Composite systems with uncertain couplings of fixed structure: Scaled Riccati equations and the problem of quadratic stability

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Abstract
We consider large scale systems consisting of a finite number of separate uncertain subsystems which interact via uncertain couplings of arbitrarily prescribed structure. The couplings are viewed as structured perturbations of the block-diagonal system representing the collection of the separate nominal subsystems (the “nominal system”). We define spectral value sets and stability radii for these time-invariant structured perturbations and derive formulas for their computation. Scaled Riccati equations are introduced to obtain explicit formulas for the stability radii with respect to time-varying and possibly nonlinear perturbations of the given structure. From these we derive necessary and sufficient conditions under which the stability radii with respect to time-invariant and time-varying perturbations are equal. These results are obtained by constructing joint quadratic Liapunov functions of optimal robustness. With their help we prove necessary and sufficient conditions for quadratic stability and sufficient conditions for the validity of a generalized Aizerman conjecture.

Keywords: Interconnected system, Riccati equation, structured perturbation, spectral value set, stability radius, time-varying perturbation, quadratic stability, Aizerman conjecture.

1 Introduction
Composite systems play a role in many different areas of application. They arise naturally where large scale systems are composed of a finite number of separate interacting subsystems. Examples of such systems are traffic, biochemical [2], and power networks [6], compartmental models in physiology and ecology [15], and systems of cooperative robotic vehicles [10], see also [3], [20]. Often in these composite systems the interconnection structure is fixed, but the magnitudes of the couplings between the subsystems are uncertain and may even change in time or be dependent on current states. The interconnection structure specifies for each subsystem $\Sigma_i$ the set of subsystems $\Sigma_j$ which can directly influence it. A structure of this type may be described by a graph, and tools from graph theory have been used by several authors to analyze such systems, see e.g. [8], [22]. Due to economic costs, reliability and availability of information flows, etc. there will also, in general, be structural constraints imposed on the control of composite systems. For

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instance, the subsystems may be separated geographically and for each one only local measurements may be available for feedback control. This leads to problems of decentralized control which have been studied extensively in the literature, see e.g. [3], [21]. As a result of such structural constraints also closed loop interconnected systems will in general not be fully coupled but have a fixed interconnection structure.

In this paper we do not deal with control aspects but our results may provide an aid to the design of decentralized feedback controls. We analyze stability properties of composite system consisting of (possibly uncertain) subsystems interacting via uncertain couplings of a fixed structure. The couplings are viewed as structured perturbations of the block-diagonal system representing the collection of the disconnected nominal subsystems. This block-diagonal system will be regarded as the “nominal system”. We assume that the individual nominal subsystems are stable or have been stabilized (e.g. via decentralized control). Then a basic question is whether the overall system remains stable provided the uncertainties (possibly time-varying) are bounded in norm by some given realistic uncertainty level.

There is an extensive literature on stability properties of composite systems. Various tools such as input-output methods [24], [3], passivity methods, scalar and vector Liapunov functions [17], [20] and (dynamic) graphs [22] have been used to obtain sufficient stability criteria. Our aim is to develop non-conservative results on the stability of uncertain composite systems with norm bounded couplings and an arbitrary fixed interconnection structure. If the interconnection structure is a priori known, the application of robustness estimates for unstructured perturbations will in general lead to conservative results. On the other hand it is well known from $\mu$-analysis that tight robustness margins for structured perturbations are not easily obtained.

In a first step we determine the spectral value sets [12] associated with uncertain composite systems of the above type, i.e. the set of eigenvalues of all the composite systems which are obtained from the block-diagonal nominal one by adding perturbations which preserve the prescribed interaction structure and are bounded in an appropriate norm by a given uncertainty level $\delta > 0$. We then derive computable formula for the stability radius [12] of the nominal system with respect to these (complex time-invariant) perturbations. Similar results, but with respect to different perturbation norms, have been obtained in the recent paper [16] and it has been shown there that for the special case of one-dimensional subsystems these results are closely related to classical spectral inclusion theorems of Linear Algebra due to Gershgorin, Brauer and Brualdi. In fact, they can be used to show that the inclusion regions of Brauer and Brualdi (see [14], [23]) are tight for the corresponding perturbation structures.

In [16] constant variations of the coupling parameters were considered. However, time-varying couplings may often occur in practice. The notion of connective stability was introduced by Siljak [20] to determine conditions under which a composite system remains stable despite time-varying perturbations whereby subsystems are disconnected and again connected during the evolution of the system. It may also occur that some of the interconnected subsystems are nonlinear or time-varying, but for simplicity they have been modelled approximately by time-invariant ones. Then the “real” subsystems can be viewed as nonlinear or time-varying perturbations of the time-invariant nominal subsystems. In the present paper we will focus on the establishment of non-conservative stability results for uncertain composite systems with such time-varying and/or state-dependent uncertainties, both in the couplings and in the individual subsystems.

In general, the determination of the stability radius with respect to time-varying structured perturbations is a difficult problem [7], [26]. But if no perturbation structure is prescribed and arbitrary complex matrix perturbations are allowed (“complex full block case”) the situation is quite simple. It is known that in the complex full block case the stability radii with respect to time-invariant and time-varying perturbations coincide [12, §5.6]. This result (which does not carry over to real perturbations) has been obtained in [11] by characterizing the complex stability
radius via a parametrized Riccati equation. By means of the Riccati equation a quadratic Liapunov function of maximal robustness can be constructed. The construction of such Liapunov functions for structured perturbations has been stated as an open problem in [16, Remark 3.2]. In this paper we will show that for uncertain block-diagonal systems with irreducible perturbation structures quadratic Liapunov functions of maximal robustness can be constructed by applying a scaling technique known from $\mu$-analysis. Via scaling, new parametrized algebraic Riccati equations are obtained whose Hermitian solutions provide us with quadratic Liapunov functions of approximately optimal robustness (optimality is achieved if the perturbation structure is irreducible). By means of these quadratic Liapunov functions one can derive computable formulas for the stability radius of uncertain block-diagonal systems with respect to time-varying perturbations of the prescribed structure. In general, the stability radii with respect to time-varying and time-invariant perturbations are different. We will derive necessary and sufficient conditions for them to be equal.

The robust stability problem with respect to time-varying perturbations is closely related to the more general stability problem for linear differential inclusions of the form $\dot{x}(t) \in Ax(t)$ where $A$ is a compact set of $n \times n$ matrices. Molchanov and Pyatnitskij [18] have shown that such a differential inclusion is asymptotically stable, if and only if there exists a joint convex positive homogeneous strict Liapunov function for all the time-invariant systems $\dot{x} = Ax$, $A \in A$. This result shows a fundamental difference between robustness issues for time-varying and for time-invariant perturbations. In general, the stability of a set of time-invariant systems $\dot{x} = Ax$, $A \in A$ does not guarantee the existence of a joint Liapunov function for all these systems. In contrast, if all the time-varying systems $\dot{x}(t) = A(t)x(t)$, $A(\cdot) : \mathbb{R}_+ \to A$ measurable, are uniformly asymptotically stable then, by Molchanov and Pyatnitskij’s theorem, such a joint Liapunov function always exists.

A more specific problem is to characterize those sets of time-invariant linear systems for which a joint quadratic Liapunov function can be found. This is the problem of quadratic stability, see [5]. It is well known that for uncertain systems with full block perturbations, quadratic stability is equivalent to asymptotic stability, see [11]. Moreover it is known that this equivalence does not, in general, hold for structured perturbations [19]. To our knowledge, there are no general quadratic stability criteria available for structured perturbations besides reformulations of the problem in terms of linear matrix inequalities, see [1]. In this paper we consider block-diagonal systems under bounded perturbations of arbitrarily prescribed structure and derive explicit necessary and sufficient criteria for the quadratic stability of these systems. To check these criteria one only has to determine the spectral radius of a certain matrix, see Theorem 7.7.

The organization of the paper is as follows. In the next section we introduce the concepts of spectral value set and stability radius and for the special case of full block perturbations recall some characterizations of them via transfer functions and parametrized algebraic Riccati equations. The section concludes with the proof of a stability theorem for nonlinear time-varying full block perturbations. After fixing, in §3, the terminology and notations for the analysis of composite systems, computable formulae for the corresponding spectral value sets and stability radii are presented in §4. To derive these formulae we apply the Perron-Frobenius theory of non-negative matrices. We do not employ tools from $\mu$-analysis, but briefly express our results in terms of $\mu$-values. In §5 we introduce scaled parametrized algebraic Riccati equations and show how they can be used to construct joint quadratic Liapunov functions of approximately optimal robustness (optimal robustness if the underlying perturbation structure is irreducible). The scaled Riccati equations introduced in §5 will be the key tool for proving the main results of this paper. In §6 we derive a computable formula for the stability radius of a block diagonal system with respect to time-varying parameter perturbations. As a consequence we obtain necessary and sufficient conditions under which the stability radii with respect to time-invariant and time-varying per-
In this section we introduce some basic concepts and fix the notation. The symbols \( \mathbb{N}, \mathbb{R}, \mathbb{R}_+, \mathbb{C} \) denote the sets of positive integers, real numbers, non-negative real numbers and complex numbers, respectively. For any \( N \in \mathbb{N} \) we set \( N := \{1, 2, \ldots, N\} \). By \( \mathbb{K}^{n \times m} \) we denote the set of \( n \) by \( m \) matrices with entries in \( \mathbb{K} \) where \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). If \( A = (a_{ij}) \in \mathbb{C}^{n \times m} \) we define \( |A| := (|a_{ij}|) \) and for real matrices \( A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times m} \) we write \( A \leq B \) if \( a_{ij} \leq b_{ij} \) for all \( i \in \mathbb{N}, j \in m \). If \( A \) is square then \( \sigma(A), \rho(A) = \mathbb{C} \setminus \sigma(A) \) and \( \alpha(A) = \max \{\Re \lambda; \lambda \in \sigma(A)\} \) denote its spectrum, resolvent set, and spectral abscissa, respectively. By \( \mathbb{I}_{n,l,q} \) we denote the set of triples of matrices \( (A, B, C) \) with \( A \in \mathbb{C}^{n \times n}, B \in \mathbb{C}^{n \times l}, C \in \mathbb{C}^{q \times n}, n, l, q \in \mathbb{N} \). The open left half-plane is denoted by \( \mathbb{C}_- = \{s \in \mathbb{C}; \Re s < 0\} \) and \( A \in \mathbb{C}^{n \times n} \) is called Hurwitz stable if \( \sigma(A) \subset \mathbb{C}_- \). We use the conventions
\[
0^{-1} = \infty, \quad \infty^{-1} = 0, \quad \inf \emptyset = \infty. \tag{1}
\]
In the following definitions we suppose that \( (A, B, C) \in \mathbb{I}_{n,l,q} \) and \( \Delta \subset \mathbb{R}^{l \times q} \) is a \( \mathbb{K} \)-linear subspace provided with a norm \( \|\cdot\|_\Delta \). For a more detailed account of the definitions and some results presented in this section see [12]. We consider perturbations of the form
\[
A \sim A_\Delta = A + B \Delta C, \quad \Delta \in \Delta. \tag{2}
\]

**Definition 2.1.** The \( \mu \)-value of a matrix \( M \in \mathbb{C}^{q \times l} \) (with respect to the normed perturbation space \( (\Delta, \|\cdot\|_\Delta) \)) is defined by
\[
\mu_\Delta(M) := \left[ \inf \{ \| \Delta\|_\Delta; \Delta \in \Delta, \det(I_l - \Delta M) \} \right]^{-1}. \tag{3}
\]

**Definition 2.2.** Given a system \( (A, B, C) \in \mathbb{I}_{n,l,q} \) and the perturbation space \( (\Delta, \|\cdot\|_\Delta) \), the spectral value set of \( A \) of level \( \delta > 0 \), with respect to perturbations of the form (2), is the following subset of the complex plane.
\[
\sigma_\Delta(A, B, C; \delta) := \bigcup_{\Delta \in \Delta, \|\Delta\|_\Delta < \delta} \sigma(A + B \Delta C). \tag{4}
\]

**Definition 2.3.** Given a system \( (A, B, C) \in \mathbb{I}_{n,l,q} \) and the perturbation space \( (\Delta, \|\cdot\|_\Delta) \), the stability radius of \( A \) with respect to perturbations of the form (2) is defined by
\[
r_\Delta(A, B, C) = \inf \{\|\Delta\|_\Delta; \Delta \in \Delta, \sigma(A + B \Delta C) \not\subset \mathbb{C}_- \}. \tag{5}
\]
It is easily seen that the infimum in (5) is in fact a minimum if \( r_\Delta(A, B, C) \) is finite. In this case the stability radius is the norm of a smallest perturbation in \( \Delta \) which destabilizes \( A \). \( r_\Delta(A, B, C) = 0 \) if and only if \( \sigma(A) \not\subset \mathbb{C}_- \). \( r_\Delta(A, B, C) = \infty \) if and only if \( \sigma(A + B \Delta C) \subset \mathbb{C}_- \) for all \( \Delta \in \Delta \). Spectral value sets and stability radii can be characterized via \( \mu \)-values as follows, see [12].

**Proposition 2.4.** Let \( (A, B, C) \in \mathbb{I}_{n,l,q} \) be a given system and \( G(s) = C(sI_n - A)^{-1}B \) the associated transfer function. Then
\[
\sigma_\Delta(A, B, C; \delta) = \sigma(A) \cup \{ s \in \rho(A); \mu_\Delta(G(s)) > \delta^{-1} \}, \quad \delta > 0; \tag{6}
\]
\[
r_\Delta(A, B, C) = \left( \sup_{\omega \in \mathbb{R}} \mu_\Delta(G(i\omega)) \right)^{-1} \quad \text{if } \sigma(A) \subset \mathbb{C}_-. \tag{7}
\]
Specializing to block-diagonal and full-block perturbations further results are known if the perturbation norm \( \| \cdot \| \) is an operator norm. Let \( \mathbb{C}^l, \mathbb{C}^q \) be endowed with arbitrary norms and \( \mathbb{C}^{l \times q} \) with the induced operator norms \( \| \cdot \|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C})} \) and \( \| \cdot \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \). In the case where \( \mathbb{C}^l, \mathbb{C}^q \) are provided with 2-norms, we write \( \| \cdot \|_2 \) for vector norms and the corresponding operator norms are denoted by \( \| \cdot \|_{2,2} \). Suppose \( \Delta \subset \mathbb{C}^{l \times q} \) is the vector space of block-diagonal perturbations of the following form provided with the corresponding operator norm

\[
\Delta = \{ \text{diag}(\Delta_1, \ldots, \Delta_N); \Delta_i \in \mathbb{C}^{l_i \times q_i}, i \in N \}, \quad \| \cdot \| = \| \cdot \|_{\mathcal{L}(\mathbb{C}^l, \mathbb{C})}
\]

where \( N \geq 1, l_i \geq 1, q_i \geq 1, i \in N \) are given such that \( \sum_{i=1}^N l_i = l, \sum_{i=1}^N q_i = q \). Then estimates of the associated \( \mu \)-values, spectral value sets and stability radii can be obtained by the following scaling method. For any scaling vector \( \gamma = (\gamma_1, \ldots, \gamma_N) \) where \( \gamma_i > 0, i \in N \), we set

\[
L_\gamma = \text{diag}(\gamma_1I_{l_1}, \ldots, \gamma_NI_{l_N}) \quad \text{and} \quad R_\gamma = \text{diag}(\gamma_1I_{q_1}, \ldots, \gamma_NI_{q_N}).
\]

Then \( L_\gamma \Delta R_\gamma^{-1} = \Delta \) for all \( \Delta \in \Delta \) and this fact implies, see [12, §4.4], that

\[
\mu(\Delta)(G) \leq \| R_\gamma G L_\gamma^{-1} \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})}, \quad G \in \mathbb{C}^{q \times l}.
\]

As a consequence we have

\[
\sigma(\Delta)(A, B, C; \delta) \subset \sigma(A) \cup \{ s \in \rho(A); \| R_\gamma G(s) L_\gamma^{-1} \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} > \delta^{-1} \}; \quad r_\Delta(A, B, C) \geq \left( \sup_{\omega \in \mathbb{R}} \| R_\gamma G(\omega) L_\gamma^{-1} \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \right)^{-1}, \quad \text{if } \sigma(A) \subset \mathbb{C}_-.
\]

In the full-block case where \( \Delta = \mathbb{R}^{l \times q} \) precise formulae are obtained without any scaling. For this case the spectral value sets and stability radii are denoted by \( \sigma_{\mathbb{R}}(A, B, C; \delta) \) and \( r_{\mathbb{R}}(A, B, C) \), respectively, and are called the complex and real spectral value sets and stability radii according to whether \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{K} = \mathbb{R} \). For the complex case \( \Delta = \mathbb{C}^{l \times q} \), let \( \| \cdot \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \), one has

\[
\mu_{\mathbb{C}^q \times \mathbb{C}}(G(s)) = \| G(s) \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \quad \text{and therefore Proposition 2.4 implies}
\]

\[
\sigma_{\mathbb{C}}(A, B, C; \delta) = \sigma(A) \cup \{ s \in \rho(A); \| G(s) \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} > \delta^{-1} \}; \quad r_{\mathbb{C}}(A, B, C) = \left( \max_{\omega \in \mathbb{R}} \| G(\omega) \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \right)^{-1}, \quad \text{if } \sigma(A) \subset \mathbb{C}_-.
\]

The formula for the real stability radius \( r_{\mathbb{R}}(A, B, C) \) (where \( A, B, C \) are supposed to be real) is more complicated, see [12, §5.3.3].

In the following we will assume that both \( \mathbb{C}^l \) and \( \mathbb{C}^q \) are provided with the 2-norm \( \| \cdot \|_2 \) so that the norms \( \| \cdot \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})}, \| \cdot \|_{\mathcal{L}(\mathbb{C}^q, \mathbb{C})} \) are both spectral norms. Then

\[
r_{\mathbb{C}}(A, B, C) = \left( \max_{\omega \in \mathbb{R}} \| G(\omega) \|_{2,2} \right)^{-1} = \| G(\cdot) \|_{H_\infty}^{-1}, \quad \text{if } \sigma(A) \subset \mathbb{C}_-.
\]

Throughout the rest of this paper we will reserve the notation \( r_{\mathbb{C}}(A, B, C) \) for the complex stability radius with respect to the spectral norm on \( \mathbb{C}^{q \times l} \). This stability radius can be characterized in terms of the following parametrized algebraic Riccati equation

\[
PA + A^*P + \rho^2 C^*C + PBB^*P = 0
\]

where \( \rho \geq 0 \). We denote the real vector space of all the Hermitian matrices in \( \mathbb{C}^{n \times n} \) by \( \mathcal{H}_n(\mathbb{C}) \) and the usual order relation on \( \mathcal{H}_n(\mathbb{C}) \) by \( \preceq \). The following results have been proved in [11].

1For any Hermitian matrix \( H \) we denote by \( \lambda_{\text{max}}(H) \) the maximal eigenvalue of \( H \). For any matrix \( M \in \mathbb{C}^{h \times k} \), \( h, k \in \mathbb{N} \) we denote by \( \| M \|_{2,2} = [\lambda_{\text{max}}(M^*M)]^{1/2} = [\lambda_{\text{max}}(MM^*)]^{1/2} \) the spectral norm of \( M \).

2Note that \( P \) is a solution of (15) if and only if \( -P \) satisfies (ARE).
Theorem 2.5. Suppose that \( (A,B,C) \in \mathbb{L}_{n,l,q} \) is a given system with \( \sigma(A) \subset \mathbb{C}_- \), and \( G(s) = C(sI_n - A)^{-1}B \) is the associated transfer matrix. Then (15) has a Hermitian solution if and only if \( \rho \leq r_C := r_C(A,B,C) \). Moreover, the following statements are equivalent.

(i) There exists a (unique) solution \( P_\rho \in \mathcal{H}_n(\mathbb{C}) \) of (15) such that \( \sigma(A + BB^* P_\rho) \subset \mathbb{C}_- \).
(ii) \( \rho < r_C \).

If \( \rho = r_C \), then (15) has a unique solution \( P_{r_C} \in \mathcal{H}_n(\mathbb{C}) \) satisfying \( \sigma(A + BB^* P_{r_C}) \subset \mathbb{C}_- \). For each \( \rho \leq r_C \), \( P_\rho \) is the smallest Hermitian solution of (15). If \( P \) is any Hermitian solution of (15) then \( P \succeq 0 \). \( P \) is positive definite if \( (A,C) \) is observable. If \( P \) is positive definite then \( \dot{V}_\rho(x) = \langle x, P x \rangle \) is a joint quadratic Liapunov function for all perturbed systems

\[
\dot{x} = A\Delta x = (A + B\Delta C)x, \quad \Delta \in \mathbb{C}^{l \times q}, \quad \|\Delta\|_{2,2} \leq \rho. \tag{16}
\]

Since there is no joint Liapunov function for all perturbed systems \( \dot{x} = A\Delta x \) with \( \|\Delta\|_{2,2} < \rho \) if \( \rho > r_C(A,B,C) \), we may call the quadratic Liapunov function \( \dot{V}_\rho(x) \) (in case \( P > 0 \)) one of maximal robustness for the perturbation space \( \Delta = \mathbb{C}^{l \times q} \) endowed with the spectral norm.

