NUMERICAL APPROACH TO AN OPTIMAL MULTI-ITEM IMPERFECT PRODUCTION CONTROL PROBLEM IN UNCERTAIN ENVIRONMENT

DIPAK KUMAR JANA
Department of Engineering Sciences, Haldia Institute of Technology, Haldia,
Purba Midnapur 721657, West Bengal, India
dipakjana@gmail.com

K. MAITY
Department of Mathematics, Bhupatinagar Gangadhar Mahavidyalaya, Mugberia,
Purba Medinipur 721425, West Bengal, India
kalipada_maity@yahoo.co.in

M. MAITI
Department of Applied Mathematics with Oceanology & Computer Programming,
Vidyasagar University, Midnapore 721102, West Bengal, India
mmaiti2005@yahoo.co.in

Received 24 May 2012
Revised 12 August 2013

In this paper, some multi-item imperfect production-inventory models without shortages for defective and deteriorating items with uncertain/imprecise holding and production costs and resource constraint have been formulated and solved for optimal production. Here, the rate of production is assumed to be a function of time and considered as a control variable. Also the demand is time dependent and known. Uncertain or imprecise space constraint is also considered. The uncertain and imprecise holding and production costs are represented by uncertain and fuzzy variables respectively. These are converted to crisp constraint/numbers using uncertain measure theory for uncertain variable and possibility/necessity measure for fuzzy variable. The multi-item production inventory model is formulated as a constrained single objective cost minimization problem with the help of global criteria method. The reduced problem is then solved using Kuhn-Tucker conditions and generalized reduced gradient (GRG-LINGO 10.0) technique. Form the general model, models for particular cases with different production and demand functions are derived. Models for a single item are also presented. The optimum results for different models are presented in both tabular and graphical forms. Sensitivity analysis of average cost for the general model with respect to the changes in holding and production costs are presented.

Keywords: Optimal control; production; uncertain measure theory; Kuhn-Tucker conditions; global criteria method.
1. Introduction

Optimal control problems have been a subject of academic research and industrial applications (Bonvin et al., Diwekar et al.). The solutions of these problems involve finding the time dependent profiles of the control variable so as to optimize a particular performance index such as maximum concentration of a desired product or minimum reaction time and its techniques are traditionally applied for the optimization of several operations. A lot of research work on inventory control system has been reported in the literature (cf. Naddor, Worell and Hall, Dhouib, Gharbi, Flapper et al., Jana et al.). In the classical inventory models, normally static lot size models where costs, production, demand etc. are independent of time are formulated. But, because of the manufacturing environment, the static models are not adequate in analyzing the behavior of such systems and in deriving the optimal policies for their control. Moreover it is usually observed in the market that sales of the fashionable goods, electronic gadgets, seasonable products, food-grains, etc., increase with time. For these reasons, dynamic models of production-inventory/inventory systems have been considered and solved by some researchers (cf. Padmanabhan and Vrat, Sana et al., Maity and Maiti, etc). In these models, demand and/or production are assumed to be continuous functions of time.

One of the weaknesses of current production-inventory models is the unrealistic assumption that all items produced are of good quality. But production of defective units is a natural phenomenon in a production process. Defective units as a result of imperfect quality production process were initially considered by Porteus, Salameh and Jaber and Yoo et al. have developed inventory models for imperfect production along with inspection processes. Chakraborty and Giri have developed a joint determination of optimal safety stocks and production policy for an imperfect production system.

In the most of the earlier inventory models, life time of an item is assumed to be infinite while it is in storage. But, in reality, many physical goods deteriorate due to dryness, spoilage, vaporization etc. and are damaged due to hoarding longer than their normal storage period. The deterioration also depends on preserving facilities and environmental conditions in warehouses. So, due to deterioration effect, a certain fraction of the items either damaged or decayed and are not in perfect condition to satisfy the future demand of customers for good items. Deterioration for such items is continuous and constant or time-dependent and/or dependent on the on-hand inventory. A large number of research papers have already been published on the above type of items by Maity and Maiti, Zhou et al. and others. Many optimization papers have been published in fuzzy and fuzzy rough environments (Das et al., Sarkar et al., Liu and Sai). Fan et al. have developed a dominance-based fuzzy rough set analysis of uncertain and possibilistic data tables.

