 6–8. For general background on designs, see [1,5].

## 2. NECESSARY CONDITIONS

In order to prove Theorem 1.4, a packing will be used. Let \( v \geq k \geq t \), a \( t-(v,k,\lambda) \) packing is a pair \((X,\mathcal{B})\), where \( X \) is a \( v \)-set of elements (points) and \( \mathcal{B} \) is a collection of \( k \)-subsets of \( X \) (blocks), such that every \( t \)-subset of points occurs in at most \( \lambda \) blocks in \( \mathcal{B} \). If \( \lambda > 1 \), then \( \mathcal{B} \) is allowed to contain repeated blocks. The packing number \( D_{\lambda}(v,k,t) \) is the maximum number of blocks in any \( t-(v,k,\lambda) \) packing. The following result was stated in [5, p. 409].

**Lemma 2.1** (First Johnson bound). \( D_{\lambda}(v,k,t) \leq U_{\lambda}(v,k,t) \), where

\[
U_{\lambda}(v,k,t) = \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \cdots \left\lfloor \frac{\lambda(v-t+1)}{k-t+1} \right\rfloor \right\rfloor \right\rfloor
\]
Further, if $\lambda(v - 1) \equiv 0 \pmod{k - 1}$ and $\lambda v(v - 1) \equiv -1 \pmod{k}$, then

$$D_\lambda(v, k, 2) \leq U_\lambda(v, k, 2) - 1.$$ 

Proof of Theorem 1.4. Since condition (2) was stated in [7], we need only to prove (1). Suppose $(\mathcal{A}, \mathcal{G}, \mathcal{B})$ is a GS$_d(t, k, v, g)$, where $\mathcal{A} = Z_v \times Z_g$, $\mathcal{G} = \{G_i : 0 \leq i \leq v - 1\}$, $G_i = \{(i) \times Z_g, 0 \leq i \leq v - 1\}$. Take $t - 1$ points $(0, 0), (1, 0), \ldots, (t - 2, 0)$ in $\mathcal{A}$, define $B(j)$ to be the unique block that contains $\{(0, 0), (1, 0), \ldots, (t - 2, 0), (t - 1, j)\}$, $0 \leq j \leq g - 1$. Let $\mathcal{B}_1 = \{B(j) : 0 \leq j \leq g - 1\}$.

Take any two blocks $B_1, B_2$ of $\mathcal{B}_1$ and let $B_1$ and $B_2$ cut through $\beta$ common groups apart from the common groups $G_j$, $0 \leq j \leq t - 1$. We compute the distance $d(B_1, B_2)$ between $B_1$ and $B_2$,

$$d(B_1, B_2) = 2(k - (\beta + t)) + \beta + 1 = 2(k - t) + 1 - \beta \geq d.$$ 

So,

$$s = 2(k - t) + 1 - d \geq \beta.$$ 

This means that $B_1$ and $B_2$ cut through at most $s$ common groups apart from the common groups $G_j$, $0 \leq j \leq t - 1$.

For any set $S = \{(x_1, j_1), (x_2, j_2), \ldots, (x_u, j_u)\}$, define the projection $P(S)$ of $S$ to be the set

$$P(S) = \{x_1, x_2, \ldots, x_u\},$$

where $(x_i, j_i) \in Z_v \times Z_g$, $1 \leq i \leq u$. Let $\mathcal{A}_2 = \{t, t + 1, \ldots, v - 1\}$, $\mathcal{B}_2 = \{P(S) : S_j = B(j) \setminus \{(0, 0), (1, 0), \ldots, (t - 2, 0), (t - 1, j)\}, 0 \leq j \leq g - 1\}$. Then, from the last paragraph, we know that any $(s + 1)$-set of $\mathcal{A}_2$ is contained in at most one block of $\mathcal{B}_2$. Thus we know that $(\mathcal{A}_2, \mathcal{B}_2)$ is an $(s + 1) - (v - t, k - t, 1)$ packing. Since $|\mathcal{B}_2| = g$, then, from Lemma 2.1, condition (1) is obtained. This completes the proof. 

Remark. For $d = 6$, $t = 2$, $k = 5$, $v = 25$ and $g = 84$, since $\delta = 1$ and Theorem 1.4 (1) does not hold, then there cannot exist a GS$_6(2, 5, 25, 84)$ although a 5-GDD(84$^{25}$) does exist from Yin et al. [16].

The following corollary answers Problem 1 explicitly.

Corollary 2.2. If there exists a GS$(t, k, v, g)$, then $v \geq k + (g - 1)(k - t)$.

Proof. Take $d = 2(k - t) + 1$ in Theorem 1.4. Since $s = 0$ and $\delta = 0$, we have $\frac{v - t}{k - t} \geq \frac{v - 1}{k - 1} \geq g$. The conclusion then follows. 

3. RECURSIVE CONSTRUCTIONS

The constructions stated in Lemma 1.2 and Lemma 1.3 have been generalized in [3] to singular indirect product (SIP) construction when $k = 3$. We will generalize the
SIP construction to general $k$ in this section. Some other constructions such as PBD construction and Filling in Hole construction are also presented. These constructions will be used to construct $\text{GS}_4(2, 4, v, g)$ in Sections 6–8.

The PBD construction was used when the block size is $k = 3$ (see [12], [3]), where PBD stands for pairwise balanced design. Here we state the PBD construction for general $k$. The following lemma is obvious.

**Lemma 3.1 (PBD).** Suppose there exists a $(v, K)$-PBD. If there exists a $\text{GS}_d(2, k, n, g)$ for every $n \in K$, then there exists a $\text{GS}_d(2, k, v, g)$.

A holey group-divisible design, $K$–HGDD, is a four-tuple $(\mathcal{V}, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called groups, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of blocks such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $\vert B \vert \leq K$ for any $B \in \mathcal{B}$. A $k$–HGDD($g^{(v, u)}$) denotes a $K$-HGDD with $v$ groups of size $g$ in $\mathcal{G}$, $u$ groups in $\mathcal{H}$ and $K = \{k\}$. In the similar way to construct a $(v, k, d)$ CWC from an $H(v, g, k, t)$, we can also construct a $(v, k, d)$ CWC from an $k$-HGDD($g^{(v, u)}$). The distance of two blocks in a $k$-HGDD($g^{(v, u)}$), is the Hamming distance of the two code words obtained from the two blocks. A holey generalized Steiner system, $HGS_d(2, k, (v, u), g)$, is a $k$–HGDD($g^{(v, u)}$) with the property that the minimum Hamming distance of related CWC is $d$. For convenience, we also say that the design has minimum Hamming distance $d$.

It is easy to see that if $u = 0$ or $u = 1$, then an $HGS_d(2, k, (v, u), g)$ is just a $\text{GS}_d(2, k, v, g)$. One can obtain a GS from an HGS by filling in hole construction stated in the following lemma.

**Lemma 3.2 (Filling in Hole).** If there exist an $HGS_d(2, k, (v, u), g)$ and a $\text{GS}_d(2, k, u, g)$, then there exists a $\text{GS}_d(2, k, v, g)$.

Before we can state SIP constructions we need the concept of IOA. Let $X = \{1, 2, \ldots, v\}$, $Y = \{v + a + 1, \ldots, v\}$. Let $L$ be an $s \times v$ matrix based on $X$, where $s = v^2 - a^2$. We say that $L$ is an incomplete orthogonal array denoted by $\text{IOA}(k, v; a)$ if each $(s \times 2)$-submatrix contains every ordered pair of $(X \times X) \setminus (Y \times Y)$ precisely once. Suppose $L = (e_{ij})$ is an $\text{IOA}(k, v; a)$, where $1 \leq i \leq v^2 - a^2$, $1 \leq j \leq k$. $R_i = (e_{i1}, \ldots, e_{ik})$ is called a vector of $L$. Suppose $L_1, L_2, \ldots, L_r$ are $r$ $\text{IOA}(k, v; a)$s on the same symbol set. The $r$ $\text{IOA}(k, v; a)$s are called simple if all the $r(v^2 - a^2)$ vectors from $L_1, L_2, \ldots, L_r$ are pairwise distinct. We can now state the following SIP constructions.

**Theorem 3.3 (SIP-1).** Let $m, n, t, u$ and $a$ be integers such that $0 \leq a \leq u$, $0 \leq a < n$, and $1 \leq t \leq n$. Suppose the following designs exist:

1. a $k$–GDD($g^m$) with the property that all its blocks can be partitioned into $t$ sets $S_0, S_1, \ldots, S_{t-1}$, such that the minimum distance in $S_r$, $0 \leq r \leq t - 1$, is $k$;
2. simple $t$ $\text{IOA}(k, n + a; a)$s;
3. an $HGS_k(2, k, (n + u, u), g)$.

Then there exists an $HGS_k(2, k, (e, f), g)$, where $f = (m - 1)a + u$ and $e = mn + f$. Further, if there exists a $\text{GS}_k(2, k, f, g)$, then there exists a $\text{GS}_k(2, k, e, g)$.

In most cases Theorem 3.3 is strong enough to use. However, in some sporadic cases we have to use its more general form which is stated below.
Theorem 3.4 (SIP-2). Let $m, t, u, h, s, w$ and $a$ be integers such that $h = sg$, $n = sw$, $0 \leq a < w$, $0 \leq sa \leq u$, $1 \leq t \leq w$. Suppose the following designs exist:

1. A $k$-GDD$(h^m)$ with the property that all its blocks can be partitioned into $t$ sets $S_0, S_1, \ldots, S_{t-1}$, and each group can be partitioned into $s$ subgroups of size $g$ such that the minimum distance in $S_r$, $0 \leq r \leq t - 1$, is $k$ with respect to the subgroups;
2. Simple $t$ IOA$(k, w + a; a)s$;
3. An HGS$_k(2, (n + u, u), g)$.

Then there exists an HGS$_k(2, (e, f), g)$, where $f = (m - 1)sa + u$ and $e = mn + f$.

Further, if there exists a GS$_k(2, (k, f), g)$, then there exists a GS$_k(2, (k, e), g)$.