We conclude this section with a theorem which extends Proposition 5.2 in [11] and will be useful for the treatment of time-varying nonlinear perturbations in §6. Since the proof in [11] works only for time-invariant perturbations, we will give a full proof.

Let \( (A,B,C) \in \mathbb{L}_{n,l,q} \) be a given system, \( \sigma(A) \subset \mathbb{C}_- \). We suppose that \( \Omega \) is an open neighbourhood of 0 in \( \mathbb{C}^n \) and consider nonlinear time-varying perturbations of \( \dot{x} = Ax \) of “output feedback form”

\[
\dot{x} = Ax + B\Delta(x,t)y, \quad y = Cx \quad \text{where} \quad \Delta(\cdot, \cdot) \in \mathcal{M}_n(\Omega). \tag{17}
\]

Here \( \mathcal{M}_n(\Omega) \) is the vector space of all bounded matrix functions \( \Delta(\cdot, \cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{C}^{l \times q} \) with the Carathéodory properties, i.e. \( \Delta(x, \cdot) : \mathbb{R}_+ \to \mathbb{C}^{l \times q} \) is measurable for each \( x \in \Omega \), \( \Delta(\cdot, t) : \Omega \to \mathbb{C}^{l \times q} \) is continuous for each \( t \in \mathbb{R}_+ \), and for each compact product set \( K \times I \subset \Omega \times \mathbb{R}_+ \) there exists an integrable \( k(\cdot) : I \to \mathbb{R}_+ \) such that

\[
\|\Delta(x,t)Cx - \Delta(\hat{x},t)C\hat{x}\|_2 \leq k(t)\|x - \hat{x}\|_2, \quad (x,t), (\hat{x},t) \in K \times I.
\]

The norm on \( \mathcal{M}_n(\Omega) \) is taken to be

\[
\|\Delta(\cdot, \cdot)\| = \sup_{x \in \Omega, t \geq 0} \|\Delta(x,t)\|_{2,2}, \quad \Delta(\cdot, \cdot) \in \mathcal{M}_n(\Omega). \tag{18}
\]

By Carathéodory’s Theorem for every \( (t_0, x^0) \in \mathbb{R}_+ \times \Omega \), there exists a unique solution \( x(t) = x(t_0, x^0) \) of (17) with \( x(t_0) = x^0 \) on some maximal semi-open interval \( [t_0, t_+ \infty) \) where \( t_+(t_0, x^0) > t_0 \), see [12, Thm. 2.1.14]. In the following theorem we will see that \( t_+(t_0, x^0) = \infty \) if \( \|\Delta(\cdot, \cdot)\| \leq r_C(A, B, C) \) and \( x^0 \) is sufficiently close to the equilibrium state \( \mathbf{r} = 0 \). Recall that a quadratic function \( V(x) = \langle x, Px \rangle \) is said to be a Liapunov function (respectively strict Liapunov function) for the nonlinear time-varying system \( \dot{x} = Ax + B\Delta(x,t)Cx \) at the origin if \( P > 0 \) and \( \dot{V}(x,t) \leq 0 \) (respectively \( \sup_{t \geq t_0} \dot{V}(x,t) < 0 \)) on some neighbourhood of the origin, \( x \neq 0 \). The existence of a quadratic Liapunov function ensures that the origin is uniformly stable, whereas the existence of a strict quadratic Liapunov function implies uniform asymptotic stability, see [12, Thm. 3.2.17]. Unfortunately, these criteria are not applicable in the present situation where \( P \) is obtained as a solution of the algebraic Riccati equation (15), since the corresponding \( V(\cdot) \) is in general neither a strict Liapunov function for the system (17) nor need it be positive definite.

Nevertheless we will see in the following proof that \( V(\cdot) \) can be used to establish asymptotic stability. For simplicity, we call the system (17) uniformly (asymptotically) stable if \( \mathbf{r} = 0 \) is a uniformly (asymptotically) stable equilibrium position of the system (17).
Theorem 2.6. Suppose $(A, B, C) \in \mathbb{L}_{n,l,q}$ is a given system, $\sigma(A) \subset \mathbb{C}_-$ and $\Omega$ is an open neighbourhood of $0$ in $\mathbb{C}^n$. Then

(i) The nonlinear system \((17)\) is asymptotically stable for all $\Delta \in \Delta_{nt}(\Omega)$ satisfying $\|\Delta(\cdot, \cdot)\| < r_{\mathcal{C}}(A, B, C)$. Moreover when this condition is satisfied we have $x(t; t_0, x^0) \to 0$ as $t \to \infty$ for all $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$ for which $t_+(t_0, x^0) = \infty$.

(ii) Suppose $\rho \leq r_{\mathcal{C}}(A, B, C)$ and let $P$ be a Hermitian solution of the algebraic Riccati equation \((15)\). Then $P \succeq 0$. If $\delta > 0$ satisfies $D_\delta = \{x : x \in \mathbb{C}^n \text{ and } (x, Px) < \delta\} \subset \Omega$, then $D_\delta$ is a joint domain of attraction of the equilibrium point $\overline{x} = 0$ for all the systems \((17)\) with $\Delta \in \Delta_{nt}(\Omega)$, $\|\rho(\cdot, \cdot)\| \leq \rho$.

(iii) Suppose $\rho \leq r_{\mathcal{C}}(A, B, C)$ and $P$ is a Hermitian solution of \((15)\). If $r < \rho$ there exists a constant $k > 0$ such that the derivative of the quadratic function $V_\rho(x) = (x, Px)$ along trajectories of \((17)\) satisfies $\dot{V}_\rho(x) \leq -k\|Cx\|^2$, $x \in \Omega$ for all $\Delta \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\| \leq r$. If $(A, C)$ is observable $V_\rho(x)$ is a joint Liapunov function at $\overline{x} = 0$ for all perturbed systems \((8)\) with $\Delta \in \Delta_{nt}(\Omega)$ satisfying $\|\Delta(\cdot, \cdot)\| \leq \rho$. In particular, \((17)\) is uniformly stable if $\Delta \in \Delta_{nt}(\Omega)$ and $\|\Delta(\cdot, \cdot)\| \leq r_{\mathcal{C}}(A, B, C)$.

Proof: We first prove the third statement. Let $\rho \leq r_{\mathcal{C}}(A, B, C)$. Since $A$ is Hurwitz stable, every Hermitian solution $P$ of \((15)\) is positive semi-definite. Multiplying \((15)\) from the right by $x \in \ker P$ and from the left by $x^*$, we see that $\ker P \subset \ker C$. Let $x(t) = x(t; t_0, x^0)$, $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$ be an arbitrary solution of \((17)\). The derivative of the time-invariant quadratic function $V_\rho(x) = (x, Px)$ along this solution is

\[
\dot{V}_\rho(x(t)) = \left[\langle (Ax(t) + B\Delta(x(t), t)Cx(t)), Px(t) \rangle + \langle x(t), P(Ax(t) + B\Delta(x(t), t)Cx(t)) \rangle \right]
= \langle (PA + A^*P)x(t), x(t) \rangle + 2\text{Re} \langle B\Delta(x(t), t)Cx(t), Px(t) \rangle
= -\rho^2\|y(t)\|_2^2 - \|B^*Px(t)\|_2^2 + 2\text{Re} \langle \Delta(x(t), t)y(t), B^*Px(t) \rangle
= -\|\Delta(x(t), t)y(t) - B^*Px(t)\|_2^2 - \rho^2\|y(t)\|_2^2 - \|\Delta(x(t), t)y(t)\|_2^2
\leq -\rho^2\|y(t)\|_2^2 - \|\Delta(x(t), t)y(t)\|_2^2, \quad t \in [t_0, t_+(t_0, x^0)),
\]

where $y(t) = Cx(t)$. Setting $k = r^2 - \rho^2$ we obtain $\dot{V}_\rho(x) \leq -k\|Cx\|^2$, $x \in \Omega$ for all $\Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\| \leq r$. If $(A, C)$ is observable, then $P > 0$ and so $\dot{V}_\rho$ is a time-invariant Liapunov function on $\Omega$ for all perturbed systems \((17)\) with $\Delta \in \Delta_{nt}(\Omega)$ satisfying $\|\Delta(\cdot, \cdot)\| \leq \rho$. In particular, setting $\rho = r_{\mathcal{C}}(A, B, C)$, the system \((17)\) is uniformly stable for all $\Delta \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\| \leq r_{\mathcal{C}}(A, B, C)$ (see [12, Thm.3.2.17]). This proves (iii).

We now abandon the observability assumption and prove (i). Suppose that $\Delta \in \Delta_{nt}(\Omega)$ and $\|\Delta(\cdot, \cdot)\| < \rho$. Then

\[
\dot{V}_\rho(x(t)) \leq 0, \quad t \in [t_0, t_+(t_0, x^0)) \quad \text{and} \quad \dot{V}_\rho(x(t)) = 0 \Rightarrow y(t) = Cx(t) = 0.
\]

Consider the decomposition

\[
x(t) = x_1(t) + x_2(t), \quad x_1(t) \in \ker P^+, \quad x_2(t) \in \ker P, \quad t \in [t_0, t_+(t_0, x^0)).
\]

Since there exists $\alpha > 0$ such that $\langle x, Px \rangle = \langle x_1, P x_1 \rangle \geq \alpha\|x_1\|_2^2$ for all $x \in \mathbb{C}^n$, we have

\[
\alpha\|x_1(t)\|_2^2 \leq \langle x_1(t), P x_1(t) \rangle \leq \langle x_1(t_0), P x_1(t_0) \rangle \leq \|P\|_{2,2}\|x_1(t_0)\|_2^2,
\]

for all $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$ and $t \in [t_0, t_+(t_0, x^0))$. But $\ker P \subset \ker C$, so it follows that there exists a constant $c > 0$, independent of $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$, such that

\[
\|y(t)\|_2 = \|Cx(t)\|_2 = \|Cx_1(t)\|_2 \leq c\|x_1(t_0)\|_2, \quad t \in [t_0, t_+(t_0, x^0)).
\]
Since $\|e^{At}\| \leq Me^{-\omega t}$ with suitable $M > 0$, $\omega > 0$, we have for all $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$

$$\|x(t)\|_2 = \|e^{A(t-t_0)}x^0 + \int_{t_0}^t e^{A(t-s)}B\Delta(x(s), s)y(s)ds\|_2 \leq Me^{-\omega(t-t_0)}\|x^0\|_2 + (M/\omega)\|B\|_{2,2} \|\Delta(\cdot, \cdot)\| \sup_{t \in [t_0, t_1+ (t_0, x^0)]} \|y(t)\|_2 \leq Me^{-\omega(t-t_0)}\|x^0\|_2 + (M/\omega)\|B\|_{2,2} \|\Delta(\cdot, \cdot)\| \|x(t_0)\|_2, \quad t \in [t_0, t_1+ (t_0, x^0)).$$

(23)

Let $\varepsilon > 0$ and $B(0, \varepsilon) = \{x \in \mathbb{C}^n; \|x\|_2 < \varepsilon\}$ be such that the closed ball $\overline{B}(0, \varepsilon)$ is contained in $\Omega$. Then by (23) there exists $\delta' > 0$ so that

$$x^0 \in B(0, \delta'), \quad t_0 \in \mathbb{R}_+, \quad t \in [t_0, t_1+ (t_0, x^0)) \implies \quad x(t) = x(t; t_0, x^0) \in B(0, \varepsilon).$$

Hence $t_1+ (t_0, x^0) = \infty$ for $(t_0, x^0) \in \mathbb{R}_+ \times B(0, \delta')$ and the system (17) is uniformly stable.

Now assume that $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$, $t_1+ (t_0, x^0) = \infty$ and $y(t) = Cx(t, t_0, x^0)$ does not tend to zero as $t \to \infty$. Then there exists $\varepsilon_1 > 0$ and a sequence $t_k \to \infty$ such that $\|y(t_k)\|_2 > 2\varepsilon_1$ for $k \in \mathbb{N}$. Now $\|x(t)\|_2, t \geq t_0$ is bounded by (23) and so via (17) we see that $\|\dot{x}(t)\|_2, t \geq t_0$ is also bounded. Hence there exists $\eta > 0$ such that

$$\|y(t)\|_2 > \varepsilon_1 \quad \text{for} \quad t \in [t_k, t_k + \eta], \quad k \in \mathbb{N}.$$

Let $\gamma := \rho^2 - \|\Delta(\cdot, \cdot)\|^2 > 0$. Then (19) implies for all $k \in \mathbb{N}$

$$V_\rho(x(t_k + \eta)) - V_\rho(x(k)) = \int_{t_k}^{t_k + \gamma} V_\rho(x(t)) dt \leq \int_{t_k}^{t_k + \gamma} (\rho^2 - \|\Delta(x(t), t)\|^2)\|y(t)\|^2 dt \leq -\gamma \varepsilon_1.$$

Since $V_\rho(x(t))$ is not increasing in $t$, this contradicts the fact that $V_\rho(x(t)) \geq 0, t \geq t_0$. Hence $\lim_{t \to \infty} \|y(t)\|_2 = 0$. Replacing $t_0$ by a sufficiently large $t_0', x^0$ by $x(t_0')$ and $t_1+ (t_0, x^0)$ by $\infty$, the first inequality in (23) becomes

$$\|x(t)\|_2 \leq Me^{-\omega(t-t_0)}\|x(t_0')\|_2 + (M/\omega)\|B\|_{2,2} \|\Delta(\cdot, \cdot)\| \sup_{t \in [t_0', \infty]} \|y(t)\|_2, \quad t \in [t_0', \infty).$$

We conclude that $x(t) \to 0$ as $t \to \infty$. Since $t_1+ (t_0, x^0) = \infty$ for all $x^0 \in B(0, \delta')$, setting $\rho = rC(A, B, C)$ completes the proof of (i).

Now suppose that $D_\delta = \{x \in \mathbb{C}^n; \langle x, Px \rangle < \delta\} \subset \Omega$ for some $\delta > 0$ and $\Delta = D_{\Delta x}(\Omega), \|\Delta(\cdot, \cdot)\| < \rho$. Let $x^0 \in D_\delta$ and choose $\delta_1 > 0$ such that $\langle x^0, Px^0 \rangle < \delta_1 < \delta$, then $D_{\Delta x}(\Omega)$ is contained in $\Omega$ and $\overline{D_{\Delta x}}$ is invariant for (17). Hence $t_1+ (t_0, x^0) = \infty$ and so, by (i), $(x(t) \to 0$ as $t \to \infty$. This proves (ii).

**Remark 2.7.** (i) If $P > 0$, then there will always exist a $\delta$ such that $D_\delta \subset \Omega$. However, this need not be the case if $\ker P \neq \{0\}$ (e.g. for bounded $\Omega$).

(ii) Suppose that $\Delta \in D_{\Delta x}(\Omega)$ and there exist $k > 0$ and a positive semi-definite $P \in \mathcal{H}_{n}(\mathbb{C})$ with $\ker P \subset \ker C$ such that the quadratic function $V(x) = \langle x, Px \rangle$ satisfies $\dot{V}(x) \leq -k\|Cx\|^2$ along the solutions of (17). Then by similar arguments as in the previous proof one can show that $\dot{x} = 0$ is an asymptotically stable equilibrium point of (17).

**3 Composite systems**

Let us introduce some additional notation. In the following, $\mathbf{q}, \mathbf{I}$ are finite $N$-tuples $\mathbf{l} = (l_1, \ldots, l_N)$, $\mathbf{q} = (q_1, \ldots, q_N)$, $l_j, q_j \in N \in \mathbb{N}$ and we set $q = \|q\| = \sum_{i=1}^m q_i$, $l = \|l\| = \sum_{j=1}^m l_j$. We denote by $\mathbb{K}^{l \times q}$ the set of $N \times N$ block matrices

$$[\Delta_{ij}] = [\Delta_{ij}]_{i,j \in N} = \begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1N} \\ \vdots & \ddots & \vdots \\ \Delta_{N1} & \cdots & \Delta_{NN} \end{bmatrix}, \quad \Delta_{ij} \in \mathbb{K}^{l \times q_i} \text{ for } (i, j) \in N \times N.$$ 

(24)
For any positive integer \( k \) we denote by \( \mathbb{K}^{k \times \mathbf{n}} \) the set of all block rows \([X_1, \ldots, X_N], X_j \in \mathbb{K}^{k \times q_j}, j \in \mathbb{N}\) and by \( \mathbb{K}^{1 \times k} \) the set of block columns \([Y_1^T, \ldots, Y_N^T]^T, Y_i \in \mathbb{K}^{1 \times k}, i \in \mathbb{N}\). The partitioned vectors in \( \mathbb{K}^{\mathbf{n}} := \mathbb{K}^{\mathbf{q} \times 1} \) are denoted by \( x = (x_i) \) where \( x_i \in \mathbb{K}^{\mathbf{n}} \) for \( i \in \mathbb{N} \). The block-diagonal matrix with blocks \( \Delta_j \in \mathbb{K}^{l_j \times q_j}, j \in \mathbb{N} \) is denoted by

\[
\Delta := \oplus_{j=1}^{\mathbb{N}} \Delta_j = \text{diag}(\Delta_1, \ldots, \Delta_N) = \begin{bmatrix}
\Delta_1 & 0 \\
0 & \Delta_2 \\
& \ddots \\
0 & & \ddots & 0 \\
\Delta_N
\end{bmatrix} \in \mathbb{K}^{\mathbf{q} \times \mathbf{q}}.
\]

Suppose that \( E \in \mathbb{R}^{N \times N} \) is a given nonnegative matrix with entries \( e_{ij} \geq 0 \) and let \( \mathcal{I} = \{(i, j) \in \mathbb{N} \times \mathbb{N}; e_{ij} > 0\} \) denote the set of positions of the positive entries of \( E \) in row \( i \). We say that \( \Delta = (\Delta_{ij}) \in \mathbb{C}^{l \times q} \) is of structure \( E \) if \( e_{ij} = 0 \) implies \( \Delta_{ij} = 0 \). Let \( \Delta_E \subseteq \mathbb{C}^{l \times q} \) be the vector space of all the block matrices \( \Delta \in \mathbb{C}^{l \times q} \) of structure \( E \).

To describe the perturbations \( A \sim A_{\Delta} \) in a concise way we will make use of the **Hadamard product** of matrices. Given \( X = (x_{ij}), Y = (y_{ij}) \in \mathbb{C}^{h \times k} \) where \( h, k \) are positive integers, the **Hadamard product** of \( X \) and \( Y \), denoted by \( X \circ Y \), is defined by \( X \circ Y := (x_{ij}y_{ij}) \in \mathbb{C}^{h \times k} \). For \( k \in \mathbb{Z} \) the \( k^{th} \) Hadamard power of \( X \) is defined by \( X^k = (x_{ij}^k) \) where \( x_{ij}^k = x_{ij}^k \) if \( x_{ij} \neq 0 \) and \( x_{ij}^0 = 0 \) if \( x_{ij} = 0 \). Given a matrix \( X = (x_{ij}) \in \mathbb{C}^{N \times N} \) and a block matrix \( Y = [Y_{jk}]_{j,k \in \mathbb{N}} \in \mathbb{C}^{l \times q} \), the **Hadamard block product** of \( X \) and \( Y \) is by definition the block matrix \( X \circ Y := [x_{ij}Y_{ij}]_{i,j \in \mathbb{N}} \).