In spite of the above mentioned development, there are some lacunas in the developments of imperfect production system for defective and deteriorating items. There are:
There are few articles dealing with both defective and deterioration together. Here amount of defective items increases with production rate which is time dependent. This assumption is more realistic than earlier ones. This phenomenon has been modelled by few researchers only.

A multi-item dynamic production system with dynamic requirement has been formulated as an optimal control problem and solved. Demand of the some goods increases very fast with time and to satisfy such a demand, production is also suitably changed to a dynamic form. This happens only for a particular time duration. This scenario is observed in case of fashionable/seasonable products.

In the literature, concept of uncertainty variable is quite new and till now, none has used it in inventory control system though there are several investigations with fuzzy parameters/variables. Here both uncertainty measure and possibility/necessity measures have been successfully used to represent the uncertain and fuzzy data respectively involving in a production-inventory system.

In this paper, an optimal dynamic imperfect multi-item production-inventory system for deteriorating item is considered in both uncertain and fuzzy environments. Here, both production and demand depend on time. The space constraint and inventory costs are uncertain/fuzzy and expressed by zigzag uncertain/fuzzy variables, then transferred to deterministic forms by uncertain and possibility/necessity measure theory respectively. The constraints and costs are expressed by zigzag uncertain/fuzzy variables. With uncertain/fuzzy imprecise holding and production costs, the proposed production inventory models are formulated as single-objective optimal control problems with uncertain/fuzzy parameters using global criteria method and then made deterministic applying uncertain/possibility/necessity measure theory appropriately. The problems are solved with the help of Kuhn-Tucker conditions and GRG technique. Several particular models are derived from the general model for different production function and demand functions. The models are illustrated by numerical data. Demand and production functions, stock levels are presented in both tabular and graphical forms. Some parametric studies on the total average cost are also presented.

2. Uncertain Variables and Measures

To better describe the subjective imprecise quantity, Liu\textsuperscript{10} proposed an uncertain measure and further developed an uncertainty theory which is an axiomatic system of normality, monotonicity, self-duality, countable subadditivity and product measure.

**Definition 1.** Let $\Gamma$ be a nonempty set. A collection $\mathcal{L}$ of subsets of $\Gamma$ is called a $\sigma$-algebra if (a) $\Lambda \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \ldots \in \mathcal{L}$, then $\Lambda_1 \cup \Lambda_2 \cup \cdots \in \mathcal{L}$. Each element $\Lambda$ in the $\sigma$-algebra $\mathcal{L}$ is called an event. Uncertain measure is a function from $\mathcal{L}$ to $[0, 1]$. In order to present an axiomatic definition of uncertain measure, it is necessary to assign to each event $\Lambda$ a number $\mathcal{M}\{\Lambda\}$ which
indicates the belief degree that the event \( \Lambda \) will occur. In order to ensure that the number \( M\{\Lambda\} \) has certain mathematical properties, Liu\(^{10}\) proposed the following three axioms:

**Axiom 1.** (Normality) \( M\{\Gamma\} = 1 \)

**Axiom 2.** (Monotonicity) \( M\{\Lambda\} + M\{\Lambda^C\} = 1 \), for any event \( \Lambda \)

**Axiom 3.** (Countable Subadditivity) For every countable sequence of events \( \Lambda_1, \Lambda_2, \ldots \), we have

\[
M\left\{ \sum_{i=1}^{\infty} \Lambda_i \right\} \leq \sum_{i=1}^{\infty} M\{\Lambda_i\}
\]

**Definition 2.** (Liu\(^{10}\)) The uncertainty distribution \( \Phi : R \to [0, 1] \) of an uncertain variable \( \hat{\xi} \) is defined by

\[
\Phi(t) = M\{\hat{\xi} \leq t\}.
\]

There are many types of uncertain variables, namely linear (Fig. 1), zigzag (Fig. 2), etc.