**Proof.** Let $(\mathcal{F}_1, \mathcal{B}_1)$ be a $k$-GDD$(h^m)$ satisfying the properties mentioned above, where

$$
\mathcal{F}_1 = Z_m \times (Z_s \times Z_g), \mathcal{B}_1 = \{\{i\} \times (Z_s \times Z_g) : i \in Z_m\} \quad \text{and} \quad \mathcal{B}_1 = \bigcup_{i=0}^{t-1} S_i.
$$

The set of subgroups from $\mathcal{B}_1$ are $\mathcal{F}_1 = \{(i, l) \times Z_g : (i, l) \in Z_m \times Z_s\}$. For any $C = \{[i_1, l_1, \alpha_1], \ldots, [i_k, l_k, \alpha_k]\} \in \mathcal{B}_1$, without loss of generality, we may assume that $i_1 < i_2 < \ldots < i_k$.

Let $(\mathcal{F}_2, \mathcal{G}_2, \mathcal{H}_2, \mathcal{B}_2)$ be an HGS$_k(2, (n + u, u), g)$, where

$$
\mathcal{F}_2 = \{(Z_v \cup \{\infty_0, \infty_1, \ldots, \infty_{a-1}\}) \times Z_g \cup \{Y_{sa}, \ldots, Y_{u-1}\} \times Z_g,
\mathcal{G}_2 = \{(j, l) \times Z_g : (j, l) \in Z_v \times Z_s\}, \quad \text{and}
\mathcal{H}_2 = \{\{\infty, l\} \times Z_g : 0 \leq j \leq a - 1, l \in Z_s\} \cup \{\{Y_j\} \times Z_g : sa \leq j \leq u - 1\}.
$$

Here, $Y_j = (\infty, j, s)$, $sa \leq j \leq u - 1$.

Let $L_0, L_1, \ldots, L_{t-1}$ be simple $t$ IOA$(k, w + a; a)s$ based on $Z_v \cup \{\infty_0, \infty_1, \ldots, \infty_{a-1}\}$. Now, we construct blocks as follows. Let $\mathcal{F}_1 = \bigcup_{r=0}^{t-1} V_r$, where

$$
V_r = \{[[i_1, j_1, l_1, \alpha_1], \ldots, [i_k, j_k, l_k, \alpha_k]] : ([i_1, l_1, \alpha_1], \ldots, [i_k, l_k, \alpha_k]) \in S_r,
(j_1, \ldots, j_k) \in L_r\}.
$$

Let $\mathcal{F}_2 = \bigcup_{i=0}^{m-1} H_i$, where

$$
H_i = \{[[i, a_1, l_1, \alpha_1], \ldots, [i, a_k, l_k, \alpha_k]] : ([a_1, l_1, \alpha_1], \ldots, [a_k, l_k, \alpha_k]) \in \mathcal{B}_2\},
$$

and if $a_j \in \{\infty_{sa}, \ldots, \infty_{u-1}\}$, then $i$ is replaced by $m$. Let

$$
F = \{(Z_m \times (Z_v \cup \{\infty_0, \ldots, \infty_{a-1}\})) \times Z_g \cup (\{m\} \times \{Y_{sa}, \ldots, Y_{u-1}\}) \times Z_g,
\mathcal{G} = \mathcal{F}_1 \cup \mathcal{F}_2,
$$

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and

\[ G = \{ (i, j, l) \times Z_g : (i, j, l) \in (Z_m \times Z_n) \times Z_s \}, \]
\[ H = \{ (i, \infty, k) \times Z_g : i \in Z_m, 0 \leq j \leq a - 1, k \in Z_s \} \cup \{ (m, Y) \times Z_g : \]
\[ Y \in \{ Y_{sa}, \ldots, Y_{u-1} \} \}.

Then, it is not difficult to see that \((G, H, \mathcal{G}, \mathcal{H})\) is a \(k\)-HGDD\((g^{(e, f)})\), where \(f = (m - 1)s + u\) and \(e = mn + f\).

We shall show that this HGDD is also an HGS\(_k\)(2, \(k\), \((e, f), g\)). If it is not, then there exist at least two blocks \(A, A' \in \mathcal{A}\) such that the distance of \(A\) and \(A'\) is \(k - 1\). Suppose that

\[ A = \{ [i_1, j_1, l_1, \alpha_1], \ldots, [i_k, j_k, l_k, \alpha_k] \}, \quad A' = \{ [i'_1, j'_1, l'_1, \alpha'_1], \ldots, [i'_k, j'_k, l'_k, \alpha'_k] \}. \]

Since the distance of \(A\) and \(A'\) is \(k - 1\), they have one common point and cut across \(k\) common groups. Since two points \(Q = [x, y, z, r]\), \(Q' = [x', y', z', r']\) are in the same group if and only if \([x, y, z] = [x', y', z']\), then,

\[ \{ [i_1, j_1, l_1], \ldots, [i_k, j_k, l_k] \} = \{ [i'_1, j'_1, l'_1], \ldots, [i'_k, j'_k, l'_k] \}. \]

Since \(i_1 < i_2 < \ldots < i_k\), and \(i'_1 < i'_2 < \ldots < i'_k\), we have that \(j_h = j'_h, 1 \leq h \leq k\). Since \((j_1, \ldots, j_{k-1})\) and \((j'_1, \ldots, j'_{k-1})\) are two vectors of the simple \(t\) IOAs, this is a contradiction. This completes the proof. \(\square\)

Take \(s = 1\) in Theorem 3.4, we obtain Theorem 3.3.

4. CONSTRUCTION USING SKEW STARTERS

We shall give a direct construction for \(\text{GS}_4(2, 4, v, 6)\) by using a skew starter. But, we start with a more general construction.

Let \(G\) be an abelian group of odd order \(v\). A starter in \(G\) is a set of unordered pairs \(S = \{ (s_i, t_i) : 1 \leq i \leq \frac{v-1}{2} \}\) which satisfies the following properties:

1. \(\{ s_i : 1 \leq i \leq \frac{v-1}{2} \} \cup \{ t_i : 1 \leq i \leq \frac{v-1}{2} \} = G \setminus \{ 0 \};\)
2. \(\{ \pm (s_i - t_i) : 1 \leq i \leq \frac{v-1}{2} \} = G \setminus \{ 0 \}.\)

A set \(A = \{ a_i : 1 \leq i \leq \frac{v-1}{2} \}\) is defined to be an adder for the starter \(S\) if the elements in \(A\) are nonzero and distinct, and the set \(S + A = \{ s_i + a_i, t_i + a_i \} : 1 \leq i \leq \frac{v-1}{2} \}\) is again a starter. The adder \(A\) is called skew if \(A \cap (-A) = \emptyset\). The starter \(S\) is called skew if \(\{ \pm (s_i + t_i) : 1 \leq i \leq \frac{v-1}{2} \} = G \setminus \{ 0 \}\).

There are known existence results on skew starters.

**Lemma 4.1.** (1) [6] Let \(q\) be a prime power such that \(q = 2^t + 1\), where \(t > 1\) is odd. Then there exists a skew starter in finite field \(GF(q)\). (2) [10] There exists a skew starter in \(Z_{16k^2 + 1}\).

To construct a \(\text{GS}_4(2, 4, v, 6)\) from a starter and a skew adder, we need an extra property.
Lemma 4.2. Suppose $G$ is an abelian group of order $v$ and there exist a starter $S = \{s_i, t_i : 1 \leq i \leq \frac{v-1}{2} \}$ and a skew adder $A = \{a_i : 1 \leq i \leq \frac{v-1}{2} \}$ of $S$. If $S$ and $A$ satisfy the following $(*-)\text{-property}$, then there exists a GS$_4(2,4,v,6)$.

$(*-)\text{-property}:$ the triples $\{s_i, t_i, -a_i \}, \{s_i + a_i, t_i + a_i, a_i \}, \{s_i - t_i, -a_i - t_i, -a_i \}$, $\{t_i - s_i, -a_i - s_i, -s_i \}, 1 \leq i \leq \frac{v-1}{2}$, are pairwise distinct.

Proof. Let $\mathcal{X} = G \times Z_6$, $\mathcal{B} = \{(g \times Z_6 : g \in G)\}$. Let $\mathcal{A}$ denote the set of blocks that are obtained by developing $B$ on group $G \times Z_6$. It is not difficult to see that $(\mathcal{X}, \mathcal{B}, \mathcal{A})$ is a 4-GDD($\ast$). We prove that the minimum distance of this design is 4. If it is not so, then there exist at least two different blocks $B = \{(x_1, h), (x_2, h), (x_3, h + 1), (x_4, h + 4)\} \in \mathcal{A}$. $B' = \{(x_1', h'), (x_2', h'), (x_3', h' + 1), (x_4', h' + 4)\} \in \mathcal{A}$, such that the distance between $B$ and $B'$ is 3. So, $B$ and $B'$ have one common point and cut across four common groups. Thus, we have that $\{x_1, x_2, x_3, x_4\} = \{x_1', x_2', x_3', x_4'\}$. Note that $S_1 = \{x_1 - x_4, x_2 - x_4\} \in S$, $S_2 = \{x_1' - x_4', x_2' - x_4'\} \in S$, and that $a_1 = x_4 - x_3 \in A$, $a_2 = x_4' - x_3' \in A$. We distinguish two cases:

1. If $x_4 = x_4'$, then we have that $\{x_1 - x_4, x_2 - x_4, x_3 - x_4\} = \{x_1' - x_4', x_2' - x_4', x_3' - x_4'\}$. It is clear that $\{x_1 - x_4, x_2 - x_4\} \cap \{x_1' - x_4', x_2' - x_4'\} \neq \emptyset$. This leads to a contradiction since $S$ is a starter.