Given \( (A_i, B_i, C_i) \in \mathbb{L}_{n_i, l_i, q_i}, i \in \mathbb{N} \), the object of this paper is to study the variation of the spectrum of the block-diagonal matrix \( A = \oplus_{i=1}^{\mathbb{N}} A_i \in \mathbb{C}^{n \times n} \) under perturbations of the form

\[
A \sim A_{\Delta} := A + B(E \circ \Delta)C, \quad \Delta \in \Delta_E,
\]

where \( \mathbf{n} = (n_1, \ldots, n_N) \) is given, \( n := |\mathbf{n}| \) and \( B, C \) are the block-diagonal matrices \( B = \oplus_{i=1}^{\mathbb{N}} B_i \in \mathbb{C}^{n \times 1}, C = \oplus_{i=1}^{\mathbb{N}} C_i \in \mathbb{C}^{q \times n} \). The non-negative matrix \( E \) has a double role. On the one hand it defines the **structure of the admissible perturbations**, i.e., the perturbation set \( \Delta_E \). On the other hand the positive entries \( e_{ij} \) of \( E \) provide **weights** for the blocks \( \Delta_{ij} \). Note that since these weights cannot, in general, be absorbed by the matrices \( B_i \) and/or \( C_j \), they provide an additional scaling flexibility. The scaled blocks \( e_{ij} \Delta_{ij} \) represent uncertain couplings between the subsystems described by the triplets \( (A_i, B_i, C_i) \in \mathbb{L}_{n_i, l_i, q_i}, i \in \mathbb{N} \). In fact, consider the system

\[
\Sigma : \quad \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)
\]

which is the direct sum of the \( N \) subsystems

\[
\Sigma_i : \quad \dot{x}_i(t) = A_ix_i(t) + B_iu_i(t), \quad y_i(t) = C_ix_i(t), \quad i \in \mathbb{N}.
\]

The transfer matrix of \( \Sigma \) is the direct sum of the transfer matrices of these subsystems

\[
G(s) = C(sI - A)^{-1}B = \oplus_{i=1}^{\mathbb{N}} G_i(s), \quad G_i(s) := C_i(sI_{n_i} - A_i)^{-1}B_i, \quad i \in \mathbb{N}.
\]

Introducing the couplings

\[
u_i(t) = \sum_{j \in \mathcal{I}_i} e_{ij} \Delta_{ij} y_j(t), \quad i \in \mathbb{N}
\]

one obtains the coupled subsystem equations

\[
\dot{x}_i(t) = A_ix_i(t) + B_i \sum_{j \in \mathcal{I}_i} e_{ij} \Delta_{ij} C_jx_j(t), \quad i \in \mathbb{N}
\]
which together describe the composite system

\[
\Sigma_\Delta : \begin{bmatrix}
\dot{x}_1 \\
\vdots \\
\dot{x}_N
\end{bmatrix} = (A + B E) C \begin{bmatrix}
x_1 \\
\vdots \\
x_N
\end{bmatrix} = A_\Delta \begin{bmatrix}
x_1 \\
\vdots \\
x_N
\end{bmatrix}.
\]

Thus the perturbed system $\Sigma_\Delta$ with system matrix $A_\Delta$ can be viewed as the composite system obtained by interconnecting the subsystems $\Sigma_i$ via the uncertain couplings (29) determined by the unknown perturbation blocks $\Delta_{ij}$. The unperturbed (“nominal”) system $\Sigma_0 : \dot{x} = Ax$ obtained by setting $\Delta = 0$ is simply the direct sum of the subsystems $\dot{x}_i = A_i x_i$.

The interconnection structure of the composite system, determined by the matrix $E$, is best illustrated by drawing the associated graph $G(E)$, [13]. The node set of this directed graph is $N$ or alternatively, in the present context of interconnected systems, the set of subsystems $\{\Sigma_i; i \in N\}$. The set of directed arcs is given by $I = \{(i, j); e_{ij} > 0\}$ where the pair $(i, j)$ denotes the arc from the node $\Sigma_j$ to the node $\Sigma_i$.

**Example 3.1.** Consider the following structure matrix $E$ and the associated graph $G(E)$.

\[
E = \begin{bmatrix}
0 & 20 & 0 & 0 \\
20 & 0 & 1 & 10 \\
10 & 0 & 0 & 5 \\
0 & 0 & 0 & 0.1
\end{bmatrix},
\]

The set of directed arcs of $G(E)$ is $I = \{(1, 2), (2, 1), (2, 3), (2, 4), (3, 1), (3, 4), (4, 4)\}$. The diagonal entries of $E$ portray a perturbation structure where the first three subsystems are unperturbed whereas the fourth one is subjected to perturbations of small weight. The off-diagonal entries model a situation where perturbation blocks of similar size will cause a strong interaction between the first two subsystems, a medium influence of the fourth on the second subsystem, a medium influence of the first on the third subsystem, a lesser influence of the fourth on the third subsystem and a comparatively small influence of the third on the second subsystem.

4 **Stability radii and spectral value sets**

In this section we derive computable formulas for the spectral value sets and stability radii of the block-diagonal system $\dot{x} = Ax$ with respect to structured perturbations of the form (25). We continue to use the set-up of the previous section and begin by extending the perturbation space from $\Delta_{E}$ to $\Delta = \mathbb{C}^{1 \times q} \supset \Delta_{E}$. Let $E^0$ be the $N \times N$-matrix obtained from $E$ by normalizing all its non-zero entries to 1, i.e. $E^0 = (e^0_{ij})$ with $e^0_{ij} = 1$ if $e_{ij} > 0$ and $e^0_{ij} = 0$ if $e_{ij} = 0$. Clearly, $E = E \circ E^0$ and we have for all $\Delta \in \mathbb{C}^{1 \times q}$

\[
\Delta_0 := E^0 \circ \Delta \in \Delta_{E} \quad \text{and} \quad E \circ \Delta = E \circ \Delta_0.
\]

The block matrix $\Delta_0 = [e^0_{ij} \Delta_{ij}]$ is obtained from $\Delta$ by replacing all the blocks $\Delta_{ij}$ in $\Delta$ for which $e_{ij} = 0$ by zero blocks of the same dimensions. Because of (32) the set of perturbed matrices $A_\Delta$ is not extended if we extend the perturbation space from $\Delta_{E}$ to $\Delta = \mathbb{C}^{1 \times q}$:

\[
\{A_\Delta; \Delta \in \Delta_{E}\} = \{A_\Delta; \Delta \in \mathbb{C}^{1 \times q}\}.
\]

Note that this is the reverse of the standard notation in graph theory. We have used our notation in order to be in harmony with the system theoretic interpretation.
Since this leads to a substantial simplification in the notation we will henceforth allow ∆ to vary in \( \mathbb{C}^{l \times q} \) and consider perturbations of the form
\[
A \sim A_\Delta := A + B(E \circ \Delta)C = A + B(E \circ \Delta_0)C, \quad \Delta \in \Delta = \mathbb{C}^{l \times q}. \tag{33}
\]

Let us introduce a norm on the extended perturbation space \( \Delta \). If \( \Delta_i = [\Delta_{i1}, \cdots, \Delta_{iN}] \) is the \( i \)-th block row of \( \Delta = [\Delta_{jk}] \in \mathbb{C}^{l \times q}, \ i \in \mathbb{N} \), then
\[
\|\Delta_i\|_{2,2}^2 = \lambda_{\max}(\Delta_i\Delta_i^*) = \lambda_{\max}(\sum_{j \in \mathbb{N}} \Delta_{ij}\Delta_{ij}^*). \tag{34}
\]

On the space of perturbation matrices \( \Delta = \mathbb{C}^{l \times q} \) we introduce the mixed norm \( \| \cdot \|_\Delta \) defined by
\[
\|\Delta\|_\Delta = \max_{i \in \mathbb{N}} \|\Delta_i\|_{2,2} = \max_{i \in \mathbb{N}} \left( \lambda_{\max}(\sum_{j \in \mathbb{N}} \Delta_{ij}\Delta_{ij}^*) \right)^{1/2}, \quad \Delta = (\Delta_{ij}) \in \Delta. \tag{35}
\]

\( \|\Delta\|_\Delta \) is the operator norm of \( \Delta \) as a linear map from \( \mathbb{C}^q \) provided with the 2-norm to \( \mathbb{C}^l \) provided with the \( (2|\infty) \)-Hölder norm \( \| \cdot \|_{2|\infty} \) defined by
\[
\|u\|_{2|\infty} = \max_{i \in \mathbb{N}} \|u_i\|_2, \quad u = (u_i) \in \mathbb{C}^l, \ u_i \in \mathbb{C}^{l_i}, \ i \in \mathbb{N}.
\]

The spectral value set (Def. 2.2) of the block-diagonal matrix \( A \) at uncertainty level \( \delta \) (with respect to the perturbations of the form (33)) is denoted by \( \sigma_\Delta(A, B, C; E; \delta) \) and the corresponding stability radius (Def. 2.3) by \( r_\Delta(A, B, C; E) \). Since for every \( \Delta \in \Delta \) we have \( \Delta = E^0 \circ \Delta \in \Delta_E \) and \( \|\Delta_0\|_\Delta \leq \|\Delta\|_\Delta \) we get from (32) and (33)
\[
\sigma_\Delta(A, B, C; E; \delta) = \bigcup_{\Delta \in \Delta, \|\Delta\|_\Delta < \delta} \sigma(A \Delta) = \bigcup_{\Delta \in \Delta_E, \|\Delta\|_\Delta < \delta} \sigma(A \Delta) = \sigma_{E^0}(A, B, C; E; \delta) \tag{36}
\]

\( r_\Delta(A, B, C, E) = \inf\{\|\Delta\|_\Delta; \Delta \in \Delta, \sigma(A \Delta) \not\subset \mathbb{C}_-\} = r_{\Delta_E}(A, B, C, E) \).

Let \( \Pi_N \) be the group of permutations of the set \( N \) and \( \pi \in \Pi_N \). For every matrix \( X = (x_{ij}) \in \mathbb{C}^{N \times N} \) and every block matrix \( Y = [Y_{ij}]_{i,j \in N} \in \mathbb{C}^{l \times q} \) we define the matrix (respectively block matrix) obtained by simultaneous permutation \( \pi \) of its rows and columns (respectively block rows and block columns) as follows
\[
\pi(X) = (x_{\pi(i)\pi(j)})_{i,j \in N} \quad \text{and} \quad \pi(Y) = [Y_{\pi(i)\pi(j)}]_{i,j \in N} \in \mathbb{C}^{\pi(l) \times \pi(\theta)} \tag{37}
\]

where \( \pi(1) = (l_{\pi(1)}, \ldots, l_{\pi(N)}) \) and \( \pi(q) = (q_{\pi(1)}, \ldots, q_{\pi(N)}) \). Then
\[
\pi(X) \circ \pi(Y) = \pi(X \circ Y), \quad X \in \mathbb{C}^{N \times N}, \ Y \in \mathbb{C}^{l \times q}, \ \pi \in \Pi_N. \tag{38}
\]

In order to determine \( \sigma_\Delta(A, B, C, E; \delta) \) and \( r_\Delta(A, B, C, E) \) we will sometimes make the assumption that the structure matrix \( E \) is irreducible. \( E \) is said to be irreducible if it is not possible to find a perturbation \( \pi \in \Pi_N \) such that \( \pi(E) = \begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{bmatrix} \), with \( E_{11} \in \mathbb{R}^{N_1 \times N_1}, \ E_{22} \in \mathbb{R}^{(N-N_1) \times (N-N_1)} \) for some \( N_1 \in [1, N) \). The irreducibility of \( E \) has a nice graph theoretical interpretation. Let \( \mathcal{G}(E) \) be the directed graph corresponding to the matrix \( E \), see Section 3. Then the matrix \( E \) is irreducible if and only if \( \mathcal{G}(E) \) is strongly connected, see [4, §2.2].

If \( E \) is reducible, it is known (see [4, §2.3]) that \( E \) can be reduced by a simultaneous row and column permutation \( \pi \in \Pi_N \) to a block-triangular form
\[
\pi(E) = \begin{bmatrix} E_{11} & 0 & \cdots & 0 \\ E_{21} & E_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ E_{s1} & E_{s2} & \cdots & E_{ss} \end{bmatrix} \tag{39}
\]
where each diagonal block $E_{hh} \in \mathbb{R}^{N_h \times N_h}$, $h \in \mathcal{S}$ is square and is either irreducible or a $1 \times 1$ null matrix. Applying the permutation $\pi$ to $A + B(E \odot \Delta)C$ we obtain

$$
\pi(A + B(E \odot \Delta)C) = \pi(A) + \pi(B)(\pi(E) \odot \pi(\Delta))\pi(C)
$$

(40)

where

$$
\pi(A) = \oplus_{h=1}^{s} A^h, \quad \pi(B) = \oplus_{h=1}^{s} B^h, \quad \pi(C) = \oplus_{h=1}^{s} C^h, \quad \pi(\Delta) = (\Delta^{hk})_{h,k \in \mathcal{S}} \in \mathbb{C}^{\pi(s) \times \pi(s)}.
$$

(41)

The super-blocks $A^h, B^h, C^h, h \in \mathcal{S}$ are block-diagonal matrices with $N_h$ diagonal blocks on the diagonal and, for any $\Delta \in \Delta_E$, $\pi(\Delta)$ is a lower block-triangular matrix with the super-blocks $\Delta^{hk}$ consisting of $N_h \times N_k$ blocks $\Delta_{ij}$ of $\Delta$, $h, k \in \mathcal{S}$. Clearly the blocks on the diagonals of $\pi(A)$, $\pi(B)$, $\pi(C)$ are a permutation of the diagonal blocks of $A$, $B$, $C$, respectively. Moreover, if $\Delta \in \Delta_E$, the super-blocks $\Delta^{hk}$ are of the structure $E_{hk}$ in the sense that if an entry $E_{hk}(i, j)$ of $E_{hk}$ is zero then the block $\Delta^{hk}(i, j) = 0$.

**Example 4.1.** Suppose we are given four subsystems $(A_i, B_i, C_i) \in L_{n_i, l_i, q_i}, i = 1, \ldots, 4$ and the interconnection matrix $E \in \mathbb{R}^{4 \times 4}$ as in Example 3.1. The graph $\mathcal{G}(E)$ has two strongly connected components with node sets $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ and $\{\Sigma_4\}$. Choose $\pi \in \Pi_4$ to be the permutation which maps $1, 2, 3, 4$ to $1, 2, 3$, respectively. Then the permuted matrix $\pi(E)$ is of the form (39) with $s = 2$:

$$
\pi(E) = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ 0 & 20 & 0 & 1 \\ 10 & 20 & 0 & 0 \end{bmatrix} = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix}, \quad E_{11} = [0.1, E_{21} = \begin{bmatrix} 0 & 10 \\ 5 & 0 \end{bmatrix}, E_{22} = \begin{bmatrix} 0 & 20 & 0 \\ 20 & 0 & 1 \\ 10 & 0 & 0 \end{bmatrix}.
$$

The block-diagonal matrices $\pi(A)$, $\pi(B)$, $\pi(C)$ consist of each of two super-blocks. For instance, $\pi(A) = A_1^1 \oplus A_2^2$ where $A_1^1 = A_1$ and $A_2^2 = A_1 \oplus A_2 \oplus A_3$. The perturbation matrices $\Delta \in \Delta_E$, respectively $\pi(\Delta) \in \Delta_E(\pi(\Delta))$ are of the following form:

$$
\Delta = \begin{bmatrix} 0 & \Delta_{12} & 0 & 0 \\ \Delta_{21} & 0 & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & 0 & 0 & \Delta_{34} \\ 0 & 0 & 0 & \Delta_{44} \end{bmatrix}, \quad \pi(\Delta) = \begin{bmatrix} \Delta_{44} & 0 & 0 & 0 \\ 0 & 0 & \Delta_{12} & 0 \\ \Delta_{24} & \Delta_{21} & 0 & \Delta_{23} \\ \Delta_{34} & \Delta_{31} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \Delta_{11} & 0 \\ \Delta_{12} & \Delta_{22} \end{bmatrix}.
$$

\[ \square \]

The spectrum and the perturbation norm remain invariant under the simultaneous permutation of block rows and block columns:

$$
\sigma(A + B(E \odot \Delta)C) = \sigma(\pi(A) + \pi(B)(\pi(E) \odot \pi(\Delta))\pi(C))
$$

$$
\|\Delta\|_\Delta = \max_{i \in \mathcal{N}_E} \left[ \lambda_{\max} \left( \sum_{j \in \mathcal{N}} \Delta_{ij} \Delta_{ij}^* \right) \right]^{1/2} = \max_{i \in \mathcal{N}_E} \left[ \lambda_{\max} \left( \sum_{j \in \mathcal{N}} \Delta_{pi(i)j} \Delta_{pi(i)j}^* \right) \right]^{1/2} = \|\pi(\Delta)\|_\pi(\Delta).
$$

Hence

$$
\sigma_\Delta(A, B, C; E; \delta) = \sigma_{\pi(\Delta)}(\pi(A), \pi(B), \pi(C), \pi(E); \delta)
$$

$$
r_\Delta(A, B, C, E) = r_{\pi(\Delta)}(\pi(A), \pi(B), \pi(C), \pi(E))
$$

(42)

and we may therefore assume, wherever convenient, that $E$ has the block-triangular structure as $\pi(E)$ in (39). Because of the block-triangular structure of $\pi(A) + \pi(B)(\pi(E) \odot \pi(\Delta))\pi(C)$ the proof of the following lemma is straightforward.

**Lemma 4.2.** Suppose that $E$ is reduced to lower block-triangular form as in (39) by a permutation $\pi \in \Pi_N$, and $\pi(A), \pi(B), \pi(C)$ and $\pi(\Delta)$, $\Delta \in \Delta$ are decomposed as in (41). If, for any pair $(h,k) \in \mathcal{S} \times \mathcal{S}$, we provide the space of super-blocks $\Delta^{hk} = \{\Delta^{hk} \in \Delta \}$ with the norm

$$
\|\Delta^{hk}\|_{\Delta^{hk}} = \max_{i \in \mathcal{N}_h} \left[ \lambda_{\max} \left( \sum_{j \in \mathcal{N}_k} \Delta^{hk}_{ij} \Delta^{hk}_{ij}^* \right) \right]^{1/2}, \quad \Delta^{hk} = (\Delta^{hk})_{i \in \mathcal{N}_h, j \in \mathcal{N}_k},
$$

(43)
then
\[
\sigma_{\Delta}(A, B, C, E; \delta) = \bigcup_{h \in \mathbb{Z}} \sigma_{\Delta_{hh}}(A^h_x, B^h_x, C^h_x, E_{hh}; \delta), \quad \delta > 0
\] (44)
and
\[
r_{\Delta}(A, B, C, E) = \min_{h \in \mathbb{Z}} r_{\Delta_{hh}}(A^h_x, B^h_x, C^h_x, E_{hh}).
\] (45)

**Remark 4.3.** If \(E_{hh}\) is a \(1 \times 1\) null matrix then \(A^h_x = [A_i]\) for some \(i \in \mathbb{N}\). It follows that the eigenvalues of \(A_i\) are fixed eigenvalues of all perturbed matrices \(A + B(E \circ \Delta)C, \Delta \in \Delta\) and \(r_{\Delta_{hh}}(A^h_x, B^h_x, C^h_x, E_{hh}) = \infty\) if \(A\) is asymptotically stable. Hence Lemma 4.2 shows that the analysis of the spectral perturbation problem under consideration can be reduced to the case where \(E\) is irreducible. \(\square\)

We illustrate Lemma 4.2 by applying it to the data of Example 4.1.

**Example 4.4.** Using the notations of Example 4.1 we obtain from (44) and (45) that
\[
\sigma_{\Delta}(A, B, C, E; \delta) = \sigma_{\Delta_{11}}(A^1_x, B^1_x, C^1_x, E_{11}; \delta) \cup \sigma_{\Delta_{22}}(A^2_x, B^2_x, C^2_x, E_{22}; \delta)
\]
\[
r_{\Delta}(A, B, C, E) = \min \{r_{\Delta_{11}}(A^1_x, B^1_x, C^1_x, E_{11}), r_{\Delta_{22}}(A^2_x, B^2_x, C^2_x, E_{22})\}
\]
where
\[
A^1_x = A_4, \quad A^2_x = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}, \quad E_{11} = [0.1], \quad E_{22} = \begin{bmatrix} 0 & 20 & 0 \\ 20 & 0 & 1 \\ 10 & 0 & 0 \end{bmatrix}
\]
and the components \(B^i_x, C^i_x, i = 1, 2\) of the block-diagonal matrices \(B, C\) have an analogous structure as those of \(A\). The perturbations of \(A^1_x\) are of the form
\[
A^1_x = A_4 + 0.1B^1_x \Delta_{11} C^1_x = A_4 + 0.1B_4 \Delta_{44} C_4
\]
whereas the perturbations of \(A^2_x\) are of the form
\[
A^2_x = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \sim A^2_x + B^2_x (E_{22} \circ \Delta_{22}) C^2_x = \begin{bmatrix} A_1 & 20A_{12} & 0 \\ 20\Delta_{21} & A_2 & \Delta_{23} \\ 10\Delta_{11} & 0 & A_3 \end{bmatrix}.
\]

Note that by the above formulas the perturbation blocks \(\Delta_{24}, \Delta_{44}\) have no influence on \(\sigma_{\Delta}(A, B, C, E; \delta)\) and \(r_{\Delta}(A, B, C, E)\). The spectrum of \(A_\Delta\) remains invariant if these blocks in the perturbation matrix \(\Delta\) are changed. \(\square\)

In order to determine the stability radius \(r_{\Delta}(A, B, C, E)\) of \(A\) with respect to perturbations of the form (33) we will make use of the Perron-Frobenius theory of non-negative matrices \(M \in \mathbb{R}^{N \times N}_+\), see [4], [13, Ch. 8].

**Lemma 4.5.** Suppose \(M, M' \in \mathbb{R}^{N \times N}_+, \alpha, \beta \in \mathbb{R}\). Then

(i) \(\varphi(M) \in \sigma(M)\) and there exists a non-negative eigenvector \(z\) of \(M\) corresponding to the eigenvalue \(\varphi(M)\) (Perron vector). If \(M\) is irreducible, the Perron vector is uniquely determined modulo multiplication by a positive scalar and all its coordinates are positive.