![Fig. 1. Linear and zigzag uncertainty variable.](image1)

![Fig. 2. Inverse linear uncertainty variable.](image2)
**Definition 3.** Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). Then the inverse function \( \Phi^{-1} \) is called the inverse uncertainty distribution of \( \xi \).

**Definition 4.** (Liu\textsuperscript{10}) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_{-\infty}^{\infty} \Phi(M(\xi \geq r))dr - \int_{-\infty}^{0} \Phi(M(\xi \leq r))dr
\]

provided that at least one of the two integrals is finite.

**Theorem 1.** (Liu\textsuperscript{12}) Let \( \xi \) be an uncertain variable with uncertainty distribution \( \Phi \). If the expected value exists, then

\[
E[\xi] = \int_{0}^{1} \Phi^{-1}(\alpha)d\alpha.
\]

**Lemma 1.** Let \( \xi \sim \mathcal{L}(a,b) \) be a linear uncertain variable. Then its inverse uncertainty distribution \( \Phi^{-1}(\alpha) = \frac{1}{2}(1 - \alpha)a + b\alpha \) and it can be expressed as

\[
E[\xi] = \int_{0}^{1} \frac{1}{2}(1 - \alpha)a + b\alpha)d\alpha = \frac{a + b}{2}.
\]

The figure of an Inverse of Linear Uncertainty variable \( \mathcal{L}(a,b) \) is depicted in Fig. 3.

![Fig. 3. Two TFN \( \tilde{a} \) and \( \tilde{b} \) and Pos (\( \tilde{a} \leq \tilde{b} \)).](image)

**Lemma 2.** Let \( \xi \sim \mathcal{Z}(a,b,c) \) be a zigzag uncertain variable. Then its inverse uncertainty distribution \( \Phi^{-1}(\alpha) = \frac{1}{2}(1 - \alpha)a + b\alpha + c\alpha \) and it can be expressed as

\[
E[\xi] = \int_{0}^{1} \frac{1}{2}(1 - \alpha)a + b\alpha + c\alpha)d\alpha = \frac{a + 2b + c}{4}.
\]

**Theorem 2.** (Liu\textsuperscript{12}) Let \( \xi \) and \( \eta \) be independent uncertain variables with finite expected values. Then for any real numbers \( a_1 \) and \( a_2 \), we have

\[
E[a_1\xi + a_2\eta] = a_1E[\xi] + a_2E[\eta].
\]

**Lemma 3.** Assume \( x_1, x_2, \ldots, x_n \) are nonnegative decision variables, and \( \xi_1, \xi_2, \ldots, \xi_n, \xi \) are independent zigzag uncertain variables \( \mathcal{Z}(a_1,b_1,c_1), \mathcal{Z}(a_2,b_2,c_2), \ldots, \).
$Z(a_n, b_n, c_n), Z(a, b, c)$, respectively. Then for any confidence level $\alpha \geq 0.5$, the chance constraint

$$
\mathcal{M}\left\{ \sum_{i=1}^{n} \hat{\xi}_i x_i \leq \hat{\xi} \right\} \geq \alpha
$$

holds if and only if

$$
\sum_{i=1}^{n} \left( (2 - 2\alpha) b_i + (2\alpha - 1) c_i \right) x_i \leq (2 - 2\alpha) a + (2 - 2\alpha) b.
$$

2.1. Uncertain programming problem

Uncertain programming is a type of mathematical programming involving uncertain variables. Since an uncertain objective function $f(x, \hat{\xi})$ cannot be directly minimized, we may minimize its expected value. Assume that $x$ is a decision vector, $\hat{\xi}$ is an uncertain vector, $f$ is an objective function and $g_j$ are constraints functions for $j = 1, 2, \ldots, p$. Let us examine

$$
\begin{cases}
\min f(x, \hat{\xi}) \\
\text{s.t.} & g_j(x, \hat{\xi}) \leq 0, j = 1, 2, \ldots, p \\
& x \in X
\end{cases}
$$