2. If $x_4 \neq x_4'$, then there is some $x_i = x_i'$, $i < 4$. Thus from $\{x_1, x_2, x_3, x_4\} = \{x_1', x_2', x_3', x_4'\}$, we obtain $\{x_i - x_i : 1 \leq i \leq 4, i \neq i \} = \{x_i' - x_i' : 1 \leq i \leq 3\}$. Since $x_i - x_i = (x_i - x_4) - (x_i - x_4)$, from $(*)\text{-property}$ we have that $S_1 = S_2$, $a_1 = a_2$. So, $(x_1 - x_4) + (x_2 - x_4) - (x_4 - x_3) = (x_1' - x_4') + (x_2' - x_4') - (x_4' - x_3')$. That is $x_1 + x_2 + x_3 - 3x_4 = x_1' + x_2' + x_3' - 3x_4'$. Again from $\{x_1, x_2, x_3, x_4\} = \{x_1', x_2', x_3', x_4'\}$, we obtain $x_1 + x_2 + x_3 + x_4 = x_1' + x_2' + x_3' + x_4'$, so, $4x_4 = 4x_4'$. Since the order of $G$ is odd, we have $x_4 = x_4'$, this is a contradiction. The proof is completed. □

Suppose $S = \{s_i, t_i : 1 \leq i \leq \frac{v-1}{2} \}$ is a skew starter in group $G$. Let $A = \{-s_i + t_i : 1 \leq i \leq \frac{v-1}{2} \}$, then $A$ is a skew adder of $S$. It is not difficult to check that $S$ and $A$ satisfy $(*)\text{-property}$. So, we have the following result:

Corollary 4.3. If there exists a skew starter in an abelian group of odd order $v$, then there exists a GS$_4(2,4,v,6)$.

Lemma 4.4. There exists a GS$_4(2,4,v,6)$ for $v \in \{15,21,39\}$.

Proof. With the help of a computer, we have found a starter $S$ and a skew adder $A$ in $Z_v$ with the $(*)\text{-property}$, which we list below. So, from Lemma 4.2, there exists a GS$_4(2,4,v,6)$ for $v \in \{15,21,39\}$.

$v = 15$

$S = \{\{1,2\}, \{3,7\}, \{4,11\}, \{5,14\}, \{6,9\}, \{8,13\}, \{10,12\}\}$.

$A = \{\{1,6,8,2,5,12,11\}\}$.

$v = 21$

$S = \{\{1,17\}, \{7,20\}, \{2,11\}, \{3,10\}, \{8,19\}, \{14,16\}, \{4,5\}, \{6,12\}, \{9,13\}, \{15,18\}\}$.

$A = \{\{2,13,12,3,20,15,11,5,17,7\}\}$. 


$v = 39$

$S = \{\{7, 13\}, \{8, 34\}, \{5, 23\}, \{2, 36\}, \{4, 29\}, \{11, 31\}, \{3, 35\}, \{10, 25\}, \{17, 18\},$
$\{20, 37\}, \{6, 33\}, \{16, 27\}, \{19, 28\}, \{21, 24\}, \{22, 32\}, \{15, 38\}, \{12, 14\},$
$\{1, 9\}, \{26, 30\}\}.$

$A = \{16, 29, 25, 34, 13, 35, 18, 1, 37, 27, 7, 17, 30, 11, 6, 19, 8, 3, 15\}.$

**Remark.** There is a starter $S$ and a skew adder $A$ in $Z_{15}$ without (*)-property, which we list below:

$$S = \{\{1, 9\}, \{2, 3\}, \{4, 6\}, \{5, 10\}, \{7, 13\}, \{8, 12\}, \{11, 14\}\},$$

$$A = \{2, 5, 8, 11, 12, 1, 6\}.$$  

It is not difficult to check that $S$ is a starter, and $A$ is a skew adder of $S$. However, $\{s_1 - t_1, -a_1 - t_1, -t_1\} = \{4, 6, 7\} = \{s_3, t_3, -a_3\}.$

### 5. RGDD CONSTRUCTION FOR SMALL HGS

To use SIP constructions, we need some small HGS. They usually come from recursive constructions. However, in some cases we can also use RGDD to get such HGS, which is discussed in this section.

**Lemma 5.1.** Suppose there exists a 3-RGDD$(g^{2u+1})$, whose $g$ parallel classes can be partitioned into $u$ sets $S_1, \ldots, S_u$, each containing $g$ parallel classes, such that the distance of any two blocks in $S_i$ is at least 4. Then there exists an HGS$_4(2, 4, (3u + 1, u), g)$.

**Proof.** Suppose $P_{i1}, \ldots, P_{ig}$ $(1 \leq i \leq u)$ are all parallel classes of the RGDD $(\forall^*, \mathcal{G}, \mathcal{A})$, where $S_i = \bigcup_{j=1}^{g} P_{ij}$. Let $\mathcal{B}_{ij}$ denote the blocks obtained by adding $\infty_{ij}$ to each block of $P_{ij}$. Let

$$\forall^* = \forall^*_{i1} \cup \{\infty_{ij} : 1 \leq i \leq u, 1 \leq j \leq g\},$$

$$\mathcal{G} = \mathcal{G}_1 \cup \{\{\infty_{i1}, \ldots, \infty_{ig}\} : 1 \leq i \leq u\},$$

$$\mathcal{A} = \bigcup_{i=1}^{u} \bigcup_{j=1}^{g} \mathcal{B}_{ij}.$$  

Then it is not difficult to see that $(\forall^*, \mathcal{G}, \mathcal{A})$ is a 4-HGDD$(g^{(3u+1,u)})$.

We shall show that this HGDD is also an HGS$_4(2, 4, (n + u, u), g)$, that is we need to show that the minimum distance of the HGDD is 4. Suppose that $A = \{a, b, c, \infty_{ij}\}$ and $A' = \{a', b', c', \infty_{ij'}\}$ are two arbitrary blocks of the HGDD. If $i = i'$, then the two blocks come from two parallel classes in the same $S_i$, and so by assumption the distance between $A$ and $A'$ is at least 4. If $i \neq i'$, then $\infty_{ij}$ and $\infty_{ij'}$ are not in the same group. Since $\{a, b, c\}$ and $\{a', b', c'\}$ are two blocks of the 3-RGDD, their distance is at least 2. Thus, the distance of $A$ and $A'$ is at least 4. This completes the proof. \qed
Lemma 5.2. We obtain from Lemma 5.1 the following result.

1. Adding Lemma 5.1, and give the partition of the parallel classes.

Proof. We take the parameters below similar to the above lemma.

where

\[ P_0 = \{\{a, 44, 53\}, \{b, 24, 35\}, \{c, 30, 49\}, \{29, 38, 51\}, \{2, 18, 26\}, \{5, 8, 12\}, \{36, 37, 43\} \}
\]

and

\[ P_i = \{\{x_1 + 2i, x_2 + 2i, x_3 + 2i\} : x_1, x_2, x_3 \in P_0\}, \quad 0 \leq i \leq 26. \]

Adding \( \infty_i \) to \( P_i \) for all \( i, 0 \leq i \leq 26 \), and partitioning all the parallel classes below,

\[ S_1 = \{P_0, P_5, P_{25}\}, S_2 = \{P_1, P_6, P_{12}\}, S_3 = \{P_2, P_{13}, P_{17}\}, \]
\[ S_4 = \{P_3, P_8, P_{14}\}, S_5 = \{P_4, P_9, P_{24}\}, S_6 = \{P_7, P_{11}, P_{18}\}, \]
\[ S_7 = \{P_{10}, P_{21}, P_{23}\}, S_8 = \{P_{15}, P_{19}, P_{26}\}, S_9 = \{P_{16}, P_{20}, P_{22}\}, \]

we obtain from Lemma 5.1 the following result.

Lemma 5.2. There exists an HGS\(_4(2, 4, (28, 9), 3)\).

We further have the following.

Lemma 5.3. There exists an HGS\(_4(2, 4, (3u + 1, u), 2)\) for \( u = 7, 10 \).

Proof. We take the parameters below similar to the above lemma.

\[ g = 2, u = 7 \]
\[ \mathcal{V}_1 = \mathbb{Z}_{28} \cup \{a, b\}, \]
\[ \mathcal{G}_1 = \{\{i, i + 14\} : 0 \leq i \leq 13\} \cup \{\{a, b\}\}. \]

where

\[ P_0 = \{\{a, 0, 1\}, \{b, 2, 5\}, \{3, 4, 6\}, \{7, 9, 19\}, \{8, 18, 27\}, \{10, 15, 23\}, \{11, 17, 22\}, \]
\[ \{12, 16, 24\}, \{13, 20, 26\}, \{14, 21, 25\}\].
and
\[ P_i = \{ \{x_1 + 2i, x_2 + 2i, x_3 + 2i\} : \{x_1, x_2, x_3\} \in P_0 \}, \quad 0 \leq i \leq 13. \]

Adding \( \infty_l \) to \( P_i \) for all \( i, 0 \leq i \leq 13 \), and partitioning
\[ S_j = \{ P_{2(j-1)}, P_{2(j-1)+1} \}, \quad 1 \leq j \leq 7, \]
we obtain from Lemma 5.1 an HGS\(_4(2, 4, (22, 7), 2)\). For the second HGS we have the following.

\[ g = 2, u = 10 \]
\[ v_1 = \mathbb{Z}_{40} \cup \{a, b\}, \]
\[ \mathcal{G}_1 = \{ \{i, i+20\} : 0 \leq i \leq 19 \} \cup \{\{a, b\}\}. \]
\[ \mathcal{A}_1 = \bigcup_{i=0}^{19} P_i, \]

where
\[ P_0 = \{\{a, 13, 34\}, \{b, 4, 39\}, \{14, 23, 36\}, \{8, 12, 29\}, \{18, 30, 32\}, \{2, 10, 35\}, \{6, 7, 9\}, \{3, 28, 38\}, \{17, 20, 33\}, \{1, 19, 27\}, \{11, 15, 21\}, \{5, 16, 22\}, \{0, 24, 31\}, \{25, 26, 37\}\}. \]
\[ P_i = \{\{x_1 + 2i, x_2 + 2i, x_3 + 2i\} : \{x_1, x_2, x_3\} \in P_0 \}, \quad 1 \leq i \leq 19. \]

Add \( \infty_l \) to \( P_i \) for all \( i, 0 \leq i \leq 19 \) and let
\[ S_j = \{ P_{2(j-1)}, P_{2(j-1)+1} \}, \quad 1 \leq j \leq 10. \]

\[ \square \]

6. PROOF OF THEOREM 1.7

In this section, we shall prove Theorem 1.7. From Lemma 1.5, the necessary conditions for the existence of a GS\(_4(2, 4, v, 3)\) are \( v \equiv 0, 1 \pmod{4} \) and \( v \geq 5 \).