(ii) If there exists \(z \in \mathbb{R}^n_+, z \neq 0\) such that \(Mz \geq \alpha z\) then \(\varphi(M) \geq \alpha\).

(iii) If there exists \(z > 0\) such that \(Mz \leq \beta z\) then \(\varphi(M) \leq \beta\).

(iv) If \(M \leq M'\) then \(\varphi(M) \leq \varphi(M')\). If \(M\) is irreducible and \(M \leq M', M \neq M'\) then \(\varphi(M) < \varphi(M')\).

(v) If \((M_k)\) is a sequence in \(\mathbb{R}^{N \times N}_+\) converging to \(M\) then \(\varphi(M_k) \to \varphi(M)\) as \(k \to \infty\).

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Proof: (i)-(iv) follow from the Perron-Frobenius theory of non-negative matrices, see [13, Thm. 8.3.1] and [13, Thm. 8.4.4] for (i), [13, Thm. 8.3.2] for (ii), [13, Cor. 8.1.29] for (iii), [13, Cor. 8.1.19] and [4, Cor. 1.3.29] for (iv).

(v) follows from the continuous dependence of the spectrum \(\sigma(M)\) on the matrix \(M\). \(\square\)

Remark 4.6. Let \(E\) be a non-negative \(N \times N\)-matrix, \(G(E)\) the associated directed graph and \(Z_0(E)\) the set of cycles of this digraph. Then

\[ \phi(E) = 0 \iff E^N = 0 \iff Z_0(E) = \emptyset. \]

The following theorem is the main result of this section and is the key tool for determining \(\sigma_\Delta(A, B, C, E; \delta)\) and \(r_\Delta(A, B, C, E)\).

**Theorem 4.7.** Suppose \(E \in \mathbb{R}^{N \times N}_+\) and \((A_i,B_i,C_i) \in \mathbb{M}_{n_i,q_i}, \ i \in \mathbb{N}\) are given. If \(A = \bigoplus_{i=1}^N A_i\) is nonsingular, \(B = \bigoplus_{i=1}^N B_i, C = \bigoplus_{i=1}^N C_i\) and \(\Delta = \mathbb{C}^{1 \times q}\) is provided with the norm (35), then the “distance of \(A\) from singularity” with respect to perturbations of the form (33) is given by

\[ d_\Delta(A,B,C,E) := \inf\{\|\Delta\|_\Delta; \Delta \in \Delta \text{ and } \det(A + B(E \circ \Delta)C) = 0\} = \frac{1}{\sqrt{\phi(D^2E^{\circ 2})}} \tag{46} \]

where \(D = \text{diag}(\|C_1A_1^{-1}B_1\|_{2,2}, \ldots, \|C_NA_N^{-1}B_N\|_{2,2})\). If additionally \(A, B, C\) are real then also

\[ \inf\{\|\Delta\|_\Delta; \Delta \in \Delta \cap \mathbb{R}^{1 \times q} \text{ and } \det(A + B(E \circ \Delta)C) = 0\} = \frac{1}{\sqrt{\phi(D^2E^{\circ 2})}}. \tag{47} \]

**Proof:** Suppose that \(A\) is nonsingular and \((A + B(E \circ \Delta)C)x = 0\) for some \(\Delta = (\Delta_{ij}) \in \Delta, x = (x_i) \in \mathbb{C}^n, x_i \in \mathbb{C}^{n_i}, x \neq 0\). Then

\[ -A_i x_i = B_i \sum_{j \in \mathbb{N}} e_{ij} \Delta_{ij} C_j x_j, \quad i \in \mathbb{N}. \tag{48} \]

Setting \(y_i = C_i x_i\) and \(z_i = \|y_i\|_2^2, i \in \mathbb{N}\) we obtain by (48) and (35)

\[ y_i = -C_i A_i^{-1} B_i \sum_{j \in \mathbb{N}} e_{ij} \Delta_{ij} y_j \quad \text{and} \quad z_i \leq \|C_i A_i^{-1} B_i\|_{2,2}^2 \|\Delta\|_\Delta^2 \sum_{j \in \mathbb{N}} e_{ij}^2 z_j, \quad i \in \mathbb{N}. \]

The \(y_i\) cannot all be zero since otherwise \(Ax = 0\) by (48) and \(x \neq 0\) would imply that \(A\) is singular. Hence \(z = (z_i) \in \mathbb{R}^\mathbb{N}_+\) satisfies \(z \leq \|\Delta\|_\Delta^2 D^2E^{\circ 2} z, z \neq 0\). By Lemma 4.5 this implies \(\|\Delta\|_\Delta^2 \leq \phi(D^2E^{\circ 2})\) and so the inequality \(\leq\) in (46). We also see that there cannot exist a \(\Delta \in \Delta\) such that \(\det(A + B(E \circ \Delta)C) = 0\) if \(\phi(D^2E^{\circ 2}) = 0\). So equality holds in (46) if \(\phi(D^2E^{\circ 2}) = 0\) (making use of the conventions (1)).

To prove the converse inequality \(\leq\) in (46) we may therefore assume \(\phi(D^2E^{\circ 2}) > 0\). We construct a perturbation matrix \(\Delta \in \Delta_\mathbb{R}\) satisfying \(\|\Delta\|_\Delta^2 = 1/\phi(D^2E^{\circ 2})\) and a vector \(x \in \mathbb{C}^n, x \neq 0\) such that \((A + B(E \circ \Delta)C)x = 0\). By the Perron-Frobenius theory there exists \(z = (z_1, ..., z_N)^T \geq 0, z \neq 0\) such that \(D^2E^{\circ 2} z = \phi(D^2E^{\circ 2}) z\) that is

\[ \|C_i A_i^{-1} B_i\|_{2,2}^2 \sum_{j \in \mathbb{N}} e_{ij}^2 z_j = \phi(D^2E^{\circ 2}) z_i, \quad i \in \mathbb{N}. \tag{49} \]

If \(\|C_i A_i^{-1} B_i\|_{2,2} = 0\) then necessarily \(z_i = 0\) by (49). Hence there exists, for every \(i \in \mathbb{N}\), a vector \(u_i \in \mathbb{C}^{n_i}\) such that \(y_i := C_i A_i^{-1} B_i u_i\) satisfies

\[ \|y_i\|_2 = \|C_i A_i^{-1} B_i\|_{2,2}^2 \|u_i\|_2^2 \quad \text{and} \quad \|y_i\|_2^2 = z_i. \tag{50} \]
Here we can choose \( u_i = 0 \) if \( z_i = 0 \). Setting \( x_i = A_i^{-1}B_iu_i \) for \( i \in \mathbb{N} \) we obtain that \( x_i = 0 \) whenever \( z_i = 0 \). Now define for \( i \in \mathbb{N} \),
\[
\Delta_{ij} = \begin{cases} 
{e_{ij}u_iu_j^*} / \left( \sum_{k \in \mathbb{N}} e_{ik}^2 \|y_k\|^2 \right) & \text{if } j \in I_i \text{ and } \sum_{k \in \mathbb{N}} e_{ik}^2 \|y_k\|^2 \neq 0 \\
0 & \text{if } j \notin I_i \text{ or } \sum_{k \in \mathbb{N}} e_{ik}^2 \|y_k\|^2 = 0.
\end{cases}
\tag{51}
\]
If \( \sum_{k \in \mathbb{N}} e_{ik}^2 \|y_k\|^2 = \sum_{k \in \mathbb{N}} e_{ik}^2 z_k = 0 \) for some \( i \in \mathbb{N} \), then \( \Delta_{ij} = 0 \) for all \( j \in \mathbb{N} \) by (51), and \( z_i = 0 \) by (49) so that \( x_i = 0 \) and
\[
A_ix_i + B_i \sum_{j \in \mathbb{N}} e_{ij} \Delta_{ij} C_j x_j = B_i \sum_{j \in \mathbb{N}} e_{ij} \Delta_{ij} y_j = 0.
\]
On the other hand, if \( \sum_{k \in \mathbb{N}} e_{ik}^2 \|y_k\|^2 \neq 0 \) then by (51)
\[
A_ix_i + B_i \sum_{j \in \mathbb{N}} e_{ij} \Delta_{ij} C_j x_j = A_ix_i - B_iu_i \left( \sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2 \right) = A_ix_i - B_iu_i = 0.
\]
We conclude that \( (A + B(E \circ \Delta))x = 0 \) where \( x := (x_i) \in \mathbb{C}^n \), \( x \neq 0 \). It remains to show that \( \|\Delta\|_\Delta = 1/\sqrt{\varrho(D^2E^{(2)})} \). Now, for every \( i \in \mathbb{N} \) such that \( \sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2 \neq 0 \), we have
\[
\sum_{j=1}^N \Delta_{ij} \Delta_{ij}^* = \sum_{j \in \mathbb{N}} \left( \sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2 \right) = \sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2
\]
and so by (50) and (49)
\[
\lambda_{\max}(\sum_{j \in \mathbb{N}} \Delta_{ij} \Delta_{ij}^*) = \frac{\lambda_{\max}(u_ii_i^*)}{\sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2} = \frac{\|u_i\|^2}{\sum_{j \in \mathbb{N}} e_{ij}^2 z_j} = \begin{cases} z_i/\|C_iA_i^{-1}B_i\|^2 & z_i \neq 0 \\
0 & z_i = 0
\end{cases} \geq 1/\varrho(D^2E^{(2)}), \quad z_i \neq 0
\]
Since \( z \neq 0 \) there exists \( i \in \mathbb{N} \) such that \( z_i \neq 0 \) and hence by (49) \( \sum_{j \in \mathbb{N}} e_{ij}^2 \|y_j\|^2 \neq 0 \). It follows that \( \|\Delta\|_\Delta = \max_{i \in \mathbb{N}} \lambda_{\max}(\sum_{j \in \mathbb{N}} \Delta_{ij} \Delta_{ij}^*) = 1/\varrho(D^2E^{(2)}) \) and this concludes the proof of (46).
If \( A, B, C \) are real, the \( u_i, i \in \mathbb{N} \) can be chosen to be real and then the perturbation matrix \( \Delta \) constructed above is real and (47) follows from the previous proof.

**Remark 4.8.** If \( \varrho(D^2E^{(2)}) = 0 \) then \( A + B(E \circ \Delta)C \) is nonsingular for all \( \Delta \in \Delta_E \). If \( \varrho(D^2E^{(2)}) > 0 \) the above proof shows how to construct a perturbation \( \Delta \in \Delta_E \) (resp. \( \Delta \in \Delta_E \cap \mathbb{R}^{1 \times q} \)) of minimum norm \( \|\Delta\|_\Delta \) such that \( A + B(E \circ \Delta)C \) is singular. In this case the “inf” can be replaced by “min” in (46). \( \square \)

**Corollary 4.9.** Suppose \( E \in \mathbb{R}_+^{N \times N} \) and \( (A_i, B_i, C_i) \in \mathbb{L}_{m, i, q_i}, i \in \mathbb{N} \) are given. If \( A = \oplus_{i=1}^N A_i, B = \oplus_{i=1}^N B_i, C = \oplus_{i=1}^N C_i, \) and \( \Delta = \mathbb{C}^{1 \times q} \) is provided with the norm (35), then
\begin{enumerate}[(i)]
\item For every \( \delta > 0 \) the spectral value set of \( A \) of level \( \delta \) with respect to perturbations of the form (33) is
\[
\sigma_\Delta(A, B, C; E; \delta) = \bigcup_{\Delta \in \Delta_E, \|\Delta\|_\Delta < \delta} \sigma(A + B(E \circ \Delta)C) = \sigma(A) \cup \{s \in \rho(A); \varrho(D(s)^2E^{(2)}) > \delta^{-2}\}
\tag{52}
\]
where \( D(s) = \text{diag}(\|G_1(s)\|_{2,2}, \ldots, \|G_N(s)\|_{2,2}), G_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i, i \in \mathbb{N} \).
\item If \( A \) is Hurwitz stable, its stability radius with respect to perturbations of the form (33) is
\[
r_\Delta(A, B, C, E) = \left[ \max_{\omega \in \mathbb{R}} \varrho \left( D(\omega)^2E^{(2)} \right) \right]^{-1/2}.
\tag{53}
\end{enumerate}
Proof: (i) By definition we have $\sigma(A) \subset \sigma_\Delta(A, B, C, E; \delta)$. Now suppose $s \in \rho(A)$. Since $s \in \sigma(A + B(E \circ \Delta)C)$ holds if and only if $\det((sI_n - A) - B(E \circ \Delta)C)) = 0$, we obtain from (46) that $s \in \sigma_\Delta(A, B, C, E; \delta)$ if and only if $1/\sqrt{\varrho(D(s)^2E^2)} < \delta$. This proves (52).

(ii) By the definition of $r_\Delta(A, B, C, E)$, the continuity of the spectrum and by (52) we have for every $\delta \geq 0$

$$\delta > r_\Delta(A, B, C, E) \iff \sigma_\Delta(A, B, C, E; \delta) \cap \mathbb{N} \neq \emptyset \iff \exists \omega \in \mathbb{R} : \varrho(D(\omega)^2E^2) > 1/\delta^2.$$

Observing that the function $\omega \mapsto \varrho(D(\omega)^2E^2)$ admits a maximum on $\mathbb{R}$ since $\lim_{|\omega| \to \infty} \varrho(D(\omega)^2E^2) = 0$, this proves (53). \qed

Remark 4.10. Suppose $A, B, C$ are real and $\omega \mapsto \varrho(D(\omega)^2E^2)$ admits its maximum on $\mathbb{R}_+$ at $\omega = 0$. Then $D(0) = D$ and it follows from Theorem 4.7 that the stability radii of $A$ with respect to complex and with respect to real perturbations of structure $E$ are equal:

$$r_\Delta(A, B, C, E) = r_\Delta(A, B, C, E) = [\varrho(D(0)^2E^2)]^{-1/2}.$$

As an illustration of Corollary 4.9 we consider an example where $E$ is of “cyclic” structure.

Example 4.11. Suppose that $(A_i, B_i, C_i) \in \mathbb{L}_{n_i, d_i, q_i}$, $i \in N$, $G_i(s) = C_i(sI_{n_i} - A_i)^{-1}B_i$, and

$$E = \begin{bmatrix}
0 & e_{12} & 0 & \cdots & 0 \\
0 & 0 & e_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e_{N-1,N} \\
e_{N,1} & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad D(s) = \text{diag} ((\|G_1(s)||_{2,2}, \ldots, \|G_N(s)||_{2,2})). \quad (54)$$

Then

$$\varrho(D(s)^2E^2) = \varrho\left(\begin{bmatrix}
0 & e_{12}^2 ||G_1(s)||_{2,2}^2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
e_{N-1,N}^2 ||G_N(s)||_{2,2}^2 & 0 & \cdots & 0
\end{bmatrix}\right)$$

and we obtain from (53) that

$$r_\Delta(A, B, C, E) = \left[\max_{\omega \in \mathbb{R}} \varrho(D(\omega)^2E^2)\right]^{-1/2} = \left[\max_{\omega \in \mathbb{R}} \sqrt{e_{12}^2 \cdots e_{N-1,N}^2 ||G_1(s)||_{2,2}^2 \cdots ||G_N(s)||_{2,2}^2}\right]^{-1}.$$

In particular, if we have e.g.

$$N = 2, \quad A = \begin{bmatrix} -1 + i & 0 \\
0 & -2 + i \end{bmatrix}, \quad B = C = I_2, \quad E = \begin{bmatrix} 0 & 1 \\
1 & 0 \end{bmatrix},$$

then $r_\Delta = r_\Delta(A, I_2, I_2, E)$ is given by

$$r_\Delta = \left[\max_{\omega \in \mathbb{R}} \sqrt{|1 + (\omega - 1)i|^{-1} \cdot |2 + (\omega - 1)i|^{-1}}\right]^{-1} = \left[\min_{\omega \in \mathbb{R}} |1 + (\omega - 1)^2| |4 + (\omega - 1)^2|\right]^{1/4} = \sqrt{2}. \quad \Box$$
Although the perturbations (33) are structured, our approach does not make use of \( \mu \)-values.

In order to explain how our results are related to \( \mu \)-analysis, we describe the perturbed system \( \dot{x} = A_\Delta x \) equivalently by an equation of the form

\[
\dot{x} = Ax + B\tilde{\Delta} \tilde{C}x
\]

where \( \tilde{C} \) is a suitable matrix and \( \tilde{\Delta} \) is a block-diagonal counterpart of \( \Delta \) with spectral norm \( \|\tilde{\Delta}\|_{2,2} = \|\Delta\|_\Delta \). Given \( \Delta = (\Delta_{ij}) \in \Delta \) we define \( \tilde{\Delta} \) by

\[
\tilde{\Delta} = \text{diag} (\Delta^1, \ldots, \Delta^N) \in \mathbb{C}^{1 \times Nq}, \quad \Delta^i = [\Delta_{i1}, \ldots, \Delta_{iN}] \in \mathbb{C}^{i \times q}.
\] (55)

Note that the spectral norm of \( \tilde{\Delta} \) satisfies

\[
\|\tilde{\Delta}\|_2^2 = \max_{i \in N} \|\Delta^i\|_2^2 = \|\Delta\|_\Delta, \quad \Delta = (\Delta_{ij}) \in \Delta.
\] (56)

Now define \( \tilde{C} \in \mathbb{C}^{Nq \times n} \) by

\[
\tilde{C} = \begin{bmatrix}
\tilde{C}^1 \\
\vdots \\
\tilde{C}^N
\end{bmatrix}, \quad \tilde{C}^i = \text{diag} (e_{i1} C_1, \ldots, e_{iN} C_N) \in \mathbb{C}^{q \times n}.
\] (57)

Note that each block row of \( \tilde{C} \) contains at most one non-zero block and hence the non-zero blocks of any two different block columns of \( \tilde{C} \) occur at different positions. As a consequence \( \tilde{C}^* \tilde{C} \) is block-diagonal,

\[
\tilde{C}^* \tilde{C} = \sum_{i \in N} \tilde{C}^i* \tilde{C}^i = \text{diag} \left( (\sum_{i \in N} e_{i1}^2) C_1^* C_1, \ldots, (\sum_{i \in N} e_{iN}^2) C_N^* C_N \right) \in \mathbb{C}^{n \times n}.
\] (58)

Moreover, we have

\[
\Delta^i \tilde{C}^i = [e_{i1} \Delta_{i1} C_1, \ldots, e_{iN} \Delta_{iN} C_N], \quad \tilde{\Delta} \tilde{C} = \begin{bmatrix}
\Delta^1 \tilde{C}^1 \\
\vdots \\
\Delta^N \tilde{C}^N
\end{bmatrix} = (E \circ \Delta) C
\] (59)

and hence

\[
A_\Delta = A + B(E \circ \Delta) C = A + B\tilde{\Delta} \tilde{C}, \quad \Delta \in \Delta.
\] (60)

Defining \( \tilde{\Delta} = \{ \tilde{\Delta}; \Delta \in \Delta \} \) it follows from (60) and (56) that

\[
\inf \{ \|\Delta\|_\Delta; \Delta \in \Delta \text{ and } \det(A_\Delta) = 0 \} = \inf \{ \|\tilde{\Delta}\|_{2,2}; \tilde{\Delta} \in \tilde{\Delta} \text{ and } \det(A + B\tilde{\Delta} \tilde{C}) = 0 \}.
\]

On the other hand since \( \det(I + UV) = \det(I + VU) \) for \( U \in \mathbb{C}^{h \times k}, V \in \mathbb{C}^{k \times h}, h, k \in \mathbb{N} \) we have the equivalence

\[
\det(A + B\tilde{\Delta} \tilde{C}) = 0 \iff \det(I + \tilde{\Delta} \tilde{C} A^{-1} B) = 0.
\]

If we endow \( \tilde{\Delta} \) with the spectral norm, we therefore obtain

\[
d_\Delta(A, B, C, E) = \inf \{ \|\Delta\|_\Delta; \Delta \in \Delta \text{ and } \det(A_\Delta) = 0 \} = 1/\mu(\tilde{G})
\] (61)

where \( \tilde{G} := \tilde{C} A^{-1} B. \)

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Remark 4.12. Especially for sparse matrices $E$ one can reduce the dimensions of $\hat{\Delta}$ and $\hat{C}$ a lot by eliminating in the rows of $\Delta$, (55) all the blocks $\Delta_{ij}$ for which $e_{ij} = 0$ and removing in $C^i$ (57) all the block rows for which $e_{ij} = 0$. However, this would considerably complicate the notation and would not provide any computational advantage since the blown-up matrices $\hat{\Delta}$ and $\hat{C}$ will not be used in calculations.

The characterization (61) expresses the distance $d_\Delta(A, B, C, E)$ in terms of the $\mu$-value of the blown-up matrix $G := \hat{C}A^{-1}B$ with respect to the set $\Delta$ of blown-up block-diagonal perturbations $\hat{\Delta}$. The question arises if it is possible to characterize $d_\Delta(A, B, C, E)$ in terms of the $\mu$-value of the more natural “transfer function at $s = 0$”, $G = CA^{-1}B$ and the given perturbation set $\Delta_E$. Since $\Delta_E$ only reflects the structure of $E$ but not the magnitude of its non-zero entries, such a characterization will only be possible, if the missing information is included in the set-up, e.g. via a suitable norm on $\Delta_E$ depending on $E$. The following corollary shows how this can be done.