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu proposed the above uncertain programming model (8) is equivalent to the crisp model:

$$
\begin{cases}
\min \mathbb{E}[f(x, \hat{\xi})] \\
\text{s.t.} & \mathcal{M}\left\{ g_j(x, \hat{\xi}) \leq 0 \right\} \geq \alpha, j = 1, 2, \ldots, p \\
& x \in X
\end{cases}
$$

By Theorem 1, the above crisp model can be expressed as

$$
\begin{cases}
\min \int_{0}^{1} f(x, \Phi^{-1}_1(\alpha), \Phi^{-1}_2(\alpha), \ldots, \Phi^{-1}_n(\alpha)) \\
\text{s.t.} & g_j(x, \Phi^{-1}_1(\alpha_j), \Phi^{-1}_2(\alpha_j), \ldots, \Phi^{-1}_n(\alpha_j)) \leq 0, j = 1, 2, \ldots, p \\
& x \in X
\end{cases}
$$

3. Possibility/Necessity Measures in Fuzzy Environment

Any fuzzy subset $\tilde{a}$ of $\mathbb{R}$ (where $\mathbb{R}$ represents a set of real numbers) with membership function $\mu_{\tilde{a}}(x) : \mathbb{R} \to [0, 1]$ is called a fuzzy number. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy quantities with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$ respectively. Then according to Dubois and Prade,

$$
\text{Pos}(\tilde{a} \ast \tilde{b}) = \{ \sup(\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \mathbb{R}, x \ast y) \}
$$

$$
\text{Nes}(\tilde{a} \ast \tilde{b}) = \{ \inf(\max(1 - \mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \mathbb{R}, x \ast y) \}
$$
where the abbreviation Pos and Nec represent possibility and necessity respectively and * is any of the relations $>,<,=,\leq,\geq$.

The dual relationship of possibility and necessity requires that

$$\text{Nes}(\tilde{a} \ast \tilde{b}) = 1 - \text{Pos}(\tilde{a} \ast \tilde{b}).$$  \hfill (11)

Also necessity measures satisfy the condition

$$\text{Min} (\text{Nes}(\tilde{a} \ast \tilde{b}), \text{Nec}(\tilde{a} \ast \tilde{b})) = 0.$$

The relationships between possibility and necessity measures satisfy also the following conditions (cf. Dubois and Prade\textsuperscript{26}):

$$\text{Pos}(\tilde{a} \ast \tilde{b}) \geq \text{Nes}(\tilde{a} \ast \tilde{b}), \text{Nes}(\tilde{a} \ast \tilde{b}) > 0 \Rightarrow \text{Pos}(\tilde{a} \ast \tilde{b}) = 1 \text{ and } \text{Pos}(\tilde{a} \ast \tilde{b}) < 1 \Rightarrow \text{Nes}(\tilde{a} \ast \tilde{b}) = 0.$$

If $\tilde{a}, \tilde{b} \in \mathbb{R}$ and $\tilde{c} = f(\tilde{a}, \tilde{b})$ where $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a binary operation then membership function $\mu_{\tilde{c}}$ of $\tilde{c}$ is defined as

$$\mu_{\tilde{c}}(z) = \sup\{\min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y)), x, y \in \mathbb{R} \text{ and } z = f(x, y), \forall z \in \mathbb{R}\} \hfill (12)$$

Imprecise constraints: Let us consider the constraint $\tilde{a} \geq \tilde{b}$. This can be represented in necessity and possibility sense as $\text{Nes}(\tilde{a} \geq \tilde{b})$ and $\text{Pos}(\tilde{a} \geq \tilde{b})$. $\text{Nes}(\tilde{a} \geq \tilde{b})(\text{Pos}(\tilde{a} \geq \tilde{b}))$ estimates that an event "$\tilde{a} \geq \tilde{b}$" will occur with the minimum (maximum) chance at least $\eta$ (say) by decision maker (DM). Hence the said constraint can be represented as $\text{Nes}(\tilde{a} \geq \tilde{b}) > \eta$ (Pos $(\tilde{a} \geq \tilde{b}) > \eta$).