Let
\[ H_{b,c(a)} = \{ v : v \equiv b, c \pmod{a}, \ v > a \}. \]

The following result was stated in [5, p. 208].

**Lemma 6.1.** \( B(5, 8, 9) = H_{0,1(4)} \setminus D \), where \( D = \{12, 13, 16, 17, 20, 24, 28, 29, 32, 33, 44, 52, 60, 68, 84, 92, 96, 100, 104, 108, 112, 113, 116, 124, 132, 140, 156, 172, 173, 192, 204, 212, 228, 244, 252, 268, 272, 300, 308, 312\} \).

**Lemma 6.2.** Suppose \( v \equiv 1 \pmod{4} \) is a prime power, then there exists a GS\(_4(2, 4, v, 3)\).
Proof. Let \( v = 4t + 1 \) and \( \xi \) be a primitive element of \( \text{GF}(v) \). Denote
\[
S = \{(g + \xi^i, g + \xi^{i+1}, g + \xi^{2i+1}, g + \xi^{3i+1}) : g \in \text{GF}(v), \ 0 \leq i \leq t - 1\}.
\]
It is readily checked that the following 4-GDD is the desired GS. The 4-GDD is based on \( \text{GF}(v) \times Z_3 \) with groups \( \{g\} \times Z_3, \ g \in \text{GF}(v) \), having block set
\[
\{\{(a, j), (b, j + 1), (c, j), (d, j + 1)\} : (a, b, c, d) \in S, \ j \in Z_3\}. \quad \Box
\]

From Lemma 6.2 we have a GS\(_4(2, 4, v, 3)\) for \( v = 5, 9 \). To apply the PBD construction we need a GS\(_4(2, 4, 8, 3)\) which is given in the next lemma.

**Lemma 6.3.** There exists a GS\(_4(2, 4, v, 3)\) for any \( v \in \{8, 12, 16, 20, 24, 28, 32, 44\} \).

Proof. We construct a GS\(_4(2, 4, v, 3)\) on \( \mathcal{V} = (\mathbb{Z}_{v-1} \cup \{\infty\}) \times Z_3 \). Let \( \mathcal{G} = \{\{a\} \times Z_3 : a \in \mathbb{Z}_{v-1} \cup \{\infty\}\} \) and \( \mathcal{B} \) be obtained by developing a set of base blocks \( \mathcal{A} \mod (v - 1, 3) \), where \( \mathcal{A} \) contains \( s = \frac{v}{4} \) base blocks. For each \( v \), we found the following base blocks by computer search.

\( v = 8 \)
\[
\{\{(2, 1), (3, 2), (4, 2), (6, 2)\}, \{(1, 1), (5, 0), (0, 2), (\infty, 1)\}\}.
\]

\( v = 12 \)
\[
\{\{(2, 1), (3, 1), (4, 2), (7, 2)\}, \{(5, 1), (8, 2), (9, 1), (0, 1)\}, \{(6, 1), (10, 0), (1, 2), (\infty, 1)\}\}.
\]

\( v = 16 \)
\[
\{\{(2, 1), (3, 1), (4, 2), (7, 1)\}, \{(5, 1), (10, 0), (11, 2), (14, 2)\}, \{(6, 1), (8, 1), (13, 2), (0, 1)\}, \{(9, 1), (12, 2), (1, 0), (\infty, 1)\}\}.
\]

\( v = 20 \)
\[
\{\{(14, 1), (6, 2), (4, 0), (7, 0)\}, \{(15, 1), (9, 1), (8, 1), (12, 0)\}, \{(10, 1), (3, 2), (18, 1), (1, 1)\}, \{(11, 1), (2, 0), (16, 1), (17, 0)\}, \{(0, 1), (5, 2), (13, 0), (\infty, 1)\}\}.
\]

\( v = 24 \)
\[
\{\{(7, 1), (13, 2), (19, 2), (0, 1)\}, \{(10, 1), (14, 1), (17, 2), (22, 1)\}, \{(6, 1), (8, 1), (11, 1), (16, 2)\}, \{(2, 1), (15, 1), (21, 0), (1, 2)\}, \{(3, 1), (4, 2), (5, 2), (12, 1)\}, \{(9, 1), (18, 0), (20, 2), (\infty, 1)\}\}.
\]

\( v = 28 \)
\[
\{\{(20, 1), (25, 1), (8, 1), (24, 1)\}, \{(17, 1), (19, 2), (11, 0), (12, 2)\}, \{(9, 1), (18, 2), (6, 1), (16, 0)\}, \{(14, 1), (0, 0), (3, 2), (23, 1)\}, \{(10, 1), (4, 1), (2, 1), (26, 0)\}, \{(21, 1), (13, 0), (7, 1), (22, 2)\}, \{(5, 1), (1, 0), (15, 2), (\infty, 1)\}\}.
\]

\( v = 32 \)
\[
\{\{(13, 1), (17, 2), (23, 2), (25, 2)\}, \{(9, 1), (15, 0), (20, 0), (1, 2)\}, \{(12, 1), (14, 0), (27, 1), (30, 2)\}, \{(2, 1), (3, 1), (6, 1), (24, 1)\}, \{(4, 1), (10, 2), (11, 0), (18, 0)\}, \{(5, 1), (7, 2), (16, 1), (21, 2)\}, \{(19, 1), (22, 0), (26, 2), (31, 1)\}, \{(8, 1), (28, 0), (29, 2), (\infty, 1)\}\}.
\]

\( v = 44 \)
\[
\{\{(12, 1), (21, 2), (27, 2), (32, 2)\}, \{(11, 1), (29, 0), (30, 0), (37, 0)\}, \{(2, 1), (3, 2), (15, 1), (31, 0)\}, \{(8, 1), (10, 1), (14, 0), (0, 0)\}, \{(17, 1), (18, 0), (28, 2), (\infty, 1)\}\}.
\]
\{(4,1), (25,0), (35,0), (38,1)\}, \{(7,1), (20,0), (23,2), (42,2)\}, \{(5,1), (9,2), (24,2), (26,1)\}, \{(16,1), (34,2), (36,0), (41,1)\}, \{(19,1), (22,1), (33,0), (39,1)\}, \{(6,1), (13,2), (40,2), (1,2)\}.

Lemma 6.4. There exists a GS4(2, 4, v, 3) for any \( v \in H_{0,1(4)} \setminus D \).

Proof. For any \( v \in H_{0,1(4)} \setminus D \), from Lemma 6.1 there is a \( (v, \{5,8,9\}) \)-PBD. Since there is a GS4(2, 4, w, 3) for \( w \in \{5,8,9\} \) from Lemma 6.2 and Lemma 6.3, then the result comes from Lemma 3.1.

In the remainder of this section, we shall deal with the values \( v \) in \( D \). By Lemma 6.2, and Lemma 6.3, we need to deal with \( v = 33 \) and even \( v \in D \setminus \{12,16,20,24,28,32,44\} \). This will be accomplished by using SIP constructions. To do so, we need certain simple \( n \) IOAs.

Lemma 6.5 ([9]). If \( v \geq 3w \), and \( (v, w) \neq (6,1) \), then there exists an IOA(4, v; w).

Lemma 6.6. If \( n \geq 2a \) and \( (n + a, a) \neq (6,1) \), then there exist simple \( n \) IOA(4, n + a; a)s.

Proof. From Lemma 6.5, there exists an IOA(4, n + a; a). Suppose \( A = (a_{ij}) \) is an IOA(4, n + a; a) based on symbol set \( \mathbb{Z}_n \cup Q \), where \( Q = \{\infty_1, \ldots, \infty_a\} \), \( 1 \leq i \leq (n + a)^2 - a^2 \), \( 1 \leq j \leq 4 \). For \( 0 \leq s \leq n - 1 \), let

\[
l^s_{ij} = \begin{cases} a_{ij}, & \text{if } 1 \leq j \leq 2, \\ a_{ij} + s \pmod{n}, & \text{if } 3 \leq j \leq 4 \text{ and } a_{ij} \in \mathbb{Z}_n, \\ a_{ij}, & \text{if } 3 \leq j \leq 4 \text{ and } a_{ij} \in Q. 
\end{cases}
\]

Let \( L_s = (l^s_{ij}) \), then it is not difficult to see that \( L_0, \ldots, L_{n-1} \) are \( n \) IOA(4, n + a; a)s. We prove that the \( n \) IOAs are simple. If it is not so, then there exist \( i \) and \( j \), \( 0 \leq i < j \leq n - 1 \), such that \( L_i \) and \( L_j \) have a common vector. Suppose the common vector is \((x_1, x_2, x_3, x_4)\). It is clear that at least one of \( x_1 \) and \( x_4 \) is not in \( Q \). Assume that \( x_i \notin Q \), \( i \in \{3, 4\} \). Then there exists an integer \( k \), \( 1 \leq k \leq (n + a)^2 - a^2 \), such that \( a_{kl} + i = x_l = a_{kl} + j \). This leads to \( i = j \), a contradiction. The proof is completed.

With the above simple \( n \) IOA(4, n + a; a)s, we can rewrite SIP-2 construction in the following working lemma.

Lemma 6.7. Let \( m, t, u, h, s, w \) and \( a \) be integers such that \( h = sg \), \( n = sw \), \( w \geq 2a \), \( 0 \leq sa \leq u \), \( 1 \leq t \leq w \) and \( (w, a) \neq (5,1) \). Suppose the following designs exist:

1. a 4-GDD(hm) with the property that all its blocks can be partitioned into \( t \) sets \( S_0, \ldots, S_{t - 1} \), and each group can be partitioned into \( s \) subgroups of size \( g \) such that the minimum distance in \( S_r \), \( 0 \leq r \leq t - 1 \), is 4 with respect to the subgroups.
2. an HGS4(2, 4, (n + u, u), g).

Then there exists an HGS4(2, 4, (e,f), g), where \( f = (m - 1)sa + u \) and \( e = mn + f \). Further, if there exists a GS4(2, 4, f, g), then there exists a GS4(2, 4, e, g).
Take $s = 1$, $a = 0$ and $u = 0$ or 1 in Lemma 6.7 to obtain the following.