Corollary 4.13. Suppose the conditions of Theorem 4.7 and $\Delta_E$ is provided with the norm

$$
\|\Delta\|_{\Delta_E} = \|E^{-1}\circ \Delta\|_{\Delta}.
$$

If $G = CA^{-1}B$ then $\mu_{\Delta_E}(G) = \sqrt{\vartheta(D^2E^{-2})}$ and

$$
\inf\{\|\Delta\|; \Delta \in \Delta \text{ and } \det(A + B(E \circ \Delta)C) = 0\} = 1/\mu_{\Delta_E}(G).
$$

Proof: For every $\Delta \in \Delta_E$ define $\Delta_E = E \circ \Delta \in \Delta_E$. Then $\|\Delta_E\|_{\Delta_E} = \|\Delta\|_{\Delta}$ for all $\Delta \in \Delta_E$ and $\Delta \mapsto \Delta_E$ is a norm preserving isomorphism from the normed space $(\Delta_E, \|\cdot\|_{\Delta_E})$ onto $(\Delta_E, \|\cdot\|_{\Delta_E})$. Moreover, we obtain from (58) that for every $\Delta \in \Delta$

$$
det(A + B(E \circ \Delta)C) = 0 \iff det(I + A^{-1}B(E \circ \Delta)C) = 0 \iff det(I + \Delta_E G) = det(I + (E \circ \Delta)CA^{-1}B) = 0.
$$

Now by definition $\mu_{\Delta_E}(G)^{-1}$ equals

$$
\inf\{\|\Delta\|; \Delta \in \Delta \text{, det}(I + \Delta_E G) = 0\} = \inf\{\|\Delta\|; \Delta \in \Delta \text{, det}(I + G) = 0\}.
$$

Hence (63) follows from (64), and $\mu_{\Delta_E}(G) = \sqrt{\vartheta(D^2E^{-2})}$ is a consequence of Theorem 4.7.

Remark 4.14. There are many different possibilities for representing the perturbed matrices $A_\Delta = A + B(E \circ \Delta)C$, $\Delta \in \Delta$ in the form $A_\Delta = A + \hat{B}\hat{\Delta}\hat{C}$, $\hat{\Delta} \in \Delta$ where $\Delta$ is a vector space of block-diagonal matrices. (60) is just one of these representations, with $\hat{B} = \hat{B}$, $\hat{C}$ defined by (57) and $\Delta$ is the set of all block-diagonal perturbations $\hat{\Delta}$ of the form (55). Another representation of $A_\Delta$ with block-diagonal perturbations $\hat{\Delta}$ is $A_\Delta = A + \hat{B}\hat{\Delta}\hat{C}$ where

$$
\hat{B} = \oplus_{i \in N} [B_{i1}, \ldots, B_{iN}], B_{ij} = B_i; \quad \hat{\Delta} = \oplus_{i \in N} \oplus_{j \in N} \Delta_{ij}; \quad \hat{C} = \begin{bmatrix} \oplus_{j \in N} e_{ij} C_j \\ \vdots \\ \oplus_{j \in N} e_{Nj} C_j \end{bmatrix}.
$$

In order to relate this perturbation structure to the Riccati equation associated with the system $(\hat{A}, \hat{B}, \hat{C})$ via Theorem 2.5, we must provide $\Delta$ with a norm $\|\cdot\|_\Delta$ such that $\|\Delta\|_\Delta = \|\hat{\Delta}\|_{2,2}$ for all $\Delta \in \Delta_E$. If we choose $\hat{B}, \hat{C}, \hat{\Delta}$ as in (65), a suitable norm on $\Delta$ would be $\|\Delta\|_\Delta := \max_{i,j \in N} \|\Delta_{ij}\|_{2,2}$. If instead the following weighted norm is introduced on $\Delta_E$

$$
\|\Delta\|_{\Delta_E} := \max_{(i,j) \in E} \epsilon_{ij}^{-1} \|\Delta_{ij}\|_{2,2}
$$

then

$$
\|E \circ \Delta\|_{\Delta_E} = \|\Delta\|_\Delta = \|\hat{\Delta}\|_{2,2}, \quad \Delta \in \Delta_E,
$$

and one can prove that, with respect to the norms $\|\cdot\|_\Delta$ and $\|\cdot\|_{\Delta_E}$ just defined,

$$
\inf\{\|\Delta\|; \Delta \in \Delta \text{ and } \det(A + B(E \circ \Delta)C) = 0\} = 1/\mu_{\Delta_E}(G)
$$

In [16, Theorem 4.2] the following formula for the $\mu$-value of any block-diagonal matrix $G = \text{diag}(G_1, \ldots, G_N) \in \mathbb{C}^{n \times 1}$ (with respect to the norm $\|\cdot\|_{\Delta_E}$ on $\Delta_E$) was derived

$$
\mu_{\Delta_E}(G) = \vartheta(ED) \quad \text{where} \quad D = \text{diag}(\|G_1\|_{2,2}, \ldots, \|G_N\|_{2,2}).
$$

An analysis of the corresponding Riccati equations and quadratic stability problem is more complicated and for lack of space cannot be dealt with in this article.
5 Riccati equation

We continue to use the set-up of Section 3. There are various parametrized algebraic Riccati equations which can be used to construct joint quadratic Liapunov functions for perturbed systems of the form $\dot{x} = A_\Delta x$ (31). If we neglected the structure matrix $E$ and considered the full block case where $A_\Delta = A + B\Delta C$, $\Delta \in \mathbb{C}^{l \times q}$ the associated Riccati equation would be

$$PA + A^*P + \rho^2 C^*C + PBB^*P = 0. \quad (67)$$

If $\sigma(A) \subset \mathbb{C}_-$, then by Theorem 2.5 this Riccati equation has a Hermitian solution if and only if

$$\rho \leq \left[ \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} \right]^{-1} = \left[ \max_{i \in N} \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} \right]^{-1} = \min_{i \in N} g_i^{-1}, \quad g_i := \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} \quad (68)$$

where $G(s)$ is the transfer function of the block-diagonal system $(A, B, C)$, see (28).

If we take the perturbation structure into account, different Riccati equations can be considered, depending on the representation of the perturbed system matrix $A_\Delta = A + B(E \circ \Delta)C$ in the form $A_\Delta = A + \tilde{B}\Delta \tilde{C}$. In this section we will focus on the block-diagonal representation (60) where the matrices $\tilde{B} = B, \tilde{C}$ and $\Delta$ are given by (57) and (55). The parametrized algebraic Riccati equation associated with the system $(A, B, C)$ is given by

$$PA + A^*P + \rho^2 \tilde{C}^*\tilde{C} + PBB^*P = 0, \quad \tilde{C}^*\tilde{C} = \text{diag} \left( \sum_{i \in N} c_{i1}^2 \gamma_i^*C_1, \ldots, \sum_{i \in N} c_{iN}^2 \gamma_i^*C_N \right). \quad (69)$$

We cannot expect to be able to characterize $r_\Delta(A, B, C, E)$ by the solvability of this Riccati equation. In fact, by Theorem 2.5, (69) has a Hermitian solution if and only if

$$\rho \leq r_C(A, B, \tilde{C}) = \inf \{ \|\tilde{\Delta}\|_{2,2} ; \tilde{\Delta} \in \mathbb{C}^{l \times Nq} \text{ and } \sigma(A + B\tilde{C}) \notin \mathbb{C}_- \} = \left[ \max_{\omega \in \mathbb{R}} \|\tilde{G}(\omega)\|_{2,2} \right]^{-1} \quad (70)$$

where $\tilde{G} = \tilde{C}(sI - A)^{-1}B$. Thus (69) is related to a much larger class of perturbations than the block-diagonal perturbations of the form (55). We will therefore expect that, in general, $r_C(A, B, \tilde{C}) < r_\Delta(A, B, C, E)$.

In order to get a better lower bound of $r_\Delta(A, B, C, E)$ we use the scaling technique described in §2. For any scaling vector $\gamma = (\gamma_1, \ldots, \gamma_N)$ where $\gamma_i > 0, i \in N$ we set $R_\gamma = \text{diag}(\gamma_1I_q, \ldots, \gamma_NI_q)$ and $L_\gamma = \text{diag}(\gamma_1I_{1}, \ldots, \gamma_NI_{N})$. Then since $\tilde{\Delta}$ is block-diagonal of the form (55) we have $\tilde{\Delta} = L_\gamma^{-1}\Delta R_\gamma$ and hence setting $B_\gamma = BL_\gamma^{-1}$, $\tilde{C}_\gamma = R_\gamma \tilde{C}$ we obtain from (60) that

$$B(E \circ \Delta)C = B\tilde{\Delta} \tilde{C} = B_\gamma \tilde{C}_\gamma. \quad (71)$$

$\tilde{C}_\gamma$ has the same structure as $\tilde{C}$ but with $c_{ij}$ replaced by $\gamma_i e_{ij}$. More precisely,

$$B_\gamma = \text{diag}(\gamma_1^{-1}B_1, \ldots, \gamma_N^{-1}B_N),$$

$$\tilde{C}_\gamma = \begin{bmatrix} \tilde{C}_\gamma^1 \\ \vdots \\ \tilde{C}_\gamma^N \end{bmatrix}, \quad \tilde{C}_\gamma^i = \text{diag}(\gamma_i e_{11}C_1, \ldots, \gamma_i e_{1N}C_N). \quad (72)$$

The Riccati equation associated with the triple $(A, B_\gamma, \tilde{C}_\gamma)$ is

$$PA + A^*P + \rho^2 \tilde{C}_\gamma^*\tilde{C}_\gamma + PB_\gamma B_\gamma^*P = 0. \quad (73)$$
Theorem 5.1. Suppose that $E \in \mathbb{R}^N \times N$, $(A_i, B_i, C_i) \in L_{m_i, d_i, q_i}$, $i \in \mathbb{N}$ and a scaling vector $\gamma = (\gamma_1, \ldots, \gamma_N) > 0$ are given, $A = \oplus_{i=1}^N A_i$, $\sigma(A) \subset \mathbb{C}_-$, $B$ and $\hat{C}_\gamma$ are defined by (72), and $\Delta$ is provided with the norm (35). Then

(i) The algebraic Riccati equation (73) has a Hermitian solution if and only if

$$\rho^2 \leq \rho_\gamma^2 := \left[ \max_{j \in \mathbb{N}} g_j^2 \sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2 \right]^{-1} \min_{j \in \mathbb{N}} \frac{\gamma_j^2}{\sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2}$$

(74)

where $g_j = \max_{\omega \in \mathbb{R}} ||G_j(\omega)||_{2, 2}$ and $G_j = C_i(sI - A_i)^{-1} B_i$, $i \in \mathbb{N}$.

(ii) If $\rho \leq \rho_\gamma$, the smallest solution $P(\rho, \gamma)$ of (73) (satisfying $\sigma(A + B \gamma B_r^* \rho) \subset \mathbb{C}_-$) is of the form $P(\rho, \gamma) = \oplus_{j \in \mathbb{N}} P_j(\rho, \gamma)$ where $P_j(\rho, \gamma), j \in \mathbb{N}$ is the smallest solution of the algebraic Riccati equation

$$P_j A_j + A_j^* P_j + \rho^2 \sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2 C_j^* C_j + \gamma_j^{-2} P_j B_j B_j^* P_j = 0.$$ 

(75)

(iii) $\sigma(A + B(E \circ \Delta) C) \subset \mathbb{C}_-$ for all $\Delta \in \mathbb{D}$ with $||\Delta||_\Delta < \rho_\gamma$.

(iv) Suppose $E$ does not have a zero column, $\rho \leq \rho_\gamma$ and $P$ is a Hermitian solution of (73).

If $r \leq \rho$ there exists a constant $k > 0$ such that the derivative of the quadratic function $V_\rho(x) = \langle x, P x \rangle$ along trajectories of $\dot{x} = A \Delta x$ (31) satisfies $\dot{V}_\rho(x) \leq -k ||Cx||^2$, $x \in \mathbb{C}^n$ for all $\Delta \in \mathbb{D}$, $||\Delta||_\Delta \leq r$. If the pairs $(A_j, C_j)$, $j \in \mathbb{N}$ are observable, then the pair $(A, \hat{C}_\gamma)$ is observable and $V_\rho(x) = \langle x, P x \rangle$ is a joint Lyapunov function for all perturbed systems $\dot{x} = (A + B(E \circ \Delta) C)x$ with $\Delta \in \mathbb{D}$, $||\Delta||_\Delta \leq \rho$.

Proof: (i) It follows from (58) and (72) that $\hat{C}_\gamma \hat{C}_\gamma$ is the block-diagonal matrix

$$\hat{C}_\gamma \hat{C}_\gamma = \text{diag} \left( \sum_{i=1}^N \gamma_i^2 \epsilon_{i1}^2 C_i^* C_i, \ldots, \sum_{i=1}^N \gamma_i^2 \epsilon_{iN}^2 C_i^* C_i \right).$$

(76)

So if $G_j(\gamma)$ is the transfer function of the system $(A, B, \hat{C}_\gamma)$ and $\Theta = \int_0^\infty e^{A_j^* t} \hat{C}_\gamma C_j e^{A_j t} dt$ is the observability Gramian of the pair $(A, \hat{C}_\gamma)$, then

$$G_j(\gamma)^* G_j(\gamma) = \text{diag} \left( \sum_{i=1}^N \gamma_i^2 \epsilon_{i1}^2 / \gamma_1^2 G_1(s) G_1(\gamma), \ldots, \sum_{i=1}^N \gamma_i^2 \epsilon_{iN}^2 / \gamma_N^2 G_N(s) G_N(\gamma) \right),$$

(77)

$$\Theta = \text{diag} \left( \sum_{i=1}^N \gamma_i^2 \epsilon_{i1}^2 \Theta_1, \ldots, \sum_{i=1}^N \gamma_i^2 \epsilon_{iN}^2 \Theta_N \right).$$

(78)

By (77) we have $||G_j(\omega)||_{x, 2}^2 = \max_{\omega \in \mathbb{R}} \left( \sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2 / \gamma_i^2 \right) ||G_j(\omega)||_{x, 2}^2$ and hence

$$\max_{\omega \in \mathbb{R}} ||G_j(\omega)||_{x, 2}^2 = \max_{\omega \in \mathbb{R}} \left( \sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2 / \gamma_i^2 \right) \max_{\omega \in \mathbb{R}} ||G_j(\omega)||_{x, 2}^2.$$ 

(79)

Therefore (i) follows from Theorem 2.5.

(ii) If $\rho \leq \rho_\gamma$ then $\rho^2 \leq \left[ \sum_{i=1}^N \gamma_i^2 \epsilon_{ij}^2 / \gamma_i^2 \right] \max_{\omega \in \mathbb{R}} ||G_j(\omega)||_{x, 2}^2$ for $j \in \mathbb{N}$ and it follows from Theorem 2.5 that every Riccati equation (75), $j \in \mathbb{N}$ has a smallest Hermitian solution $P_j = P_j(\rho, \gamma)$ satisfying $\sigma(A_j + \gamma_j^{-2} B_j B_j^* P_j) \subset \mathbb{C}_-$.

Because of the block-diagonal structure of $A_j, \hat{C}_\gamma^* \hat{C}_\gamma$ and $B_j B_j^*$, the blockdiagonal Hermitian matrix $P := \oplus_{j \in \mathbb{N}} P_j(\rho, \gamma)$ is a solution of (73) satisfying $\sigma(A + B \gamma B^* P) \subset \mathbb{C}_-$. Since there exists only one such solution of (73) (which is the smallest one), (ii) follows.
are observable and $z$ is square and is either irreducible or a null matrix. Then, for every $y > 0$ converging towards $M$, we formulate the result for these data.

**Theorem 5.2.** Suppose $E \in \mathbb{R}_+^{N \times N}$, $g_1, \ldots, g_N \geq 0$ are given and, for any scaling vector $\gamma = (\gamma_1, \ldots, \gamma_N) > 0$, the number $\rho_\gamma$ is defined by (74). Then $\hat{\rho} = \sup_{\gamma \in (0, \infty)^N} \rho_\gamma$ is determined by

$$
\hat{\rho}^2 = (E^{\omega 2}D_g^2)^{-1} = (D_g^2E^{\omega 2})^{-1} \text{ where } D_g = \text{diag}(g_1, \ldots, g_N).
$$

For the proof of this theorem we will make use of the following lemma.

**Lemma 5.3.** For every non-negative matrix $M = (m_{ij}) \in \mathbb{R}_+^{N \times N}$

$$
\varrho(M) = \inf_{y > 0} \max_{j \in N} \frac{(y^\top M)_j}{y_j} = \inf_{y > 0} \max_{j \in N} \frac{1}{y_j} \sum_{i \in N} y_i m_{ij}.
$$

Moreover, the following conditions are equivalent:

(i) $\varrho(M) = \min_{y > 0} \max_{j \in N} \frac{(y^\top M)_j}{y_j}$.

(ii) There exists a positive row vector $z > 0$ such that $\max_{j \in N} \frac{1}{z_j} \sum_{i \in N} z_i m_{ij} = \varrho(M)$.

(iii) There exists a positive row vector $z > 0$ such that $zM \leq \varrho(M)z$.

(iv) Let $\pi \in \Pi_N$ be a permutation such that $\pi(M) = (M_{kk})_{h,k \in \mathbb{Z}}$ is of block-triangular structure where each diagonal block $M_{hh}$ is square and is either irreducible or a null matrix. Then, for every $k \in \mathbb{Z}$,

$$
\varrho(M_{kk}) = \varrho(M) \Rightarrow M_{kk} = 0 \text{ for all } i = k + 1, \ldots, s.
$$

In particular, conditions (i)-(iii) hold if $M$ is irreducible.

**Proof:** Let $(M_k)$ be a decreasing sequence of positive matrices $M_k > 0$ converging towards $M$. By [13, Cor. 8.1.31]

$$
\varrho(M_k) = \min_{y > 0} \max_{j \in N} \frac{(y^\top M_k)_j}{y_j}, \quad k \in \mathbb{N}.
$$

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Hence it follows by continuity and monotonicity of the spectral radius on $\mathbb{R}^{N \times N}_+$ (see Lemma 4.5) that
\[
\varrho(M) = \inf_{k \in \mathbb{N}} \varrho(M_k) = \inf_{k \in \mathbb{N}} \min_{y \geq 0} \max_{j \in \mathcal{N}} \frac{(y^\top M_k)_j}{y_j} = \inf_{y \geq 0} \min_{k \in \mathbb{N}} \max_{j \in \mathcal{N}} \frac{(y^\top M)_j}{y_j}.
\]
This proves (81). Let us now show the equivalence of the conditions (i)-(iv). (i) $\Rightarrow$ (ii) follows directly from (81). The equivalence of (ii) and (iii) follows because $zM \leq \varrho(M)z$ and $z > 0$ imply $\max_{j \in \mathcal{N}} (zM)_j / z_j = \varrho(M)$, again by (81).

Since conditions (i)-(iii) hold for $M$ if and only if they hold for any permutation $\pi(M)$, we may assume that $M = (M_{ij})$ is already of the block-triangular form described in (iv).

(iii) $\Rightarrow$ (iv): Suppose that $z = (z_1, \ldots, z_N) > 0$ satisfies $zM \leq \varrho(M)z$ and $\varrho(M_{kk}) = \varrho(M)$ for some $k \in \mathcal{N}$. If we partition $z$ in a compatible way with $M$, $z = (z^1, \ldots, z^s)$, $z^1 \in \mathbb{R}^{1 \times \nu_i}$, then $zM \leq \varrho(M)z$ implies
\[
z^k M_{kk} \leq \sum_{i=k}^s z^i M_{ik} \leq \varrho(M) z^k = \varrho(M_{kk}) z^k.
\]

On the other hand, it follows from (81) applied to $M_{kk}$ that $z^k M_{kk} \geq \varrho(M_{kk}) z^k$. Therefore $z^k M_{kk} = \sum_{i=k}^s z^i M_{ik}$ and so $\sum_{i=k+1}^s z^i M_{ik} = 0$ which implies $M_{ik} = 0$ for $i = k+1, \ldots, s$.