Let $\tilde{a} = (a_1, a_2, a_3)$ and $\tilde{b} = (b_1, b_2, b_3)$ be two triangular fuzzy numbers. Then for these fuzzy numbers, following Wang and Shu\textsuperscript{27} the Lemmas 5 and 6 can be derived.

**Lemma 5.** $\text{Pos}(\tilde{a} \leq \tilde{b}) > \eta$ iff $\delta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} > \eta$, $(a_2 > b_2, \ b_3 > a_1)$

**Proof.** Let, we have $\text{Pos}(\tilde{a} \leq \tilde{b}) > \eta$. From Fig. 4, it is clear that

$$\text{Pos}(\tilde{a} \leq \tilde{b}) = \begin{cases} 1 & \text{for } a_2 \leq b_2 \\ \delta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} & \text{for } a_2 > b_2, \ b_3 > a_1 \\ 0 & \text{for } a_1 \geq b_3 \end{cases}$$

Hence, $\text{Pos}(\tilde{a} \leq \tilde{b}) > \eta$ iff $\delta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} > \eta$, $(a_2 > b_2, \ b_3 > a_1)$. \hfill $\square$

**Lemma 6.** $\text{Nes}(\tilde{a} > \tilde{b}) > \eta$ iff $\frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} < 1 - \eta$, $(a_2 > b_2, \ b_3 > a_1)$

**Proof.** Let, we have $\text{Nes}(\tilde{a} > \tilde{b}) > \eta$. \hfill $\square$
From Fig. 2, it is clear that
\[\text{Pos}(\tilde{a} \leq \tilde{b}) = \begin{cases} 1 & \text{for } a_2 \leq b_2 \\ \zeta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} & \text{for } a_2 > b_2, \ b_3 > a_1 \\ 0 & \text{for } a_1 \geq b_3 \end{cases}\]

Hence, \(\text{Nes}(\tilde{a} > \tilde{b}) > \eta \Rightarrow (1 - \text{Pos}(\tilde{a} \leq \tilde{b})) > \eta\).

Therefore, \(\text{Nes}(\tilde{a} > \tilde{b}) > \eta\) iff \(\zeta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} < 1 - \eta, \ (a_2 > b_2, \ b_3 > a_1)\).

### 3.1. Expected value of a fuzzy variable

Based on the credibility measure, Liu and Liu\(^{24}\) presented the expected value operator of a fuzzy variable as follows.

**Definition 5.** Let \(\tilde{X}\) be a normalized fuzzy variable the expected value of the fuzzy variable \(\tilde{X}\) is defined by
\[
E[\tilde{X}] = \int_0^\infty C_r(\tilde{X} \geq r)dr - \int_{-\infty}^0 C_r(\tilde{X} \leq r)dr
\]
(13)

When the right hand side of (13) is of form \(\infty - \infty\), the expected value is not defined. Also, the expected value operation has been proved to be linear for bounded fuzzy variables, i.e., for any two bounded fuzzy variables \(\tilde{X}\) and \(\tilde{Y}\), we have \(E[a\tilde{X} + b\tilde{Y}] = aE[\tilde{X}] + bE[\tilde{Y}]\) for any real numbers \(a\) and \(b\).

**Lemma 7.** (Liu and Liu\(^{24}\)) The expected value of triangular fuzzy variable \(\tilde{A} = (a_1, a_2, a_3)\) is defined as
\[
E[\tilde{A}] = \frac{1}{2}[(1 - \rho)a_1 + a_2 + \rho a_3]
\]
\[= \frac{1}{4}[a_1 + 2a_2 + a_3], \text{ taking } \rho = 0.5.\]

### 4. Convexity of a Function

#### 4.1. Kuhn-Tucker’s necessary and sufficient conditions

Minimize \(Z = f(x)\)

subject to \(g_i(x) \geq 0, \ i = 1, 2, \ldots, m\), where \(x = [x_1, x_2, \ldots, x_n]^T\). (14)
For (14), Kuhn-Tucker’s necessary conditions are
\[
\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(x)}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n
\]
\[
\lambda_i g_i(x) = 0
\]
\[
g_i(x) \geq 0
\]
\[
\lambda_i \leq 0
\]
and sufficient conditions are \( f(x) \) should be convex and all \( g_i(x) \)'s concave functions of \( x \).