**Lemma 6.8.** Let $m, t, u, n$ be integers such that $u = 0$ or 1, $1 \leq t \leq n$, $n \not\in \{2,6\}$. Suppose the following designs exist:

1. a 4-GDD($g^n$) with the property that all its blocks can be partitioned into $t$ sets $S_0, S_1, \ldots, S_{t-1}$, such that the minimum distance in $S_r$, $0 \leq r \leq t - 1$, is 4;
2. a GS$_4(2, 4, n + u, g)$.

Then there exist both a GS$_4(2, 4, mn + u, g)$ and an HGS$_4(2, 4, (mn + u, n + u), g)$.

Further take $t = 1$ in Lemma 6.8 to obtain the following.

**Lemma 6.9.** Let $m, n, u$ be integers such that $u = 0$ or 1, $n \not\in \{2,6\}$. Suppose there exist both a GS$_4(2, 4, m, g)$ and a GS$_4(2, 4, n + u, g)$. Then there exist both a GS$_4(2, 4, mn + u, g)$ and an HGS$_4(2, 4, (mn + u, n + u), g)$.

We are now in a position to prove Theorem 1.7.

**Proof of Theorem 1.7.** As mentioned earlier, we need only to deal with $v = 33$ and $v$ even in $D \setminus \{12, 16, 20, 24, 28, 32, 34\}$.

With suitable $m$ and $n$, Lemma 6.9 can be used to obtain a GS$_4(2, 4, mn + u, 3)$ for $u = 0, 1$. This takes care of $v \in \{33, 60, 92, 104, 108, 140, 156, 172, 204, 246, 300, 312\}$.

Since a 4-GDD($3^4$) has 9 blocks, we may apply Lemma 6.7 with $t = 9$, $m = 4$, $n \geq 9$ and $u = 0$ or 1 to obtain a GS$_4(2, 4, 4n + u, 3)$. This takes care of the values $v \in \{52, 68, 84, 96, 100, 112, 116, 132, 192, 212, 228, 244, 272, 308\}$.

Finally, we deal with the remaining two values of $v = 124, 268$. From Lemma 5.2, there exists an HGS$_4(2, 4, (28, 9), 3)$. Take $m = 5, n = 19, u = 9, s = t = 1, a = 5$ in Lemma 6.7, there exists a GS$_4(2, 4, 124, 3)$ since there exists a GS$_4(2, 4, 29, 3)$ from Lemma 6.2. Take $m = 4, n = 16, u = 0, t = 9$ in Lemma 6.8, there exists an HGS$_4(2, 4, (64, 16), 3)$. Further take $m = 5, n = 48, u = 16, s = t = 1, a = 3$ in Lemma 6.7, there exists a GS$_4(2, 4, 268, 3)$ since there exists a GS$_4(2, 4, 28, 3)$ from Lemma 6.3. This completes the proof.

## 7. PROOF OF THEOREM 1.8

From Lemma 1.5, the necessary condition for the existence of a GS$_4(2, 4, v, 6)$ is $v \geq 6$. Let $K_n = \{v : v \geq n\}$, the following result is stated in [5, p. 211].

**Lemma 7.1.** $K_6 = B(E)$, where $E = \{6, 7, \ldots, 41, 45, 46, 47\}$.

We shall focus on the values $v \in E$.

**Lemma 7.2.** There exists a GS$_4(2, 4, 9, 6)$.

**Proof.** Let $\mathcal{V} = Z_{54}$ and $G_i = \{i + 9j : 0 \leq j \leq 5\}$, $\mathcal{G} = \{G_i : 0 \leq i \leq 8\}$. Then, $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ is a GS$_4(2, 4, 9, 6)$, where $\mathcal{B}$ is generated modulo 54 from the following four base blocks: $\{0, 13, 42, 44\}$, $\{0, 4, 15, 32\}$, $\{0, 1, 7, 21\}$, $\{0, 3, 8, 38\}$. \(\Box\)

**Lemma 7.3.** There exists a GS$_4(2, 4, v, 6)$ for each $v \in \{6, 8, 10, 12, 14, 16, 18, 20, 22, 28, 30, 34, 38\}$.
Proof. Let $\mathcal{V} = Z_{6v}$, $G_i = \{i + vj : 0 \leq j \leq 5\}$ and $\mathcal{G} = \{G_i : 0 \leq i \leq v - 1\}$. Let $\mathcal{A} = \{B_1, \ldots, B_{v-1}\}$ and $\mathcal{B} = \{B + 2j : B \in \mathcal{A}, 0 \leq j \leq 3v - 1\}$. With the aid of computer search we found a set of base blocks $B$ such that $(\mathcal{V}, \mathcal{A}, \mathcal{B})$ is a $\text{GS}_4(2, 4, v, 6)$. For convenience, we write $\mathcal{A} = \bigcup_{i=0}^{v-1} \{\{x, y, z\} : \{x, y, z\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding $S_i, 0 \leq i \leq 1$, which are listed below.

$v = 6$
$S_0 : \{5, 10, 27\}, \{14, 16, 23\}, \{11, 13, 21\}$.
$S_1 : \{2, 17, 34\}, \{4, 5, 12\}$.

$v = 8$
$S_0 : \emptyset$.
$S_1 : \{22, 42, 48\}, \{3, 4, 21\}, \{23, 37, 46\}, \{6, 11, 36\}, \{8, 12, 43\}, \{5, 20, 34\}$,
\{28, 38, 40\}.

$v = 10$
$S_0 : \{6, 7, 18\}, \{5, 39, 47\}$.
$S_1 : \{24, 45, 47\}, \{34, 36, 50\}, \{7, 8, 16\}, \{25, 28, 57\}, \{13, 30, 58\}, \{6, 23, 42\}$,
\{26, 48, 52\}.

$v = 12$
$S_0 : \{7, 11, 51\}, \{45, 47, 61\}, \{30, 50, 67\}, \{31, 40, 53\}, \{38, 44, 59\}$.
$S_1 : \{16, 55, 72\}, \{31, 32, 65\}, \{21, 24, 67\}, \{38, 46, 48\}, \{8, 11, 54\}, \{50, 64, 68\}$.

$v = 14$
$S_0 : \{63, 67, 74\}, \{8, 26, 30\}, \{24, 53, 69\}, \{13, 47, 82\}$.
$S_1 : \{48, 82, 83\}, \{21, 31, 39\}, \{7, 10, 26\}, \{34, 41, 73\}, \{44, 64, 76\}, \{6, 25, 54\}$,
\{23, 74, 80\}, \{14, 37, 60\}, \{2, 42, 59\}.

$v = 16$
$S_0 : \{19, 43, 50\}, \{4, 67, 73\}, \{17, 62, 77\}, \{26, 68, 78\}, \{36, 56, 81\}, \{22, 24, 35\}$,
\{83, 84, 87\}.
$S_1 : \{38, 69, 95\}, \{15, 58, 67\}, \{18, 48, 56\}, \{26, 47, 55\}, \{6, 12, 13\}, \{21, 39, 42\}$,
\{11, 60, 74\}, \{23, 57, 92\}.

$v = 18$
$S_0 : \{17, 33, 68\}, \{69, 79, 93\}, \{20, 39, 45\}, \{42, 65, 70\}, \{63, 71, 101\}, \{50, 82, 84\}$.
$S_1 : \{20, 28, 34\}, \{13, 45, 94\}, \{4, 48, 60\}, \{29, 54, 102\}, \{43, 104, 108\}$,
\{10, 41, 96\}, \{32, 42, 53\}, \{3, 14, 23\}, \{49, 83, 106\}, \{22, 59, 63\}, \{2, 18, 80\}$.

$v = 20$
$S_0 : \{10, 4281\}, \{55, 74, 96\}, \{8, 31, 58\}, \{47, 49, 84\}, \{66, 104, 117\}, \{34, 53, 99\}$,
\{12, 109, 114\}.
$S_1 : \{18, 110, 112\}, \{27, 33, 97\}, \{10, 31, 83\}, \{8, 13, 64\}, \{14, 55, 86\}$,
\{11, 54, 106\}, \{2, 78, 92\}, \{37, 45, 120\}, \{29, 58, 62\}, \{5, 103, 118\}$,
\{15, 60, 87\}, \{17, 79, 104\}.

$v = 22$
$S_0 : \{122, 126, 127\}, \{30, 90, 98\}, \{85, 103, 112\}, \{76, 78, 93\}, \{67, 73, 104\}$,
\{23, 31, 43\}, \{83, 87, 111\}, \{77, 84, 109\}, \{16, 61, 86\}$.
$S_1 : \{64, 73, 76\}, \{41, 71, 122\}, \{62, 80, 94\}, \{47, 57, 114\}, \{3, 20, 100\}$,
\{42, 79, 92\}, \{95, 106, 130\}, \{27, 75, 91\}, \{2, 15, 51\}, \{34, 74, 81\}$,
\{16, 54, 112\}, \{35, 78, 104\}.
\(v = 28\)
\(S_0 : \{21, 68, 123\}, \{2, 103, 127\}, \{64, 106, 136\}, \{44, 52, 141\}, \{98, 137, 139\}, \{50, 142, 164\}, \{37, 134, 157\}, \{12, 19, 107\}\).
\(S_1 : \{10, 76, 87\}, \{50, 137, 160\}, \{20, 93, 119\}, \{126, 153, 166\}, \{16, 47, 52\}, \{8, 88, 102\}, \{63, 75, 152\}, \{56, 134, 139\}, \{22, 60, 108\}, \{4, 64, 133\}, \{58, 71, 104\}, \{73, 110, 163\}, \{26, 36, 111\}, \{15, 115, 168\}, \{9, 125, 135\}, \{109, 129, 147\}, \{5, 124, 144\}, \{2, 18, 65\}, \{112, 118, 136\}.

\(v = 30\)
\(S_0 : \{52, 84, 167\}, \{53, 62, 102\}, \{82, 87, 149\}, \{158, 159, 160\}, \{35, 55, 112\}, \{3, 59, 69\}, \{38, 111, 133\}, \{6, 10, 43\}, \{26, 75, 153\}, \{104, 135, 146\}, \{24, 36, 137\}, \{7, 66, 81\}, \{51, 124, 132\}, \{93, 126, 139\}, \{19, 70, 91\}, \{14, 106, 122\}, \{46, 125, 162\}, \{100, 109, 117\}\).
\(S_1 : \{25, 37, 41\}, \{19, 24, 154\}, \{7, 26, 49\}, \{4, 123, 140\}, \{85, 111, 179\}, \{76, 101, 153\}, \{89, 96, 143\}, \{51, 142, 170\}, \{36, 65, 99\}, \{15, 30, 124\}, \{33, 77, 116\}\).