(iv) $\Rightarrow$ (iii) is proved by induction on $s$. If $s = 1$, then $M$ is irreducible or $N = 1$ and $M = [0]$. In the first case (iii) holds by Lemma 4.5 (i). In the second case $\varrho(M) = 0$ and (iii) holds for any $z = (z_1) > 0$. Now suppose the implication (iv) $\Rightarrow$ (iii) has been proved for $s = 1, \ldots, k-1$ for some $k \geq 2$ and $M$ is of the form (39) with $s = k$. Assume that (iv) holds. $M$ can be written as a triangular $2 \times 2$ block matrix
\[
M = \begin{bmatrix} M_{11} & 0_{\nu_1 \times (N-\nu_1)} \\ M_{21} & M_{22} \end{bmatrix}, \quad M_{11} = M_{111}, \quad M_{21} = \begin{bmatrix} M_{21} \\ \vdots \\ M_{k1} \end{bmatrix}, \quad M_{22} = \begin{bmatrix} M_{22} & 0 & \cdots & 0 \\ M_{32} & M_{33} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ M_{k2} & M_{k3} & \cdots & M_{kk} \end{bmatrix}.
\]

Let $\varrho = \varrho(M)$, $\varrho_1 = \varrho(M_{11})$, $\varrho_2 = \varrho(M_{22})$, $I_1 = \{1, \ldots, \nu_1\}$, $I_2 = \{\nu_1 + 1, \ldots, N\}$. Then $\varrho = \max\{\varrho_1, \varrho_2\}$. First suppose $\varrho_1 = \varrho$. Then $M_{11}$ is either irreducible or $M_{11} = [0]$, $\nu_1 = 1$ and $\varrho_1 = 0$. In either case there exists a left Perron vector $z^1 = (z_1, \ldots, z_{\nu_1}) > 0$ such that $z^1 M_{11} = \varrho_1 z^1$. If $\varrho_2 = \varrho$, then $M_{22}$ satisfies condition (iv) with $M$ replaced by $M_{22}$ and so there exists, by assumption of induction, a row vector $z^2 = (z_{\nu_1+1}, \ldots, z_N) > 0$ such that $z^2 M_{22} \leq \varrho z^2$.

Since $M_{11} = 0$ for $i = 2, \ldots, k$ it follows that $z = (z^1, z^2) = (z_1, \ldots, z_N)$ satisfies $zM \leq \varrho z$. If $\varrho_2 < \varrho$ then we may apply (81) to conclude that there exists $z^2 = (z_{\nu_1+1}, \ldots, z_N) > 0$ such that $z^2 M_{22} < \varrho z^2$. Again we obtain $zM \leq \varrho' z$ for $z = (z^1, z^2)$.

Finally suppose that $\varrho_1 < \varrho$. As above there exists a row vector $z^1 = (z_1, \ldots, z_{\nu_1}) > 0$ such that $z^1 M_{11} = \varrho_1 z^1$. On the other hand we necessarily have $\varrho_2 = \varrho$. Hence $M_{22}$ satisfies condition (iv) with $M$ replaced by $M_{22}$ and so there exists, by assumption of induction, a row vector $z^2 = (z_{\nu_1+1}, \ldots, z_N) > 0$ such that $z^2 M_{22} \leq \varrho z^2$. For any $\alpha > 0$ define $z(\alpha) = (z^1, \alpha z^2) = (z_1, \ldots, z_{\nu_1}, \alpha z_{\nu_1+1}, \ldots, \alpha z_N) > 0$. Since $\varrho_1 < \varrho$ we obtain $z(\alpha) M \leq \varrho z(\alpha)$ if $\alpha > 0$ is chosen sufficiently small. We conclude that in both cases, $\varrho_1 < \varrho$ and $\varrho_1 < \varrho$, there exists a row vector $z > 0$ such that $zM \leq \varrho(M)z$. This proves (iv) $\Rightarrow$ (iii).

**Proof of Theorem 5.2** First note that the second equality in (80) follows from the general fact that $\varrho(ED) = \varrho(DE)$ for any pair of square matrices $D, E$. If $\gamma > 0$ is any scaling vector we have by (74)
\[
\varrho(gD, gE)^{-2} = \max_{j \in \mathcal{N}} \frac{1}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e^2_{ij} g_j = \max_{j \in \mathcal{N}} \frac{\gamma^2 e_{ij}^2 D_j^2}{\gamma_j^2} = \max_{j \in \mathcal{N}} \frac{\gamma^2 e_{ij}^2 D_j^2}{\gamma_j^2} = \sum_{i=1}^N \frac{\gamma^2 e_{ij}^2 D_j^2}{\gamma_j^2}.
\]
Hence, applying (81) to \( M = E^2D_g^2 \) with \( y = \gamma^2 \) we obtain

\[
\hat{\rho}^{-2} = \left[ \sup_{\gamma \in (0, \infty)^N} \rho_(D_g, E) \right]^{-2} = \inf_{\gamma \in (0, \infty)^N} \rho_(D_g, E)^{-2} = \inf_{\gamma \in (0, \infty)^N} \max_{j \in \mathbb{N}} \frac{\gamma_j^2 E^{2N}D_g^2}{\gamma_j^2} = \rho(E^{2N}D_g^2)
\]

and this proves (80).

The following remark indicates that generically \( \hat{\rho} < r_\Delta(A, B, C, E) \).

**Remark 5.4.** It follows from Corollary 4.9 and Theorem 5.2 that

\[
r_\Delta(A, B, C, E) = \hat{\rho} \iff \max_{\omega \in \mathbb{R}} \rho(E^{2N}D(\omega)^2) = \rho(E^{2N}D_g^2).
\]  

(85)

Now suppose that \( E \) is irreducible and no \( G_i(s) \) vanishes identically. Then \( \hat{\rho}^2 = r_\Delta(A, B, C, E) \) if and only if there exists a joint maximum \( \omega_0 \in \mathbb{R} \) of the \( N \) functions \( \omega \mapsto \|G_i(\omega)\|_{2,2} \), i.e. such that \( \|G_i(\omega_0)\|_{2,2} = \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} \) for all \( i \in \mathbb{N} \). In fact, if this condition is not satisfied then \( 0 \leq D(\omega) \leq D_g, D(\omega) \neq D_g \) and so \( \rho(D(\omega)^2E^{2N}) < \rho(E^{2N}D_g^2) \) for all \( \omega \in \mathbb{R} \) by [4, Cor.2.1.5] since \( E^{2N}D_g^2 \) is irreducible. Conversely, if the condition is satisfied then the right hand equality in (85) follows directly from \( \|G_i(\omega_0)\|_{2,2} = \rho_\Delta(A, B, C, E) \).

Now suppose that \( E \) is reducible. Then \( \hat{\rho} = r_\Delta(A, B, C, E) \) if and only if there exists a strongly connected component of \( G_E \) with node set \( J \subset \mathbb{N} \) and a joint maximum \( \omega_0 \in \mathbb{R} \) such that \( \|G_i(\omega_0)\|_{2,2} = \max_{j \in J} \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} \) for all \( i \in J \).

We will now investigate under which conditions there exists an optimal scaling vector, i.e. \( \gamma > 0 \) such that \( \rho_\gamma = \hat{\rho} \).

**Theorem 5.5.** Suppose \( E \in \mathbb{R}^{N \times N}_+ \), \( g_1, \ldots, g_N \geq 0 \) are given, \( D_g = \text{diag}(g_1, \ldots, g_N) \) and, for any scaling vector \( \gamma > 0 \), the number \( \rho_\gamma = \rho_\gamma(D_g, E) \) is defined by (74) and \( \rho(D_g, E) = \sup_{\gamma \in (0, \infty)^N} \rho_\gamma \). Then the following conditions are equivalent for every scaling vector \( \hat{\gamma} = (\hat{\gamma}_1, \ldots, \hat{\gamma}_N) > 0 \):

(i) \( \hat{\gamma} \) is optimal, i.e. \( \rho_\gamma(D_g, E) = \hat{\rho}(D_g, E) \).

(ii) \( \hat{\gamma}^2 = (\hat{\gamma}_1^2, \ldots, \hat{\gamma}_N^2) \) satisfies \( \max_{j \in \mathbb{N}} \frac{\gamma^2E^{2N}D_g^2}{\gamma_j^2} = \rho(E^{2N}D_g^2) \). In particular, there exists an optimal scaling vector if and only if \( M = E^{2N}D_g^2 \) satisfies one of the equivalent conditions (i)-(iii) of Lemma 5.3.

**Proof:** By (74) and (80) condition (i) holds if and only if \( \hat{\gamma}^2 \) satisfies

\[
\max_{j \in \mathbb{N}} \frac{1}{\gamma_j^2} \sum_{i=1}^N \gamma_i^2 e_i g_j^2 = \rho_\gamma(D_g, E)^{-2} = \hat{\rho}(D_g, E)^{-2} = \rho(E^{2N}D_g^2).
\]

This proves (i) \( \Leftrightarrow \) (ii). Setting \( z = \hat{\gamma}^2 \) condition (ii) is identical with condition (ii) of Lemma 5.3 for \( M = E^{2N}D_g^2 \) and this yields the last statement of the theorem.

As a consequence of Theorem 5.5 there always exists an optimal scaling vector if \( E^{2N}D_g^2 \) is irreducible.

### 6 Nonlinear and/or time-varying perturbations

Throughout this section we suppose the following

\[
(A_i, B_i, C_i) \in \mathbb{L}_{n_i, a_i, q_i}, \quad \sigma(A_i) \subset \mathbb{C}_-, \quad G_i(s) = C_i(sI - A_i)^{-1}B_i, \quad i \in \mathbb{N},
\]

\[
A = \bigoplus_{i=1}^N A_i, \quad B = \bigoplus_{i=1}^N B_i, \quad C = \bigoplus_{i=1}^N C_i, \quad D(s) = \bigoplus_{i=1}^N \|G_i(s)\|_{2,2},
\]

\[
g_i := \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2}, \quad i \in \mathbb{N}, \quad D_g = \text{diag}(g_1, \ldots, g_N), \quad E \in \mathbb{R}^{N \times N}_+.
\]

(86)
Let $\Omega$ be an open neighbourhood of 0 in $\mathbb{C}^n$ and consider time-varying nonlinear perturbations of $\dot{x} = Ax$ of the form
\[ \dot{x} = Ax + B(E \circ \Delta(x,t))y, \quad y = Cx. \] (87)
where $\Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$. Here $\Delta_{nt}(\Omega)$ is the vector space of all bounded block matrix valued functions $\Delta(\cdot, \cdot) = (\Delta_{ij}(\cdot, \cdot)) : \Omega \times \mathbb{R}_+ \to \mathbb{C}^{l \times q}$ with the Carathéodory properties (see Section 2) provided with the norm
\[ \|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} = \sup_{x \in \Omega, t \geq 0} \|\Delta(x,t)\|, \quad \Delta(\cdot, \cdot) = (\Delta_{ij}(\cdot, \cdot)) \in \Delta_{nt}(\Omega). \] (88)

Note that for every $\Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$, $\Delta^F(\cdot, \cdot) = E \circ \Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$ is of structure $E$, i.e.
\[ \forall (x,t) \in \Omega \times \mathbb{R}_+: \quad e_{ij} = 0 \Rightarrow \Delta^F_{ij}(x,t) = 0. \]

By Carathéodory’s Theorem, for every $\Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$, $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$, there exists a unique solution $x_{\Delta}(t) = x_{\Delta}(t_0, x^0)$ of (87) with $x_{\Delta}(t_0) = x^0$ on some maximal semi-open interval $[t_0, t^*_\Delta(t_0, x^0))$ where $t^*_\Delta(t_0, x^0) > t_0$, see [12, Thm. 2.1.14]. In the following theorem we will see that $t^*_\Delta(t_0, x^0) = \infty$ if $\|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} < \rho(D_g, E)$ and $x^0$ is sufficiently close to the equilibrium state $\pi = 0$. For simplicity, we call the nonlinear system (87) uniformly (asymptotically) stable if $\pi = 0$ is a uniformly (asymptotically) stable equilibrium position of the system (87).

**Theorem 6.1.** Suppose (86). Then:

(i) The nonlinear system (87) is asymptotically stable for all $\Delta \in \Delta_{nt}(\Omega)$ satisfying
\[ \|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} < \rho(D_g, E) = \sup_{\gamma > 0} \rho_\gamma(D_g, E). \] (89)
Moreover if (89) holds, every trajectory $x_{\Delta}(t) = x_{\Delta}(t_0, x^0)$, $(t_0, x^0) \in \mathbb{R}_+ \times \Omega$ of (87) with an infinite life span $[t_0, \infty)$ tends to 0 as $t \to \infty$.

(ii) Suppose $\rho \leq \rho_\gamma(D_g, E)$ for some scaling vector $\gamma > 0$ and $P$ is a Hermitian solution of the associated algebraic Riccati equation (73). If $\delta > 0$ is such that $D_\delta = \{x \in \mathbb{C}^n; \langle x, Px \rangle < \delta \} \subset \Omega$, then $D_\delta$ is a joint domain of attraction of the equilibrium point $\pi = 0$ for all the systems (87) with $\Delta \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} < \rho$.

(iii) Suppose $E$ does not have a zero column, $\rho \leq \rho_\gamma(D_g, E)$ for some scaling vector $\gamma > 0$ and $P$ is a Hermitian solution of (73). If $r < \rho$, there exists a constant $k > 0$ such that the derivative of the quadratic function $V_\rho(x) = \langle x, Px \rangle$ along trajectories of (87) satisfies $V_\rho(x) \leq -k\|Cx\|^2$, $x \in \Omega$ for all $\Delta \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} \leq r$. If the pairs $(A_j, C_j)$, $j \in \mathbb{N}$ are observable, then $V_\rho(x)$ is a joint Lyapunov function at $\pi = 0$ for all perturbed systems (87) with $\Delta \in \Delta_{nt}(\Omega)$, $\|\Delta(\cdot, \cdot)\|_{\Delta_{nt}} \leq \rho$.

**Proof:** For any $\Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)$ define $\tilde{\Delta}(\cdot, \cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{C}^{l \times Nq}$ by
\[ \tilde{\Delta}(x, t) = \text{diag}(\Delta^1(x,t), \ldots, \Delta^N(x,t)), \quad (x, t) \in \Omega \times \mathbb{R}_+, \]
where $\Delta^i(x,t) \in \mathbb{C}^{l \times q}$, $i \in \mathbb{N}$ is the $i^{th}$ block row of $\Delta(x,t)$. We provide the vector space $\Delta_{nt}(\Omega)$ of all these $\tilde{\Delta}(\cdot, \cdot)$ with the norm (see (56))
\[ \|\tilde{\Delta}(\cdot, \cdot)\|_{\Delta_{nt}} := \sup_{x \in \Omega, t \geq 0} \|\tilde{\Delta}(x,t)\|_{2,2} = \sup_{x \in \Omega, t \geq 0} \|\Delta(x,t)\|_{\Delta} = \|\Delta(\cdot, \cdot)\|_{\Delta_{nt}}, \quad \tilde{\Delta}(\cdot, \cdot) \in \Delta_{nt}(\Omega). \] (90)
The map \( \Delta(\cdot, \cdot) \mapsto \tilde{\Delta}(\cdot, \cdot) \) is an (isometric) isomorphism from \( \Delta_{nt}(\Omega) \) onto \( \tilde{\Delta}_{nt}(\Omega) \). By (59) we have \( B(E \circ \Delta(x, t))C = B\tilde{\Delta}(x, t)\tilde{C} \) where \( \tilde{C} \) is as in (57). Since \( \Delta(x, t) \) is block-diagonal we may apply scaling to obtain from (71) that for all \( \gamma = (\gamma_1, \ldots, \gamma_N) > 0 \)
\[
B(E \circ \Delta(x, t))C = B\tilde{\Delta}(x, t)\tilde{C}_x, \quad (x, t) \in \Omega \times \mathbb{R}_+, \quad \Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega)
\] (91)
where \( B_{\gamma_1}, \tilde{C}_x \) are as in (72). Hence the uncertain time-varying nonlinear system (87) with \( \Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega) \) can equivalently be described by
\[
\dot{x} = Ax + B\tilde{\Delta}(x, t)\tilde{C}_x x
\] (92)
where \( \tilde{\Delta}(\cdot, \cdot) \in \tilde{\Delta}_{nt}(\Omega) \) is the associated block-diagonal perturbation.

The parametrized Riccati equation (73) is of the form (15) with \( (A, B, C) \) replaced by \( (A, B_{\gamma_1}, \tilde{C}_x) \). Hence, by Theorem 2.5, there exist Hermitian solutions \( P \) of (73) if

\[
\rho \leq \rho_1 = \max_{\omega \in \mathbb{R}} \| G_{\gamma}(\omega) \|_{2,2}^{-1} = r_C(A, B_{\gamma_1}, \tilde{C}_x) \text{ where } G_{\gamma}(s) = \tilde{C}_x(sI_n - A)^{-1}B_{\gamma}.
\]

By (91) the perturbed system (87) is of the form (17) with \( (A, B, C) \) replaced by \( (A, B_{\gamma_1}, \tilde{C}_x) \) and \( \Delta(\cdot, \cdot) \in \Delta_{nt}(\Omega) \) replaced by \( \tilde{\Delta}(\cdot, \cdot) \in \tilde{\Delta}_{nt}(\Omega) \). By Theorem 5.1 \( (A, \tilde{C}_x) \) is observable if the pairs \( (A_j, C_j) \), \( j \in N \) are observable and \( E \) does not have a zero column. Hence (ii) and (iii) follow by application of Theorem 2.6 to \( (A, B_{\gamma_1}, \tilde{C}_x) \) making use of (91) and the fact that there exist \( \alpha > 0, \beta > 0 \) such that \( \alpha C^*C \leq \tilde{C}_x^*\tilde{C}_x \leq \beta C^*C \). Finally (i) follows from Theorem 2.6 (i) since for every \( \Delta \in \Delta_{nt}(\Omega) \) satisfying \( \| \tilde{\Delta}(\cdot, \cdot) \|_{\Delta_{nt}} < \rho_1(D_\gamma, E) \) there exists \( \gamma > 0 \) such that \( \| \tilde{\Delta}(\cdot, \cdot) \|_{\tilde{\Delta}_{nt}} = \| \Delta(\cdot, \cdot) \|_{\Delta_{nt}} < \rho_1(D_\gamma, E) = r_C(A, B_{\gamma_1}, \tilde{C}_x). \)

We now examine whether or not (89) is a tight robustness estimate. In order to do this we introduce a stability radius with respect to time-varying linear and nonlinear perturbations. Consider the following time-varying linear system
\[
\dot{x}(t) = Ax(t) + B(E \circ \Delta(t))Cx(t)
\] (93)
where \( \Delta(\cdot) \in \Delta_{tv} \) and \( \Delta_{tv} \) is the vector space of all bounded measurable block matrix valued functions \( \Delta(\cdot) : \mathbb{R}_+ \to \mathbb{C}^{l \times q} \). We provide \( \Delta_{tv} \) with the \( tv \)-norm
\[
\| \Delta(\cdot) \|_{\Delta_{tv}} = \sup_{t \geq 0} \| \Delta(t) \|_{\mathbb{A}}, \quad \Delta \in \Delta_{tv}.
\] (94)
Similarly, we denote by \( \Delta_n \) the vector space of all perturbations \( \Delta \in \Delta_{nt} := \Delta_{nt}(\mathbb{C}^n) \) which are independent on time, i.e. \( \Delta(x, t) = \Delta(x) \), and provide it with the norm induced from \( \Delta_{nt} \):
\[
\| \Delta(\cdot) \|_n = \sup_{x \in \mathbb{C}^n} \| \Delta(x) \|_{\mathbb{A}}, \quad \Delta \in \Delta_n.
\] (95)
Note that, with the obvious embeddings \( \Delta \subset \Delta_{tv} \subset \Delta_{nt} \), the norm \( \| \cdot \|_{\Delta_{tv}} \) is the restriction of the norm \( \| \cdot \|_{\Delta_{nt}} \) to \( \Delta_{tv} \) and the norm \( \| \cdot \|_{\Delta} \) is the restriction of the norm \( \| \cdot \|_{\Delta_{tv}} \) to \( \Delta \).

**Definition 6.2.** Given \( A \in \mathbb{C}^{n \times n} \) the stability radius of \( A \) with respect to complex time-varying linear perturbations \( \Delta(\cdot) \in \Delta_{tv} \) is defined by
\[
r_{\Delta_{tv}}(A, B, C, E) = \inf \{ \| \Delta(\cdot) \|_{\Delta_{tv}} : \Delta(\cdot) \in \Delta_{tv} \text{ and } (93) \text{ is not asymptotically stable} \}.
\]
The stability radius of \( A \) with respect to complex nonlinear (resp. nonlinear time-varying) perturbations, \( r_{\Delta_n}(A, B, C, E) \) and \( r_{\Delta_{nt}}(A, B, C, E) \), are defined analogously.
In the full block case we have (see [12, §5.6])
\begin{equation}
r_{\Delta_{nt}}(A, B, C) = r_{\Delta_{nt}}(A, B, C) = r_{\Delta_{tv}}(A, B, C) = r_{\Delta}(A, B, C).
\end{equation}
We will see that for structured perturbations the last equality does, in general, not hold.