4.2. Global criteria method
Here, the ideal objective vector is used as a reference point and \( L_p \) metrics are used for measuring. In this case, the \( L_p \)-problem is
\[
\text{Minimize } \left( \sum_{i=1}^{k} \left| f_i(x) - f_i^* \right|^p \right)^{\frac{1}{p}} \tag{15}
\]
subject to \( x \in X \).

From the definition of the ideal objective vector \( f^* \), it is known that
\[
f_i(x) \geq f_i^*, \quad (i = 1, 2, \ldots, k).
\]
This is why no absolute values are needed if the global ideal objective vector is known. Unfortunately, because of incommensurability among the objectives, it is impossible to directly use the above family of distance function. Sometimes, it is necessary to normalize the distance family by using reference points to remove the effects of the incommensurability. The problem is then to solve the following auxiliary problem:
\[
\text{Minimize } \left\{ \sum_{i=1}^{k} \left( \frac{f_i(x) - f_i^*}{f_i^*} \right)^p \right\}^{\frac{1}{p}} \tag{16}
\]
subject to \( x \in X \).

The exponent \( \frac{1}{p} \) may be dropped. Problems with or without the exponent \( \frac{1}{p} \) are equivalent for \( 1 \leq p < \infty \), since \( L_p \) problem (16) is an increasing function of the corresponding problem without the exponent. An usual value of \( p \) is 2. This method is also sometimes called Compromise Programming. However, in our function, we have formulated the model in the form (15).

**Theorem 3.** The solution of \( L_p \) problem (16) (where \( 1 \leq p < \infty \)) is Pareto optimal (c.f., Miettinen\(^{23}\)).
5. Assumptions and Notations

The model under consideration is developed with the following assumptions and notations.

5.1. Assumptions

For $i$-th ($i = 1, 2, \ldots n$) item, it is assumed that:

1. Deterioration rates are known and constant.
2. Shortages are not allowed.
3. The inventory level, production and demand are continuous function of time.
4. Holding and production costs are uncertain in nature.
5. Initial stock and demand are known.
6. This is a single period inventory model with finite time horizon.
7. Defective rate is known and constant.

5.2. Notations

$n =$ number of items.
$C_{ui} =$ production cost per unit item which is uncertain/imprecise in nature.
$A =$ maximum space available for storage which is uncertain/imprecise in nature.
$T =$ time length of the cycle.
$\lambda =$ Lagrange multiplier.

For the $i$-th ($i = 1, \ldots, n$) item,

$D_{i}(t) = d_{i0} + d_{i1}t + \cdots + d_{im}t^r$ = demand rate at time $t$ where $d_{i0}, d_{i1}, \ldots, d_{ir}$ are known.
$U_{i}(t) = u_{i0} + u_{i1}t + \cdots + u_{im}t^m$ = production rate at time $t$ where $u_{i0}, u_{i1}, \ldots, u_{im}$ are unknown.
$x_{i}(t) =$ the inventory level at time $t$.
$I_0 =$ the desired inventory level.
$x_{i}(0) =$ initial stock level.
$p_0 =$ the satisfactory production rate(units/unit time).
$\alpha_i =$ rate of deterioration.
$\delta_i =$ defective rate ($= 1 - q_i$).
$\bar{h}_{i} =$ holding cost per unit item per unit time which is uncertain/imprecise in nature.
$\bar{a}_{i} =$ storage area for per unit item which is uncertain/imprecise in nature.
($^\hat{\text{}}$ and $^\tilde{\text{}}$ represented uncertain and fuzzy notations respectively).