\(v = 34\)
\(S_0 : \{39, 103, 125\}, \{40, 50, 198\}, \{3, 161, 171\}, \{36, 43, 149\}, \{83, 118, 196\}, \{131, 143, 156\}, \{27, 101, 109\}, \{54, 84, 153\}, \{123, 176, 188\}, \{70, 147, 199\}, \{25, 63, 65\}, \{87, 115, 124\}, \{23, 55, 82\}, \{64, 178, 180\}, \{92, 133, 144\}, \{67, 106, 121\}, \{19, 107, 197\}, \{58, 111, 130\}, \{94, 138, 173\}, \{9, 96, 128\}\).
\(S_1 : \{109, 124, 181\}, \{127, 144, 157\}, \{63, 81, 201\}, \{2, 22, 95\}, \{129, 135, 158\}, \{51, 93, 160\}, \{50, 64, 101\}, \{45, 86, 191\}, \{17, 200, 204\}, \{21, 176, 194\}, \{57, 134, 172\}, \{4, 46, 146\}, \{48, 70, 145\}\).

\(v = 38\)
\(S_0 : \{70, 147, 154\}, \{86, 97, 108\}, \{112, 143, 178\}, \{49, 91, 103\}, \{32, 128, 141\}, \{139, 194, 198\}, \{8, 9, 10\}, \{15, 175, 225\}, \{20, 87, 115\}, \{26, 79, 136\}, \{44, 177, 207\}, \{27, 64, 212\}, \{57, 102, 215\}, \{72, 134, 170\}, \{71, 73, 211\}, \{146, 186, 214\}, \{18, 21, 65\}\).
\(S_1 : \{53, 100, 146\}, \{7, 78, 123\}, \{68, 74, 193\}, \{47, 157, 171\}, \{80, 149, 173\}, \{102, 165, 190\}, \{92, 144, 181\}, \{41, 70, 194\}, \{42, 210, 222\}, \{50, 200, 224\}, \{67, 75, 196\}, \{35, 97, 204\}, \{23, 62, 143\}, \{24, 209, 225\}, \{103, 112, 129\}, \{16, 106, 147\}, \{32, 137, 197\}, \{20, 76, 131\}, \{135, 168, 219\}, \{6, 128, 151\}\).

To treat the remaining values in \(E\), we shall use SIP construction. Thus we need some 4-GDDs to start with.

**Lemma 7.4.** There exists a 4-GDD(6\(^5\)) whose blocks can be partitioned into two sets \(S_0\), \(S_1\), such that the minimum distance of either \(S_0\), or \(S_1\) is 4.

**Proof.** Let \(\mathcal{X} = Z_{30}\), \(\mathcal{G} = \{i, i + 5, i + 10, i + 15, i + 20, i + 25\} : 0 \leq i \leq 4\).
\(B_1 = \{0, 1, 2, 8\}, \ B_2 = \{1, 3, 14, 25\}, \ B_3 = \{0, 3, 7, 21\}, \ B_4 = \{0, 4, 13, 16\}\). Let \(S_0\) denote the blocks obtained by developing \(B_1\) and \(B_2\) \(+2\) mod 30. Similarly, we get \(S_1\) from \(B_3\) and \(B_4\). Here \(+2\) mod 30 means cyclically adding 2 to the base blocks. Let \(\mathcal{B} = S_0 \cup S_1\). Then it is readily verified that \((\mathcal{X}, \mathcal{G}, \mathcal{B})\) is the desired design.

**Lemma 7.5.** There exists a 4-GDD(4\(^4\)) whose groups can be partitioned into two subgroups of size 2 each and its blocks can be partitioned into two sets \(S_0\)
and $S_1$ such that the minimum distance of $S_i$, $0 \leq i \leq 1$, is 4 with respect to the subgroups.

**Proof.** Let $\mathcal{X} = \mathbb{Z}_{16}$, $\mathcal{G} = \{\{i, i+4, i+8, i+12\} : 0 \leq i \leq 3\}$. The subgroups are $\mathcal{S}\mathcal{G} = \{\{i, i+4\} : 0 \leq i \leq 3\} \cup \{\{i+8, i+12\} : 0 \leq i \leq 3\}$. $\mathcal{B} = S_0 \cup S_1$, where $S_0 = \{\{0, 1, 2, 3\}, \{0, 9, 10, 11\}, \{1, 4, 6, 11\}, \{3, 4, 9, 14\}, \{1, 8, 10, 15\}, \{2, 7, 8, 9\}, \{1, 7, 12, 14\}, \{6, 9, 12, 15\}\}$. $S_1 = \{\{0, 5, 6, 7\}, \{0, 13, 14, 15\}, \{2, 4, 5, 15\}, \{4, 7, 10, 13\}, \{5, 8, 11, 14\}, \{3, 6, 8, 13\}, \{3, 5, 10, 12\}, \{2, 11, 12, 13\}\}$.

Then $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ is the desired design. □

**Lemma 7.6.** There exists a 4-GDD$(12^4)$ with the property that each group can be partitioned into two subgroups of size 6 and its blocks can be partitioned into two sets $T_0$, $T_1$ such that the minimum distance of either $T_0$, or $T_1$ is 4 with respect to the subgroups.

**Proof.** From Lemma 7.5, there exists a 4-GDD$(4^4)$ with similar property. Suppose $(\mathcal{X}_1, \mathcal{G}_1, \mathcal{A}_1)$ is such a GDD, $\mathcal{A}_1 = S_0 \cup S_1$ and $\mathcal{S}\mathcal{G}_1$ is the set of subgroups. Take $n = 3$, $a = 0$ in Lemma 6.6, there exist simple 3 OA$(4, 3)$s. Suppose $O_1$ and $O_2$ form simple 2 OA$(4, 3)$s. We construct the desired GDD as follow. Define

$$T_i = \{\{[x], [y], [z], [l], [w], [m]\} : \{x, y, z, w\} \in S_i, (j, k, l, m) \in O_i\}, \quad 0 \leq i \leq 1.$$ 

Let

$$\mathcal{X} = \mathcal{X}_1 \times \mathbb{Z}_3, \quad \mathcal{G} = \{G \times \mathbb{Z}_3 : G \in \mathcal{G}_1\}, \quad \mathcal{A} = T_0 \cup T_1.$$ 

Then, it is not difficult to see that $(\mathcal{X}, \mathcal{G}, \mathcal{A})$ is the desired GDD, where the subgroup is $\mathcal{S}\mathcal{G} = \{G \times \mathbb{Z}_3 : G \in \mathcal{S}\mathcal{G}_1\}$. This completes the proof. □

We are now in a position to prove Theorem 1.8.

**Proof of Theorem 1.8.** From Lemma 7.1 and the PBD construction, we need only to deal with the values $v \in E$. Odd prime powers are dealt with by Lemma 7.2 and by skew starter construction of Corollary 4.3 and Lemma 4.1. By Lemma 4.4 and Lemma 7.3, we have the remaining values of $v \in \{24, 26, 32, 33, 35, 36, 40, 45, 46\}$.

For $v \in \{26, 35, 36, 40, 45, 46\}$, write $v = 5n + u$, where $u = 0$ or $1$, $n \in \{5, 7, 8, 9\}$. With a 4-GDD$(6^5)$ from Lemma 7.4, we apply Lemma 6.8 to obtain a GS$_4(2, 4, v, 6)$.

For $v \in \{24, 32, 33\}$, write $v = 4n + u$ and apply Lemma 6.7 with $m = 4$, $a = 0$, $s = 2$, $g = 6$, $t = 2$, $w \in \{3, 4\}$ and $u \in \{0, 1\}$. With the 4-GDD$(12^4)$ from Lemma 7.6 and the known GS$_4(2, 4, r, 6)$ for $r = 6, 8, 9$, we obtain the desired GS$_4(2, 4, v, 6)$. This completes the proof. □

8. **PROOF OF THEOREM 1.6**

From Lemma 1.5, the necessary conditions for the existence of a GS$_4(2, 4, v, 2)$ are $v \equiv 1 \pmod{3}$ and $v \geq 7$. We first consider the case $v \equiv 1 \pmod{6}$. As stated in
[5, III, Tables 3.18, 3.19], let

\[ Q_{1 \mod a} = \{ v > a : v \text{ is a prime power and } v \equiv 1 \pmod{a} \}. \]

\[ H_{b(a)} = \{ v > a : v \equiv b \pmod{a} \}. \]

**Lemma 8.1** ([5]). \( B(Q_{1 \mod a}) = H_{1(a)} \setminus V \), where \( V = \{ 55, 115, 145, 205, 235, 265, 319, 355, 391, 415, 445, 451, 493, 649, 667, 685, 697, 745, 781, 799, 805, 1315 \}. \)

**Lemma 8.2.** Suppose \( v > 1 \) is a prime power, and \( v \equiv 1 \pmod{6} \), then there exists a GS\(_4(2, 4, v, 2)\).

**Proof.** Take a primitive element \( \alpha \) in GF\((v)\) and let \( \beta = \alpha^{(v-1)/3} \). Denote

\[ P = \{ \alpha^i, \alpha^i \beta, \alpha^i \beta^2 : 1 \leq i \leq (v-1)/6 \}. \]

It is readily checked that the following 4-GDD is the desired GS. The 4-GDD is based on GF\((v)\) \times \(Z_2\) with groups \( \{ g \} \times \(Z_2\), \( g \in \text{GF}(v)\), having block set

\[
\{(g,j), (g+x,j+1), (g+y,j+1), (g+z,j+1) : g \in \text{GF}(v), \ x,y,z \in P, j \in Z_2\}. \]

We need to construct a GS\(_4(2, 4, v, 2)\) for any \( v \in V \). Take \( a = 0, u = 0 \) or 1 in Lemma 6.7 to obtain the following.