**Theorem 6.3.** Suppose the standing assumption of this section. Then
\begin{equation}
r_{\Delta_{nt}}(A, B, C, E) = r_{\Delta_{tv}}(A, B, C, E) = \varrho(E^{02}D_g^2)^{-1/2} = \rho(D_g, E).
\end{equation}

**Proof:** It follows from the definitions and the isometric embeddings $\Delta_{tv} \subset \Delta_{nt}$ that $r_{\Delta_{nt}}(A, B, C, E) \leq r_{\Delta_{tv}}(A, B, C, E)$. On the other hand Theorem 6.1 (with $\Omega = \mathbb{C}^n$) and Theorem 5.2 imply that $r_{\Delta_{tv}}(A, B, C, E) \geq \rho(D_g, E) = \sup_{r > 0} \rho_r(D_g, E) = \varrho(E^{02}D_g^2)^{-1/2}$. To prove (96) it therefore suffices to show $r_{\Delta_{nt}}(A, B, C, E) \leq \varrho(E^{02}D_g^2)^{-1/2}$. Suppose
\begin{equation}
\omega_i \in \mathbb{R} \text{ is such that } \|C_i(\omega_i I_{n_i} - A_i)^{-1}B_i\|_{2,2} = g_i, \ i \in \mathbb{N}.
\end{equation}
Replacing $A_i$ with $A_i - \omega_i I_{n_i}$, $i \in \mathbb{N}$ in Theorem 4.7 we see that $D$ is replaced by $D_g = \text{diag}(g_1, ..., g_N)$. So just as in the proof of Theorem 4.7 one can find $\Delta = (\Delta_{ij}) \in \Delta_E$ with $\|\Delta\|_\Delta = \varrho(E^{02}D_g^2)^{-1/2}$ such that
\begin{equation}
\det(A_\omega + B(E \circ \Delta)C) = 0, \ A_\omega = \text{diag}(A_1 - \omega_1 I_{n_1}, ..., A_N - \omega_N I_{n_N}).
\end{equation}
So there exists a non-zero $x = (x_i)_{i \in \mathbb{N}} \in \mathbb{C}^n$
\begin{equation}
x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \text{ such that } \begin{bmatrix} \omega_1 x_1 \\ \vdots \\ \omega_N x_N \end{bmatrix} = (A + B(E \circ \Delta)C) \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}.
\end{equation}
Let
\begin{equation}
\Delta(t) = (\Delta_{ij}(t)), \ \Delta_{ij}(t) = \Delta_{ij}e^{(\omega_i - \omega_j)t}, \ x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{bmatrix} = \begin{bmatrix} e^{\omega_1 t} x_1 \\ \vdots \\ e^{\omega_N t} x_N \end{bmatrix}.
\end{equation}
Then for each $i \in \mathbb{N}$
\begin{equation}
\dot{x}_i(t) = \omega_i e^{\omega_i t} x_i = e^{\omega_i t} \left[ A_i x_i + B_i \sum_{j=1}^N e_{ij} \Delta_{ij} C_j x_j \right] = A_i x_i(t) + B_i \sum_{j=1}^N e_{ij} \Delta_{ij}(t) C_j x_j(t).
\end{equation}
Hence $x(t)$ satisfies $\dot{x}(t) = A x(t) + B(E \circ \Delta(t)) C x(t)$ and $\|\Delta(\cdot)\|_{\Delta_{tv}} = \|\Delta\|_\Delta = \varrho(E^{02}D_g^2)^{-1/2}$. Since $x(t)$ does not tend to 0 as $t \to 0$, it follows that $\dot{x}(t) = A x(t) + B(E \circ \Delta(t)) C x(t)$ is not asymptotically stable and so $r_{\Delta_{tv}}(A) \leq \varrho(E^{02}D_g^2)^{-1/2}$. This completes the proof. \hfill $\Box$

**Remark 6.4.** The above theorem shows that the robustness bound (89) is tight. Moreover we have seen in the proof that there exists a minimum norm perturbation in $\Delta_{tv} \subset \Delta_{nt}$ which destroys asymptotic stability, that is there is a $\Delta(\cdot) \in \Delta_{tv}$ of tv-norm $r_{\Delta_{tv}}(A, B, C, E) = r_{\Delta_{nt}}(A, B, C, E)$ for which the system (93) is not asymptotically stable.

Now suppose that there is a scaling vector $\hat{\gamma} > 0$ satisfying $\rho_i = \hat{\rho} = \varrho(D_g, E)$ and $\hat{P} > 0$ is a Hermitian solution of the algebraic Riccati equation (73) with $\rho = \hat{\rho}$. Then $V_{\rho}(x) = \langle x, \hat{P} x \rangle$ is a joint Liapunov function of maximal robustness for the uncertain system (87). Then $V_{\hat{P}}(x)$ is a Liapunov function for every system (87) with $\Delta \in \Delta_{nt}$, $\|\Delta(\cdot)\|_{\Delta_{nt}} \leq \hat{\rho} = \varrho(E^{02}D_g^2)^{-1/2}$, and one can prove for every $\rho > \varrho(E^{02}D_g^2)^{-1/2}$ that there does not exist a Liapunov function for all the systems (87) with $\Delta \in \Delta_{tv}$, $\|\Delta(\cdot)\|_{\Delta_{tv}} \leq \rho$. The proof proceeds similarly to the proof of Theorem 6.3 showing that there exists a time-varying perturbation $\Delta \in \Delta_{tv}$ with $\varrho(E^{02}D_g^2)^{-1/2} < \|\Delta(\cdot)\|_{\Delta_{tv}} < \rho$ and a solution $x(t)$ of (93) such that $\lim_{t \to \infty} \|x(t)\|_2 = \infty$. \hfill $\Box$
Corollary 6.5. Suppose (86) and consider the following statements:

(i) \( r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta} (A, B, C, E) \).

(ii) \( r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta} (A, B, C, E) \).

(iii) \( \max_{\omega \in \mathbb{R}} g(E^{\omega} D(\omega)^{2}) = \rho(E^{\omega} D_{g}^{2}) \).

(iv) There exists a joint maximum \( \omega_{0} \in \mathbb{R} \) of the \( N \) functions \( \omega \mapsto \|G_{i}(\omega)\|_{2,2}, i \in \mathbb{N} \).

Then (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iv). If \( E \) is irreducible and \( g_{i} > 0 \) for all \( i \in \mathbb{N} \), then the four statements are all equivalent.

Proof: The equivalence (i) \( \iff \) (ii) follows from Theorem 6.3 because by definition \( r_{\Delta_{a_{i}}} (A, B, C, E) \leq r_{\Delta_{a_{i}}} (A, B, C, E) \leq r_{\Delta} (A, B, C, E) \). The equivalence (ii) \( \iff \) (iii) follows from Theorem 6.3 and (53).

(iv) \( \implies \) (i): Suppose that \( \omega_{0} \in \mathbb{R} \) is a joint maximum of the \( N \) functions \( \omega \mapsto \|G_{i}(\omega)\|_{2,2}, i \in \mathbb{N} \). Then Theorem 6.3 and Lemma 4.5 (iv) imply

\[ r_{\Delta_{a_{i}}} (A, B, C, E) = \rho(E^{\omega} D_{g}^{2})^{-1/2} = \rho(E^{\omega} D(\omega_{0})^{2})^{-1/2} = \left[ \max_{\omega \in \mathbb{R}} \rho(E^{\omega} D(\omega)^{2}) \right]^{-1/2} \]

and hence (i) by (53).

(i) \( \implies \) (iv): Now suppose that \( E \) is irreducible, \( g_{i} > 0 \) for all \( i \in \mathbb{N} \), and there does not exist a joint maximum of the functions \( \omega \mapsto \|G_{i}(\omega)\|_{2,2}, i \in \mathbb{N} \). Let \( \omega_{0} \in \mathbb{R} \) be a maximum of \( \omega \mapsto \rho(E^{\omega} D(\omega)^{2}) \). Then \( D(\omega_{0}) \leq D_{g} \) and \( D(\omega_{0}) \neq D_{g} \). Since \( E \) and hence \( E^{\omega} D_{g}^{2} \) are irreducible, it follows from [4, Cor. 2.1.5] that

\[ r_{\Delta} (A, B, C, E)^{-2} = \rho(E^{\omega} D(\omega_{0})^{2})^{-2} < \rho(E^{\omega} D_{g}^{2}) = r_{\Delta_{a_{i}}} (A, B, C, E)^{-2}. \]

This concludes the proof. \( \square \)

Remark 6.6. (i) It is noteworthy that by the last statement in the previous corollary the equality \( r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta} (A, B, C, E) \) depends only on the \( N \) isolated subsystems \((A_{i}, B_{i}, C_{i})\), \( i \in \mathbb{N} \) and not on their specific interconnection, provided that \( E \) is irreducible.

(ii) Suppose \( A, B, C \) are real and each function \( \omega \mapsto \|G_{i}(\omega)\|_{2,2}, i \in \mathbb{N} \) admits its maximum on \( \mathbb{R}_{+} \) at \( \omega = 0 \). Then \( \omega \mapsto \rho(D(\omega)^{2} E^{\omega}) \) admits its maximum on \( \mathbb{R}_{+} \) at \( \omega = 0 \) and we conclude from Corollary 6.5, Theorem 6.3 and Remark 4.10 that

\[ r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta_{a_{i}}} (A, B, C, E) = r_{\Delta} (A, B, C, E) = r_{\Delta} (A, B, C, E) = \rho(E^{\omega} D_{g}^{2})^{-1/2}. \]

In particular, the stability radii of \( A \) with respect to time-invariant and time-varying real perturbations of structure \( E \) are equal. \( \square \)

In the following example we determine the stability radii of a system composed of two interacting oscillators with respect to time-invariant and time-varying complex perturbations. We also determine the stability radius with respect to time-invariant real perturbations of the same structure.

Example 6.7. Consider a composite system consisting of two harmonic oscillators

\[ \Sigma_{i} : \quad \dot{x}_{i}(t) = \begin{bmatrix} 0 & 1 \\ -\nu_{i}^{2} & -2\xi_{i}\nu_{i} \end{bmatrix} x_{i}(t) + \begin{bmatrix} 0 \\ b_{i} \end{bmatrix} u_{i}(t), \quad y_{i}(t) = [c_{i} \ 0] x_{i}(t), \quad i = 1, 2, \]

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interconnected via $u_1 = \delta_{12} y_2, u_2 = \delta_{21} y_1$. We assume $\nu_i, \xi_i > 0$, $b_i, c_i \in \mathbb{R} \setminus \{0\}, \delta_{12}, \delta_{21} \in \mathbb{C}$. The coupled system is of the form (31) with $N = 2$, $I = \mathbf{q} = (1, 1)$ and matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\nu_1^2 & -2\xi_1 \nu_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\nu_2^2 & -2\xi_2 \nu_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (101)$$

The associated perturbation space $\mathbf{\Delta}_E$ is given by $\mathbf{\Delta}_E = \{ \begin{bmatrix} 0 & \delta_{12} \\ \delta_{21} & 0 \end{bmatrix}; \delta_{12}, \delta_{21} \in \mathbb{C} \}$ and provided with the norm $\|\mathbf{\Delta}\| := \|\mathbf{\Delta}\|_E = \max\{|\delta_{12}|, |\delta_{21}|\}, \mathbf{\Delta} \in \mathbf{\Delta}_E$. The transfer functions of the two subsystems $\Sigma_i$ are $G_i(s) = c_i b_i / (s^2 + 2\xi_i \nu_i s + \nu_i^2)$ and so

$$G_i(\omega) = c_i b_i / (\nu_i^2 - \omega^2 + 2\xi_i \nu_i \omega), \quad g_i = \min_{\omega \in \mathbb{R}} \left| G_i(\omega) \right| = \max_{\omega \in \mathbb{R}} \left| G_i(\omega) \right| = |c_i b_i| / |(\nu_i^2 - \omega^2)^2 + 4\xi_i^2 \nu_i^2 \omega^2|^{1/2}. \quad (102)$$

We conclude that $r_{\mathbf{\Delta}_{12}} = r_{\mathbf{\Delta}_{21}} (A, B, C, E)$ and $r_{\mathbf{\Delta}} = r_\mathbf{\Delta} (A, B, C, E)$ are determined by

$$r_{\mathbf{\Delta}_{12}} = \vartheta(E^2 D^2) 
^{1/2} = \vartheta \left( \begin{bmatrix} 0 & g_1^2 \\ g_1^2 & 0 \end{bmatrix} \right)^{1/2} = \frac{1}{\sqrt{g_1 g_2}}, \quad r_{\mathbf{\Delta}} = \frac{1}{\max_{\omega \in \mathbb{R}} \sqrt{|G_1(\omega)| |G_2(\omega)|}}. \quad (103)$$

A simple calculation gives

$$|G_i(\omega)| = \frac{|c_i b_i|}{\sqrt{(\nu_i^2 - \omega^2)^2 + 4\xi_i^2 \nu_i^2 \omega^2}}, \quad g_i = \begin{cases} \frac{|c_i b_i|}{\nu_i^2}, & if 1 \leq 2\xi_i^2, \\ \frac{|c_i b_i|}{2\nu_i^2 \xi_i \sqrt{1 - \xi_i}}, & if 1 > 2\xi_i^2, \end{cases} \quad i = 1, 2. \quad (104)$$

Since $E$ is irreducible, the equality $r_\mathbf{\Delta} = r_{\mathbf{\Delta}_{12}}$ holds if and only if there exists a joint minimum of the two even functions $f_i : \omega \rightarrow (\nu_i^2 - \omega^2)^2 + 4\xi_i^2 \nu_i^2 \omega^2, i = 1, 2$, on $\mathbb{R}$, see Corollary 6.5. An easy calculation shows that $f_1$ has a unique (local and global) minimum at $\omega_0 = 0$ if $1 \leq 2\xi_1^2$, and has exactly two minima at $\omega_{0i} = \pm\nu_i \sqrt{1 - 2\xi_i^2}$ if $1 > 2\xi_i^2, i = 1, 2$. Hence $r_\mathbf{\Delta} = r_{\mathbf{\Delta}_{12}}$ holds if and only if either $1 \leq 2\xi_1^2$ and $1 \leq 2\xi_2^2$ or $\nu_1^2 (1 - 2\xi_2^2) = \nu_2^2 (1 - 2\xi_1^2) > 0$.

The frequencies $\omega_{0i}$ maximize the amplitude (gain) responses of the two subsystems, so it is not surprising that they play critical roles in the stability analysis: we have seen in the proof of Theorem 6.3 periodic perturbations at these frequencies can be constructed which lead to non-decaying oscillations. The critical values $\xi_i = \sqrt{1/2}, i = 1, 2$ are those values of the damping for which engineers regard the subsystems $\Sigma_i$ to have only one significant overshoot.

We now consider real perturbations of the same structure, i.e. $\delta_{12}, \delta_{21} \in \mathbb{R}$. Then

$$A + B(E \circ \Delta)C = A + B \Delta C = \begin{bmatrix} \nu_1^2 & -2\xi_1 \nu_1 & 0 & 0 \\ 0 & -\nu_1^2 & 0 & 0 \\ 0 & 0 & -\nu_2^2 & 0 \\ b_2 \delta_{21} c_1 & 0 & b_1 \delta_{12} c_2 & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & \delta_{12} \\ \delta_{21} & 0 \end{bmatrix} \in \mathbf{\Delta}_E \cap \mathbb{R}^{2 \times 2}. \quad (105)$$

$\Delta$ is marginally destabilizing if there exists $\omega \in \mathbb{R}$ such that $\det(\omega I - A - B \Delta C) = 0$, i.e.

$$[(\nu_1^2 - \omega^2) + 2\xi_1 \nu_1 \omega][(\nu_2^2 - \omega^2) + 2\xi_2 \nu_2 \omega] = b_1 b_2 c_1 c_2 \delta_{12} \delta_{21}. \quad (106)$$

But this can only be the case if

$$2\xi_1 \nu_1 \omega (\nu_2^2 - \omega^2) + 2\xi_2 \nu_2 \omega (\nu_1^2 - \omega^2) = 0,$$

i.e. $\omega = 0$ or $\omega^2 (\xi_1 \nu_1 + \xi_2 \nu_2) = \nu_1 \nu_2 (\xi_1 \nu_2 + \xi_2 \nu_1)$. We denote this latter value of $\omega^2$ by $\omega^2_c$. In order to minimize $\|\Delta\| = \max\{|\delta_{12}|, |\delta_{21}|\}$ in (102) we must choose $\delta_{12} = \pm \delta_{21}$. Hence if $\omega = 0$, we have $|b_1 b_2 c_1 c_2| \|\Delta\|^2 = \nu_1^2 \nu_2^2$ and if $\omega = \omega^2_c$, we have

$$|b_1 b_2 c_1 c_2| \|\Delta\|^2 = (-\nu_1^2 - \omega^2_c)(\nu_2^2 - \omega^2_c) + 4\xi_1 \xi_2 \nu_1 \nu_2 \omega^2_c. \quad (107)$$

It is easy to see that

$$(\xi_1 \nu_1 + \xi_2 \nu_2)(\nu_1^2 - \nu_2^2) = \xi_1 \nu_1 (\nu^2 - \nu_2^2), \quad (\xi_1 \nu_1 + \xi_2 \nu_2)(\nu_2^2 - \omega^2_c) = \xi_2 \nu_2 (\nu^2 - \nu_2^2).$$

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Hence
\[(\xi_1\nu_1 + \xi_2\nu_2)^2|b_1b_2c_1c_2||\Delta|^2 = \xi_1\xi_2\nu_1\nu_2[(\nu_2^2 - \nu_1^2)^2 + 4\nu_1\nu_2(\xi_1\nu_1 + \xi_2\nu_2)(\xi_1\nu_2 + \xi_2\nu_1)].\]

We conclude that
\[|b_1b_2c_1c_2|r_{\Delta_2}^2 = \min \left\{ \nu_1^2\nu_2^2, \frac{\xi_1\xi_2\nu_1\nu_2[(\nu_2^2 - \nu_1^2)^2 + 4\nu_1\nu_2(\xi_1\nu_1 + \xi_2\nu_2)(\xi_1\nu_2 + \xi_2\nu_1)]}{(\xi_1\nu_1 + \xi_2\nu_2)^2} \right\}.

As an example consider two identical oscillators so that \(\xi_1 = \xi_2 = \xi, \nu_1 = \nu_2 = \nu\) and for simplicity we choose \(b_1 = b_2 = c_1 = c_2 = 1\). Then \(r_{\Delta_2} = \min\{\nu^2, 2\nu^2\}\). So \(r_{\Delta_2} = \nu^2\) if \(\xi \geq 1/2\) and \(r_{\Delta_2} = 2\nu^2\) if \(\xi < 1/2\). This is to be compared with the above results for complex perturbations that \(r_{\Delta} = r_{\Delta_{tv}} = \nu^2\) if \(2\xi^2 \geq 1\) and \(r_{\Delta} = r_{\Delta_{tv}} = 2\nu^2\sqrt{1 - \xi^2}\) if \(2\xi^2 < 1\). So we see that \(r_{\Delta} < r_{\Delta_2}\) if \(\xi < 1/\sqrt{2}\).

7 Special cases and concluding remarks

In this final section we illustrate our results by applying them to special classes of systems. In particular, we will prove that if the subsystems \(\Sigma_i\) are all real and one-dimensional or if they are all positive then there is a joint maximum at \(\omega = 0\) of the \(N\) functions \(\omega \mapsto ||G_i(\omega)||_{2,2}, i \in \mathbb{N}\), and consequently the stability radii with respect to time-varying and time-invariant parameter perturbations are equal. In a second part of this section we will discuss the relationship between our results and two classical problems of robust stability connected with the existence of joint Liapunov functions for a given set of time-invariant linear systems.