**Lemma 8.3.** Let \( m, t, u, h, s \) and \( w \) be integers such that \( h = sg, n = sw, w \not\in \{2, 6\}, 1 \leq t \leq w, u \in \{0, 1\} \). Suppose the following designs exist: (1) a 4-GDD\((h^m)\) with the property that all its blocks can be partitioned into \( t \) sets \( S_0, S_1, \ldots, S_{t-1} \) and each group can be partitioned into \( s \) subgroups of size \( g \) such that the minimum distance in \( S_r, 0 \leq r \leq t-1 \), is 4 with respect to the subgroups; (2) a GS\(_4(2, 4, n+u, g)\). Then there exist both an HGS\(_4(2, 4, (mn + u, n+u), g)\) and a GS\(_4(2, 4, mn + u, g)\).

**Lemma 8.4.** Suppose that \( m, n \) are integers, \( m \geq 5, n \neq 2 \), and there exists a GS\(_4(2, 4, 3n+1, 2)\), then there exists a GS\(_4(2, 4, 3mn + 1, 2)\).

**Proof.** From Theorem 1.7, there exists a GS\(_4(2, 4, v, 6)\) for all \( m, m \geq 6 \). For \( m = 5 \), from Lemma 7.4, there exists a 4-GDD\((6^5)\) with the property that the blocks of the GDD can be partitioned into 2 sets, such that the minimum distance of each set is 4. So, for all \( m \geq 5 \), there exists a 4-GDD\((6^m)\) with the property that the blocks of the GDD can be partitioned into at most 2 sets, such that if we partition each group of the GDD into 3 subgroups of size 2, then the minimum distance of each set is 4 with respect to the subgroups. If \( n \neq 2, 6 \), then take \( s = 3, g = 2, w = n, u = 1 \) and \( t = 2 \) in Lemma 8.3, there exists a GS\(_4(2, 4, 3mn+1, 2)\) since there exists a GS\(_4(2, 4, 3n+1, 2)\). If \( n = 6 \) and \( m \neq 6 \), we can interchange \( m \) and \( n \), from the above process to obtain a GS\(_4(2, 4, 3mn+1, 2)\). If \( m = n = 6 \), we can take \( m = 9 \) and \( n = 4 \) to get a GS\(_4(2, 4, 3mn+1, 2)\). This completes the proof.

**Lemma 8.5.** There exists a GS\(_4(2, 4, v, 2)\) for any \( v \in A \), where \( A = \{10, 13, 25, 31, 70, 109\} \).
Proof. For \( v \in A \setminus \{10, 70\} \), from Lemma 8.2, there exists a \( \text{GS}_4(2, 4, v, 2) \). A \( \text{GS}(2, 4, 10, 2) \) exists from [7]. There exists a \( \text{GS}_4(2, 4, 7, 2) \) from Lemma 8.2. Take \( m = 10, n = 7, u = 0 \) in Lemma 6.9, there exists a \( \text{GS}_4(2, 4, 70, 2) \). This completes the proof. \( \square \)

**Lemma 8.6.** There exists a \( \text{GS}_4(2, 4, v, 2) \) for any \( v \in B \), where \( B = \{55, 145, 205, 265, 391, 415, 445, 451, 493, 649, 685, 697, 745, 781, 799, 805\} \).

**Proof.** For each \( v \in B \), we can write \( v = 3mn + 1 \), where \( m \in \{6, 11, 12, 13, 15, 17, 19, 37, 41, 57, 58, 62, 65, 67\} \), \( n \neq 2 \) and \( 3n + 1 \in A \). So, from Lemma 8.4, there exists a \( \text{GS}_4(2, 4, v, 2) \). \( \square \)

**Lemma 8.7.** There exists a \( \text{GS}_4(2, 4, v, 2) \) for \( v = 16, 115 \).

**Proof.** The existence of a \( \text{GS}_4(2, 4, 16, 2) \) will be shown in Lemma 8.10. For \( v = 115 \), a \( \text{GS}_4(2, 4, 115, 2) \) is based on \( Z_{115} \times Z_2 \) with group set \( \{\{g\} \times Z_2 : g \in Z_{115}\} \) and block set

\[
\{ (g,j), (g + x, j + 1), (g + y, j + 1), (g + z, j + 1) : g \in Z_{115}, \{x, y, z\} \in P, j \in Z_2 \},
\]

where the set \( P \) has been found by a computer search, and listed below.

\[
\{22, 36, 57\}, \{7, 40, 45\}, \{10, 18, 35\}, \{5, 41, 83\}, \{2, 47, 102\}, \{15, 39, 62\},
\{6, 72, 81\}, \{8, 24, 51\}, \{11, 67, 78\}, \{16, 29, 90\}, \{17, 27, 56\}, \{12, 65, 77\},
\{23, 26, 54\}, \{44, 46, 66\}, \{9, 55, 73\}, \{4, 30, 87\}, \{14, 20, 21\}, \{19, 63, 82\},
\{3, 84, 114\}. \square
\]

Take \( s = 1 \) in Lemma 6.7 to obtain the following.

**Lemma 8.8.** Let \( m, n, t, u, g \) and \( a \) be integers such that \( 0 \leq a \leq u, n \geq 2a, 1 \leq t \leq n, \) and \( (n,a) \neq (5, 1) \). Suppose the following designs exist: (1) a 4–GDD(\( g^m \)) with the property that all its blocks can be partitioned into \( t \) sets \( S_0, S_1, \ldots, S_{t-1} \), such that the minimum distance in \( S_r, 0 \leq r \leq t - 1 \), is \( 4r \); (2) an HGS_4(2, 4, (n + u, u), g). Then there exists an HGS_4(2, 4, (e,f), g), where \( f = (m - 1)a + u \) and \( e = mn + f \). Further, if there exists a \( \text{GS}_4(2, 4, f, g) \), then there exists a \( \text{GS}_4(2, 4, e, g) \).

**Lemma 8.9.** There exists a \( \text{GS}_4(2, 4, v, 2) \) for any \( v \equiv 1 \ (\text{mod} \ 6) \), \( v \geq 7 \).

**Proof.** From Lemma 3.1 and Lemma 8.1, we need only to show that there exists a \( \text{GS}_4(2, 4, v, 2) \) for any \( v \in V \). From Lemma 8.6 and Lemma 8.7, we need only to deal with \( v \in V \setminus (B \cup \{115\}) \).

With \( m = 13, 37, 73 \), \( n = 18 \) and \( u = 1 \), Lemma 6.9 can be used to obtain a \( \text{GS}_4(2, 4, mn + 1, 2) \). This takes care of \( v \in \{235, 667, 1315\} \).

Next, we deal with \( v = 319 \). Since there exists a \( \text{GS}_4(2, 4, 7, 2) \), taking \( m = n = 7 \) and \( u = 0 \) in Lemma 6.9 gives an HGS_4(2, 4, (49, 7, 2)). Take \( m = 7, n = 42, u = 7, a = 3 \) and \( t = 1 \) in Lemma 8.8, there exists a \( \text{GS}_4(2, 4, 319, 2) \) since a \( \text{GS}_4(2, 4, 25, 2) \) exists from Lemma 8.2.

Finally, we deal with the remaining value of \( v = 355 \). From Lemma 7.5, there exists a 4-GDD(\( 4^4 \)) with the property that if we partition each group into 2 subgroups of size 2, then the blocks of the GDD can be partitioned into 2 sets, such that the minimum distance of each set is 4 with respect to the subgroups. Take \( m = 4, g = 2, \).
Lemma 8.10. \( f \) and \( \hat{f} \) are related. \( g \) and \( \hat{g} \) are related.

In what follows, we shall consider the case \( v \equiv 4 \pmod{6} \).

To construct a GS\(_4(2, 4, v, 2) \) in \( Z_{2v} \), it suffices to find a set of base blocks, \( \mathcal{A} = \{B_1, \ldots, B_s\}, \) such that \((\mathcal{V}, \mathcal{B}, \mathcal{A})\) forms a GS\(_4(2, 4, v, 2) \), where \( \mathcal{V} = Z_{2v}, G = \{G_0, G_1, \ldots, G_{v-1}\}, G_i = \{i, i + v\}, 0 \leq i \leq v - 1, \) and \( \mathcal{B} = \{B + 2j : B \in \mathcal{A}, 0 \leq j \leq v - 1\} \). For convenience, we write \( \mathcal{A} = \bigcup_{i=0}^{v-1} \{i, x, y, z\} : \{x, y, z\} \in S_i \}. \) So, for each \( \mathcal{A} \) we need only display the corresponding \( S_0 \) and \( S_1 \).

**Lemma 8.10.** There exists a GS\(_4(2, 4, v, 2) \) for any \( v \in \{16, 22, 28, 34, 52, 58, 94\} \).

**Proof.** With the aid of a computer, we have found a set of base blocks, for which we shall list \( S_0 \) and \( S_1 \) below.

- \( v = 16 \):
  - \( S_0 : \{5, 25, 28\}, \{8, 15, 26\}, \{1, 2, 22\}; \)
  - \( S_1 : \{10, 27, 29\}, \{6, 9, 19\}; \)

- \( v = 22 \):
  - \( S_0 : \{7, 21, 28\}, \{13, 24, 41\}, \{8, 12, 14\}, \{1, 3, 18\}, \{19, 31, 39\}; \)
  - \( S_1 : \{2, 27, 36\}, \{5, 34, 39\}. \)

- \( v = 28 \):
  - \( S_0 : \{17, 21, 34\}, \{40, 41, 52\}, \{8, 26, 50\}, \{23, 35, 54\}, \{5, 15, 31\}; \)
  - \( S_1 : \{2, 38, 49\}, \{6, 15, 39\}, \{44, 51, 54\}, \{3, 23, 30\}. \)

- \( v = 34 \):
  - \( S_0 : \{12, 19, 38\}, \{3, 35, 62\}, \{15, 18, 66\}, \{21, 51, 61\}, \{8, 31, 32\}, \{4, 5, 14\}; \)
  - \( S_1 : \{22, 44, 55\}, \{16, 45, 56\}, \{6, 43, 51\}, \{3, 7, 23\}, \{13, 26, 42\}. \)

- \( v = 52 \):
  - \( S_0 : \{4, 10, 40\}, \{37, 50, 63\}, \{8, 23, 35\}, \{20, 66, 90\}, \{51, 59, 61\}; \)
  - \( S_1 : \{36, 69, 98\}, \{87, 94, 96\}, \{33, 38, 56\}, \{31, 47, 80\}, \{28, 49, 60\}, \{22, 63, 66\}, \{23, 29, 48\}, \{58, 61, 101\}, \{15, 16, 81\}, \{35, 55, 86\}, \{76, 88, 104\}, \{18, 74, 100\}. \)

- \( v = 58 \):
  - \( S_0 : \{7, 68, 91\}, \{28, 60, 111\}, \{2, 15, 92\}, \{18, 38, 95\}, \{9, 72, 80\}, \{19, 63, 74\}, \{30, 97, 100\}, \{14, 41, 54\}, \{35, 52, 101\}; \)
  - \( S_1 : \{28, 71, 101\}, \{36, 86, 111\}, \{96, 100, 106\}, \{5, 13, 81\}, \{2, 39, 84\}, \{21, 30, 52\}, \{27, 114, 115\}, \{23, 57, 75\}, \{8, 55, 112\}, \{11, 25, 48\}. \)

- \( v = 94 \):
  - \( S_0 : \{12, 26, 36\}, \{18, 40, 107\}, \{106, 134, 163\}, \{34, 65, 111\}, \{19, 20, 153\}, \{98, 128, 184\}, \{63, 104, 127\}, \{114, 121, 151\}, \{2, 75, 119\}, \{76, 81, 144\}, \{39, 42, 48\}, \{50, 172, 173\}, \{83, 109, 142\}; \)
  - \( S_1 : \{35, 54, 162\}, \{22, 77, 81\}, \{89, 111, 138\}, \{25, 121, 127\}, \{104, 142, 174\}, \{19, 24, 102\}, \{49, 92, 144\}, \{39, 105, 177\}, \{12, 74, 153\}, \{15, 90, 154\}, \{91, 131, 133\}, \{21, 52, 168\}, \{33, 107, 120\}, \{119, 129, 136\}, \{40, 53, 98\}, \{86, 178, 181\}, \{17, 172, 180\}, \{29, 58, 146\}. \)
Lemma 8.11. Suppose $v = 10r \equiv 4 \pmod{6}$, and there exists a $GS_4(2, 4, r, 2)$, then there exists a $GS_4(2, 4, v, 2)$.

Proof. Since $v \equiv 4 \pmod{6}$, it is easy to see that $r \equiv 1, 4 \pmod{6}$ and $r \not\in \{2, 6\}$. So, the result comes from Lemma 6.9 by taking $m = 10$, $n = r$ and $u = 0$.

Lemma 8.12. Suppose $v = 8r \equiv 4 \pmod{6}$, $r \not= 2$, and there exists a $GS_4(2, 4, 2r, 2)$, then there exists a $GS_4(2, 4, v, 2)$.

Proof. Since $v \equiv 4 \pmod{6}$, it is easy to see that $r \not= 6$ and $2r \equiv 4 \pmod{6}$. From Lemma 7.5, there exists a $4$-GDD($4^4$) with the property that if we partition each group into 2 subgroups of size 2, then the blocks of the GDD can be partitioned into 2 sets, such that the minimum distance of each set is 4 with respect to the subgroups. Take $m = 4$, $g = 2$, $s = 2$, $w = r$ and $u = 0$ in Lemma 8.3, there exists a $GS_4(2, 4, v, 2)$. This completes the proof.

Lemma 8.13. There exists a $GS_4(2, 4, v, 2)$ for all $v, v \equiv 4 \pmod{6}$, $10 \leq v \leq 82$.

Proof. From Lemma 8.5 and Lemma 8.10, we need only to consider the cases $v \in \{40, 46, 64, 76, 82\}$. For $v = 40$, take $r = 5$ in Lemma 8.12, there exists a $GS_4(2, 4, 40, 2)$. For $v \in \{46, 64, 76, 82\}$, take $m \in \{5, 7, 9\}$ and $n \in \{3, 5\}$ in Lemma 8.4, there exists a $GS_4(2, 4, v, 2)$ since a $GS_4(2, 4, 3n + 1, 2)$ exists from Lemma 8.5 and Lemma 8.10. This completes the proof.

Lemma 8.14. There exists a $GS_4(2, 4, v, 2)$ for any $v \in F_1 = \{88, 112, 130, 160, 184, 250, 304, 310, 328, 340\}$.

Proof. Write $v = mr$, where $r \in S = \{11, 14, 23, 38, 41\}$ and $m = 8$, or $r \in S' = \{13, 16, 25, 31, 34\}$ and $m = 10$. From Lemma 8.13, there exists a $GS_4(2, 4, 2r, 2)$ for any $r \in S$. From Lemma 8.9 and Lemma 8.13, there exists a $GS_4(2, 4, r, 2)$ for any $r \in S'$. So, from Lemma 8.11 and Lemma 8.12, there exists a $GS_4(2, 4, v, 2)$. This completes the proof.

Lemma 8.15. There exists a $GS_4(2, 4, v, 2)$ for any $v \in F_2 = \{124, 142, 178, 202, 214, 238, 268, 292\}$

Proof. For $v = 124, 142$, a $GS_4(2, 4, v, 2)$ exists by taking $m = 7$, $n = 15$, $u = 7$, $t = 1$ and $a \in \{2, 5\}$ in Lemma 8.8, where the input designs HGS$_4(2, 4, (15 + 7t, 7), 2)$ and GS$_4(2, 4, b, 2)$ for $b \in \{19, 37\}$ come from Lemma 5.3 and Lemma 8.2 respectively.

For $v = 202, 214$, applying Lemma 8.8 with $m = 13$, $n = 15$, $u = 7$, $t = 1$ and $a \in \{0, 1\}$ gives a $GS_4(2, 4, v, 2)$ since a $GS_4(2, 4, w, 2)$ exists for $w = 19, 31$ from Lemma 8.2.

For $v = 178$, we may start with a $GS_4(2, 4, 8, 6)$ from Theorem 1.7 and apply Lemma 8.3 with $m = 8$, $h = 6$, $g = 2$, $s = 3$, $w = 7$, $u = 10$ and $t = 1$ to obtain a $GS_4(2, 4, v, 2)$.

From Lemma 8.3, there exists an HGS$_4(2, 4, (21 + 10, 10), 2)$. Take $m = 10$, $n = 21$, $u = 10$, $t = 1$, $a = 2$ or $8$ in Lemma 8.8, there exists a $GS_4(2, 4, v, 2)$ for $v = 238$ or 292 since there exists a $GS_4(2, 4, f, 2)$ for $f = (m - 1)a + u \in \{28, 82\}$ from Lemma 8.13.
Finally, from Lemma 7.4, there exists a 4-GDD(6³) with the property that the blocks of the GDD can be partitioned into 2 sets, such that the minimum distance of each set is 4. Take $m = 5$, $h = 6$, $g = 2$, $s = 3$, $w = 3$, $u = 1$ and $t = 2$ in Lemma 8.3, there exists an HGS$_4(2, 4, (46, 10), 2)$. Take $m = 7$, $n = 36$, $u = 10$, $t = 1$ and $a = 1$ in Lemma 8.8, there exists a GS$_3(2, 4, 268, 2)$ since there exists a GS$_4(2, 4, f, 2)$ for $f = (m - 1)a + u = 16$ from Lemma 8.10. This completes the proof.

Lemma 8.16. There exists a GS$_4(2, 4, v, 2)$ for any $v \equiv 4 \pmod{6}$, $10 \leq v \leq 358$.

Proof. Write $v = 3mn + 1$, since $v \equiv 4 \pmod{6}$, we have that $m, n \equiv 1 \pmod{2}$. So, $3n + 1 \equiv 4 \pmod{6}$. If $m \geq 5$, then we have that $3n + 1 \leq 70$, so, there exists a GS$_4(2, 4, 3n + 1, 2)$ from Lemma 8.13. Since $n \neq 2$, then from Lemma 8.4, there exists a prime. For $v = 3p + 1$, since $v \in F_1 \cup F_2$, then there exists a GS$_4(2, 4, v, 2)$ from Lemma 8.14 and Lemma 8.4. If $v = 3(3p) + 1$ and $p \geq 5$, then let $m = p$, $n = 3$. Lemma 8.4 works since there exists a GS$_4(2, 4, 10, 2)$ from Lemma 8.5. So, we only leave $v = 3(3 \times 3) + 1 = 28$ in the case $v = 3(3p) + 1$. From Lemma 8.10, there exists a GS$_4(2, 4, 28, 2)$. This completes the proof. □

For convenience, let $[x, y]_6^4$ denote the set of integers $v$ for $x \leq v \leq y$ and $v \equiv 4 \pmod{6}$.

Lemma 8.17. There exists a GS$_4(2, 4, v, 2)$ for any $v \equiv 4 \pmod{6}$, $v \geq 10$.

Proof. Suppose there exists a GS$_4(2, 4, 6c + 4, 2)$, then we may take $m = 4$, $h = 4$, $g = 2$, $s = 2$, $w = 3c + 2$ and $u = 0$ in Lemma 8.3 to obtain an HGS$_4(2, 4, (46c + 4, 6c + 4), 2)$. Take $m = 4$, $h = 4$, $g = 2$, $s = 2$, $w = 3(3c + 2)$ and $u = 6c + 4$ in Lemma 6.7. If there exists a GS$_4(2, 4, 6c + 4, 2)$, then there exists a GS$_4(2, 4, v, 2)$, where $v = 4(3(6c + 4)) + 6c + a + 4 = 78c + 52 + 6a$, $0 \leq a \leq 3c + 2$. We have proved that the existence of GS$_4(2, 4, v, 2)$ for $v \in [6c + 4, 24c + 16]_6^4$ implies the existence for $v \in [78c + 52, 96c + 64]_6^4$.

It is not difficult to check that $\bigcup_{c \geq 4} [78c + 52, 96c + 64]_6^4 = [354, \infty]_6^4$. From Lemma 8.16, there exists a GS$_4(2, 4, v, 2)$ for each $v \in [10, 358]_6^4$. So, the conclusion comes from induction. □

We are now in a position to prove Theorem 1.6.

Proof of Theorem 1.6. Combine Lemma 8.9 and Lemma 8.17. □

REFERENCES