One-dimensional subsystems

Suppose that each subsystem \(\Sigma_i\) is one-dimensional, i.e. \(l_i = n_i = q_i = 1\) and \((A_i, B_i, C_i) = (a_i, b_i, c_i) \in \mathbb{L}_{1,1,1}, i \in \mathbb{N}\). Since \(B(E \circ \Delta)C = (b_ie_i\delta_{ij}c_j)\) for \(\Delta = (\delta_{ij}) \in \mathbb{C}^{N \times N}\), we can incorporate the diagonal entries of \(B = \text{diag}(b_1, \ldots, b_N), C = \text{diag}(c_1, \ldots, c_N)\) into \(E \in \mathbb{R}^{N \times N}\) and suppose \(B = C = I_N\) without restriction of generality. Hence let

\[A = \text{diag}(a_1, \ldots, a_N) \in \mathbb{C}^{N \times N}, \quad B = I_N, \quad C = I_N, \quad E \in \mathbb{R}^{N \times N}.\] \hspace{1cm} (103)

We consider perturbations of the form

\[A \sim A_{\Delta} = A + E \circ \Delta, \quad \Delta \in \Delta = \mathbb{C}^{N \times N}.\] \hspace{1cm} (104)

Note that \(E \circ \Delta \in \Delta = \{(\delta_{ij}) \in \mathbb{C}^{N \times N}; \delta_{ij} = 0\text{ if } e_{ij} = 0\}\) for every \(\Delta \in \Delta\). \(\Delta\) is provided with the norm

\[||\Delta|| = \max_{i \in \Delta} \left| \sum_{j \in N} |\delta_{ij}|^2 \right|^{1/2}, \quad \Delta = (\delta_{ij}) \in \Delta.\]

We write \(\sigma_{\Delta}(A, E; \delta)\) for \(\sigma_{\Delta}(A, I_N, I_N, E; \delta)\), \(r_{\Delta}(A, E)\) for \(r_{\Delta}(A, I_N, I_N, E)\) and similarly for the other stability radii, \(r_{\Delta_{tv}}, r_{\Delta_{tv}}\) and \(r_{\Delta_{tv}}\). If \(\text{Re}\ a_i \neq 0, i \in \mathbb{N}\), the transfer functions \(G_i(s)\), the scalars \(g_i\) defined in (68) and the diagonal matrix \(D_g = \text{diag}(g_1, \ldots, g_N)\) are given by

\[G_i(s) = (s - a_i)^{-1}, \quad g_i = \max_{\omega \in \mathbb{R}} |G_i(\omega)| = |\text{Re} a_i|^{-1}, \quad i \in \mathbb{N}, \quad D_g = |\text{Re} A|^{-1}.\] \hspace{1cm} (105)

**Theorem 7.1.** Suppose (103). Then

(i) For every \(\delta > 0\) the spectral value set of \(A\) at level \(\delta > 0\) with respect to perturbations of the form (104) is

\[\sigma_{\Delta}(A, E; \delta) = \bigcup_{\Delta \in \Delta, ||\Delta|| < \delta} \sigma(A + E \circ \Delta) = \sigma(A) \cup \{\lambda \in \rho(A); \rho(|\lambda I_N - A|^{-2}E^{\nu^2}) > \delta^{-2}\}.\] \hspace{1cm} (106)
(ii) If $A$ is Hurwitz stable, its stability radius with respect to perturbations of the form (104) is

$$
\rho(\Delta, A, E) = \left[ \max_{\omega \in \mathbb{R}} \left( |\omega I_N - A|^2 \right) \right]^{-1/2}.
$$

If additionally $A$ is real, then

$$
\rho(\Delta, A, E) = \rho(\Delta, (A, E)) = \rho(\Delta, (A, E)) = \rho(|A|^2 E^2)^{-1/2}.
$$

(iii) If $A$ is Hurwitz stable, its stability radius with respect to time-varying linear or nonlinear perturbations of structure $E$ are given by

$$
\rho(\Delta, (A, E)) = \rho(\Delta, (A, E)) = \rho(|A|^2 E^2)^{-1/2}.
$$

**Proof:** (i) and (107) follow from Corollary 4.9 since

$$
D(s)^2 = \left[ \text{diag} \left( \left( (s - a_1)^{-1}, \ldots, \left( (s - a_N)^{-1} \right) \right) \right) \right]^2 = \left( (s I_N - A)^{-2} \right), \quad s \in \rho(A).
$$

If additionally $A$ is real then the functions $\omega \mapsto |G_i(\omega)| = |\omega - a_i|^{-1}$ have a joint maximum at $\omega = 0$ and $D_g = |A|$. Hence (108) follows directly from Remark 6.6 (ii).

(iii) The first two equalities of (109) follow from Theorem 6.3 because of (105). The last equality follows from (108).

**Remark 7.2.** Let $A$ be any complex diagonal Hurwitz matrix and suppose that $E$ is irreducible. The functions $f_i : \omega \mapsto |G_i(\omega)| = |\omega - a_i|^{-1}$ have a unique maximum at $\omega_i = \text{Im} a_i$. Hence it follows from Corollary 6.5 that $\rho(\Delta, (A, E)) = \rho(\Delta, (A, E))$ if and only if all the diagonal elements $a_i$ of $A$ have the same imaginary part, $i \in \mathbb{N}$. The “if” holds without the assumption that $E$ is irreducible.

**Positive subsystems**

Positive systems occur frequently in the modelling of real processes where the state coordinates represent variables which cannot take negative values. They are used to model phenomena in such diverse fields as economics, population dynamics and biology, see [9]. We will now show that positive block-diagonal systems have similar properties to the real scalar diagonal ones. A system $(A, B, C) \in \mathbb{L}_{n,t,q}$ is called **positive** if $B \geq 0, C \geq 0$ and $A$ is a Metzler matrix, i.e. $A + r I_n \geq 0$ for some $r > 0$. We use the same notation as in Section 6.

**Theorem 7.3.** Suppose (86) and that the subsystems $(A_i, B_i, C_i) \in \mathbb{L}_{n_i,t_i,q_i}$ are positive. If $\mathbf{n} = (n_1, \ldots, n_N)$ and $\Delta_{\mathbb{R}} = \Delta \cap \mathbb{R}^{n \times n}$, then

$$
\rho(\Delta, (A, B, C, E)) = \rho(\Delta, (A, B, C, E)) = \rho(\Delta, (A, B, C, E)) = \rho(\Delta, (A, B, C, E)) = \rho(|A|^2 E^2 D_g^2)^{-1/2}
$$

where $D_g = \text{diag} \left( \left( \|C_1 A_1^{-1} B_1\|_2, \ldots, \|C_N A_N^{-1} B_N\|_2 \right) \right)$.

**Proof:** Since $(A_i, B_i, C_i)$ is positive, we have $g_i = \max_{\omega \in \mathbb{R}} \|G_i(\omega)\|_{2,2} = \|C_i A_i^{-1} B_i\|_{2,2}$ for $i \in \mathbb{N}$, see [12, Ex. 5.3.22]. Thus the $N$ functions $\omega \mapsto \|G_i(\omega)\|_{2,2}, i \in \mathbb{N}$ have a joint maximum at $\omega_0 = 0$. Hence the result follows from Remark 6.6 (ii).
Quadratic and absolute stability

We conclude the paper by applying our results to two classical robustness issues, the problems of *quadratic* and of *absolute* stability. As in the previous section we consider systems with structured norm-bounded uncertainties of the following kind:

\[
\begin{align*}
\dot{x} &= Ax + B(E \circ \Delta)y, & y &= Cx, & \Delta \in \Delta(r), \\
\dot{x} &= Ax + B(E \circ \Delta(t))y, & y &= Cx, & \Delta \in \Delta_{tv}(r), \\
\dot{x} &= Ax + B(E \circ \Delta(x))y, & y &= Cx, & \Delta \in \Delta_n(r), \\
\dot{x} &= Ax + B(E \circ \Delta(x,t))y, & y &= Cx, & \Delta \in \Delta_{nt}(r),
\end{align*}
\]  

(110)

(111)

(112)

(113)

where the data \((A, B, C, E)\) are given as in (86), \(\Delta = C^{l \times q}, \Delta_{tv}, \Delta_n, \Delta_{nt}\) are defined as in the previous section, and for any uncertainty level \(r > 0\), \(\Delta\) varies within the following sets of norm bounded perturbations

\[
\begin{align*}
\Delta(r) &= \{ \Delta \in \Delta; \| \Delta \| \leq r \}, \\
\Delta_n(r) &= \{ \Delta \in \Delta_n; \| \Delta \|_n \leq r \}, \\
\Delta_{tv}(r) &= \{ \Delta \in \Delta_{tv}; \| \Delta \|_{tv} \leq r \}, \\
\Delta_{nt}(r) &= \{ \Delta \in \Delta_{nt}; \| \Delta \|_{nt} \leq r \}.
\end{align*}
\]

The following definition specifies different concepts of robust stability.

**Definition 7.4.** Let \(r > 0\). The uncertain system \((A, B, C, E)\) is said to be

(i) *asymptotically stable at level* \(r\) if all the systems (110) are asymptotically stable,

(ii) *tv-stable at level* \(r\) if all the systems (111) are asymptotically stable,

(iii) *absolutely stable at level* \(r\) if all the systems (112) are asymptotically stable,

(iv) *nt-stable at level* \(r\) if all the systems (113) are asymptotically stable.

For any uncertainty level \(r > 0\) the following implications are either trivial or easily proved:

\[ nt\text{-stable } \Rightarrow \text{tv-stable } \Rightarrow \text{absolutely stable } \Rightarrow \text{asymptotically stable}. \]  

(114)

In the sequel we will discuss the relationships of these concepts with the notion of quadratic stability. Here we use a modification of the usual definition, compare [5].

**Definition 7.5.** We say that the uncertain system \((A, B, C, E)\) is *quadratically stable* at level \(r\) if there is a quadratic function \(V(x) = \langle x, Px \rangle\), with a positive definite Hermitian matrix \(P\), such that the derivative of \(V\) along the trajectories of (110)

\[
\dot{V}(x) = \langle Px, Ax + B(E \circ \Delta)y \rangle + \langle Ax + B(E \circ \Delta)y, Px \rangle, \\
y = Cx
\]

(115)

satisfies \(\dot{V}(x) \leq -k\|y\|^2\) for some \(k > 0\) and all \(\Delta \in \Delta(r)\).

The usual definition of quadratic stability requires that \(V(x)\) is a *strict* Liapunov function, i.e. \(\dot{V}(x)\) is negative definite. Then it is well known that quadratic stability implies all the four properties of robust stability defined in the preceding definition. However, the quadratic Liapunov functions constructed from Riccati equations of the form (15) are usually not strict but only satisfy \(\dot{V}(x) \leq -k\|y\|^2\). So as a direct consequence of Liapunov theory we are only able to conclude stability but not asymptotic stability. However, a similar argument as in the proof of Theorem 2.6 can be used to show that the condition \(\dot{V}(x) \leq -k\|y\|^2\) in fact suffices to prove asymptotic stability of the uncertain systems (110). This motivates our definition of a slightly weaker concept of quadratic stability.

In the following we will apply our previous results to the following two problems.
Problem of Quadratic Stability: Under which conditions does any one of the above properties of robust stability imply quadratic stability?

Aizerman Problem: Under which conditions does the asymptotic stability of the uncertain system \((A, B, C, E)\) at level \(r > 0\) imply that it is absolutely stable at level \(r\)? (If this implication holds for every \(r > 0\), we say that the \textit{structured complex version of the generalized Aizerman conjecture} holds true for \((A, B, C, E)\).)

Remark 7.6. Note that the original Aizerman conjecture was stated for the real single input single output case where \((A, B, C) \in L_{n,1,1}\) is real and \(\Delta = \mathbb{R}\). In this real case the Aizerman conjecture is not true: there are counterexamples of \((A, B, C) \in L_{n,1,1}\) for which every matrix in the set \(\{A + B\Delta C; \Delta \in \mathbb{R}, |\Delta| \leq 1\}\) is asymptotically stable but \((A, B, C)\) is not absolutely stable at level 1, see [25, §7.3].

Assuming observability it is well known that in the full-block case all the above concepts of robust stability are equivalent. Note that the full block case is a very special case of the situation considered in this paper, namely \(N = 1, n = (n), l = (l), q = (q), (A, B, C) = (A_1, B_1, C_1) \in L_{n,l,q}\). In this special case we have that an uncertain observable system \((A, B, C)\) is asymptotically stable at level \(r > 0\) if and only if it is quadratically stable at level \(r > 0\), see [12, §5.6]. Moreover, the complex version of the Aizerman conjecture is true in the full block case, [12, Thm. 5.6.22].

For structured perturbations these results do, in general, no longer hold. Counterexamples have been given in [19]. Besides reformulations of the property in terms of linear matrix inequalities or \(\mu\)-analysis there are apparently no general necessary and sufficient criteria available for quadratic stability of systems with structured uncertainties, compare [1]. In our framework where the nominal system is block-diagonal and the perturbations \(E \circ \Delta \in \Delta_F\) have an arbitrarily prescribed zero structure (defined by \(E\)) we can solve the problem of quadratic stability and establish its precise relationship with the other notions of robust stability. Moreover, we obtain computable tests for deciding whether a given uncertain system \((A, B, C, E)\) is quadratically stable at level \(r\) or not. Finally we obtain systematic procedures for the construction of counterexamples of uncertain systems which are asymptotically stable at a level \(r > 0\) but neither tv-stable nor quadratically stable at this level.

Theorem 7.7. Suppose (86). Then for any \(r > 0\):

(i) The uncertain system \((A, B, C, E)\) at level \(r\) is tv-stable if and only if \(r < g(E^{o2}D_g^2)^{-1/2}\). Moreover it is tv-stable if and only if it is nt-stable.

(ii) The asymptotic stability of the uncertain system \((A, B, C, E)\) at level \(r\) implies the tv-stability at level \(r\) if and only if \(\max_{\omega \in \mathbb{R}} g(D(\omega)^2E^{o2}) = g(E^{o2}D_g^2)\). In this case the structured complex version of the generalized Aizerman conjecture is valid for \((A, B, C, E)\).

(iii) Suppose the pairs \((A_j, C_j), j \in \mathbb{N}\) are observable and \(E\) does not have a zero column. Then the uncertain system \((A, B, C, E)\) is quadratically stable at level \(r\) if and only if it is tv-stable at level \(r\) (or, equivalently, \(r < g(E^{o2}D_g^2)\)).

Proof: In the proof of the three statements we implicitly make use of the fact that the uncertain system \((A, B, C, E)\) at level \(r\) is tv-stable (respectively nt-stable, respectively asymptotically stable) if and only if \(r < r_{\Delta_{nt}}(A, B, C, E)\) (respectively \(r < r_{\Delta_{nt}}(A, B, C, E)\), respectively \(r < r_{\Delta}(A, B, C, E)\)). This follows from the definitions of \(r_{\Delta_{nt}}, r_{\Delta_{tv}}, r_{\Delta}\) and the fact that there is a \(\Delta(\cdot) \in \Delta_{tv} \subset \Delta_{nt}\) with \(\|\Delta(\cdot)\|_{\Delta_{nt}} = r_{\Delta_{nt}} \) (resp. \(\Delta \in \Delta\) with \(\|\Delta\|_\Delta = r_\Delta\)) such that the corresponding perturbed system \(\dot{x} = Ax + B(E \circ \Delta(t))Cx\) (resp. \(\dot{x} = Ax + B(E \circ \Delta)Cx\)) is not asymptotically stable, see Remark 6.4.

(i) This follows directly from (96).
(ii) The asymptotic stability of the uncertain system \((A, B, C, E)\) at level \(r\) implies the tv-stability at level \(r\) if and only if \(r_{Δ_ω}(A, B, C, E) = r_{Δ}(A, B, C, E)\). Hence the first statement in (ii) follows from Corollary 6.5. Furthermore, if \(\max_{ω∈R} g(D(ω)^2E^2) = g(E^{22}D^2_γ)\) then \(r_{Δ_ω}(A, B, C, E) = r_{Δ}(A, B, C, E)\) by Corollary 6.5 and therefore the structured complex version of the generalized Aizerman conjecture holds true for \((A, B, C, E)\).

(iii) Suppose that the uncertain system \((A, B, C, E)\) at level \(r\) is tv-stable. Then \(r < r_{Δ_ω}(A, B, C, E) = ̂ρ(D_γ, E)\) by Theorem 6.3 and (80) and there exists a scaling vector \(γ > 0\) such that \(r < ̂ρ_γ(D_γ, E)\). Let \(P\) be any Hermitian solution of the scaled algebraic Riccati equation (73) with \(ρ = ̂ρ_γ(D_γ, E)\). By Theorem 6.1(iii) \(V(x) = \langle x, Px \rangle\) is a quadratic Liapunov function for (110) and therefore the structured complex version of the generalized Aizerman conjecture holds true for \((A, B, C, E)\).

To illustrate Theorem 7.7 we return to Example 6.7 and construct a joint Liapunov function for the system consisting of two harmonic oscillators with uncertain couplings of norm \(r\). For simplicity of presentation we assume that

\[ A_i = \begin{bmatrix} 0 & 1 \\ -ν_i^2 & -2ξ_iν_i \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix}, \quad C_i = \begin{bmatrix} c_i & 0 \end{bmatrix}, \quad i = 1, 2; \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Δ = \{0 \begin{bmatrix} Δ_{12} \\ Δ_{21} \end{bmatrix}; Δ_{12}, Δ_{21} ∈ \mathbb{C}\} \]

and consider the (4-dimensional) uncertain interconnected system described by

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A + B(E ∘ Δ)C \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad Δ ∈ Δ_E, \quad (116) \]

see (101). For simplicity of presentation we assume that \(ν_i = 1, i = 1, 2\). With \(γ = (γ_1, γ_2) > 0\) and \(ρ ≤ ̂ρ_γ\) the Riccati equations (75) take the form

\[ P_1A_1 + A_1^∗P_1 + ρ^2γ_1^2C_1^∗C_1 + γ_1^2P_1B_1B_1^∗P_1 = 0 \]
\[ P_2A_2 + A_2^∗P_2 + ρ^2γ_2^2C_2^∗C_2 + γ_2^2P_2B_2B_2^∗P_2 = 0. \]

In Example 6.7 we have shown that

\[ r_{Δ_ω} = \rho\left(\begin{bmatrix} 0 & g_2^2 \\ g_1^2 & 0 \end{bmatrix}\right)^{-1/2} = \frac{1}{\sqrt{g_1g_2}} \quad \text{where} \quad g_i = \begin{cases} |c_i|b_i | & \text{if } 1 ≤ 2ξ_i^2, \\
\frac{|c_i|b_i}{2ξ_i\sqrt{1 - ξ_i^2}} & \text{if } 1 > 2ξ_i^2, \end{cases}, \quad i = 1, 2. \]

By Theorem 5.2 we have \( ̂ρ = (g_1g_2)^{-1/2} \). To find an optimal scaling vector \(γ > 0\) we determine a positive Perron vector \(γ^2\) of \(E^{22}D^2_γ\):

\[ E^{22}D^2_γ \begin{bmatrix} γ_1^2 \\ γ_2^2 \end{bmatrix} = \begin{bmatrix} 0 & g_2^2 \\ g_1^2 & 0 \end{bmatrix} \begin{bmatrix} γ_1^2 \\ γ_2^2 \end{bmatrix} = g_1g_2 \begin{bmatrix} γ_1^2 \\ γ_2^2 \end{bmatrix}, \quad \text{i.e.} \quad γ_1^2g_2 = γ_2^2g_1. \]
Choosing \( \hat{\gamma}_1 = \sqrt{g_1}, \hat{\gamma}_2 = \sqrt{g_2} \) we obtain \( \rho_i = \hat{\rho} \) by Theorem 5.5. Then if \( P_1 = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \), the Riccati equation for the first subsystem with \( \rho = \hat{\rho} \) and \( \gamma = \hat{\gamma} \) takes the form

\[
\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2\xi_1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -2\xi_1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + g_1^{-1} \begin{bmatrix} \xi_1^2 & 0 \\ 0 & 0 \end{bmatrix} + g_1^{-1} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & b_1^2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0.
\]

A similar equation holds for \( P_2 \) with \( \xi_1 \) replaced by \( \xi_2 \) and \( g_1 \) replaced by \( g_2 \). Note that although the Riccati equations (117) and (118) are coupled through the parameters \( \rho \) and \( \gamma \), they are decoupled at the optimal parameter values. The Riccati equation for \( P_1 \) can be decomposed to

\[
-2p_2 + g_1^{-1}c_1^2 + g_1^{-1}b_1^2p_2^2 = 0 \\
p_1 - 2\xi_1p_2 - p_3 + g_1^{-1}b_1^2p_2p_3 = 0 \\
2p_2 - 2\xi_1p_3 + g_1^{-1}b_1^2p_3^2 = 0.
\]

By solving the first quadratic equation for \( p_2 \), then the third quadratic equation for \( p_3 \), and finally the middle one for \( p_1 \), one obtains the following positive definite solution of (117) (with \( \rho = \hat{\rho} \) and \( \gamma = \hat{\gamma} \))

\[
P_1 = \frac{|c_1|}{|b_1|} \begin{bmatrix} 2\xi_1 & 1 \\ 1 & 2\xi_1 + \sqrt{4\xi_1^2 - 2} \end{bmatrix} \quad \text{if } 1 \leq 2\xi_1^2 \quad \text{and} \quad P_1 = \frac{|c_1|}{|b_1|\sqrt{1 - \xi_1^2}} \begin{bmatrix} 1 & \xi_1 \\ \xi_1 & 1 \end{bmatrix} \quad \text{if } 1 > 2\xi_1^2.
\]

Replacing \( \xi_1 \) by \( \xi_2 \) and \( g_1 \) by \( g_2 \) one obtains an analogous formula for a solution \( P_2 \succ 0 \) of (117) (with \( \rho = \hat{\rho} \) and \( \gamma = \hat{\gamma} \)). By Theorem 5.1 \( V(x) = \langle x_1, P_1x_1 \rangle + \langle x_2, P_2x_2 \rangle \) is a joint Liapunov functions for all the perturbed systems (116) with

\[
\Delta = \begin{bmatrix} 0 & \delta_{12} \\ \delta_{21} & 0 \end{bmatrix} \in \Delta_E, \quad ||\Delta|| = \max\{||\delta_{12}||, ||\delta_{21}||\} \leq \rho_5 = (g_1g_2)^{-1/2} = r_{\Delta_E}.
\]

By Remark 6.4 \( V(\cdot) \) is a quadratic Liapunov function of maximal robustness for the uncertain system (116).

\[\square\]

References