6. Mathematical Formulation of Proposed Production Inventory Model

6.1. Model in uncertain environment

An imperfect production-inventory system for $n$ deteriorating items with warehouse capacity constraint and dynamic demand is considered. Here, the items are
produced at a variable rate $U_i(t)$ and deteriorate at a constant rate, $\alpha_i$. Demand of the items is time dependent and the stock level at time, $t$ decreases due to deterioration and consumption. Shortages are not allowed.

The differential equation for $i^{th}$ item representing above system during a fixed time-horizon $T$ is

$$\dot{x}_i(t) = (1 - \delta_i)U_i(t) - D_i(t) - \alpha_i x_i(t) \quad (17)$$

with end conditions $x_i(0) = 0$.

Here, the desired inventory cost level $I_0$ and satisfactory total production cost level $p_0$ are assumed for the system to be the reference points for state ($x_i$) and control ($U_i$) variables respectively.

If $\int_0^T \bar{h}_i x_i dt$ is the $i^{th}$ holding cost for the time period $T$, then the total holding cost is given by

$$\bar{G}_0 = \sum_{i=1}^n \int_0^T \bar{h}_i x_i(t) dt = \int_0^T \sum_{i=1}^n \bar{h}_i x_i dt. \quad (18)$$

Similarly, the total production cost is given by

$$\bar{H}_0 = \sum_{i=1}^n \int_0^T \bar{C}_{ui} U_i(t) dt = \int_0^T \sum_{i=1}^n \bar{C}_{ui} U_i(t) dt. \quad (19)$$

With these values, we apply **global criteria method** for the optimization of present optimal control problem. In this case, the corresponding problem is

$$\text{Minimize } J = \left[ \left| \int_0^T \sum_{i=1}^n \bar{h}_i x_i dt - I_0 \right|^p + \left| \int_0^T \sum_{i=1}^n \bar{C}_{ui} U_i(t) dt - p_0 \right|^p \right]^\frac{1}{p} \quad (20)$$

subject to the constraint $(17)$, $\bar{A} - \sum_{i=1}^n x_i(t) \bar{a}_i \geq 0 \quad (21)$

and $U_i(t) \geq 0, D_i(t) \geq 0, 0 \leq t \leq T$, where $x_i(0) = 0 \quad (22)$

and the parameters with “$-$” on the top either uncertain (or fuzzy) quantities.

In this formulation, usual value of $p$ is 2. This method is also sometimes called **compromise programming**.
6.2. Equivalent crisp representation of the inventory model

Taking expectation over (18), (19) using uncertain measures for uncertain quantities and possibility/necessity measures for fuzzy parameters, are reduced to the following deterministic forms:

Then the total expected holding cost is given by

\[ E[\bar{G}_0] = \int_0^T \sum_{i=1}^n \bar{h}_i x_i(t) dt \]

using Lemma 2. \hspace{1cm} (23)

Similarly, the total expected production cost is given by

\[ E[\bar{H}_0] = \int_0^T \sum_{i=1}^n \left( \frac{C_{ui1} + 2C_{ui2} + C_{ui3}}{4} \right) U_i(t) dt. \] \hspace{1cm} (24)

6.2.1. Model with uncertain parameters \( \hat{h}_i, \hat{C}_{ui}, \hat{a}_i, \hat{A} \)

Here holding cost, set-up costs, unit required area and total available space are uncertain quantities. According to Liu \(^{10}\) and using Eqs. (23) and (24), the above production inventory problems (20), (21) in uncertain environment can be converted to crisp model as:

Minimize \( E[\hat{J}] = \left[ \left| \int_0^T \sum_{i=1}^n \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) x_i(t) dt - I_0 \right|^p \right]^p \)

subject to (17), (22) and \( M \left\{ \sum_{i=1}^n \hat{a}_i x_i \leq \hat{A} \right\} \geq \alpha . \) \hspace{1cm} (25)

Let \( \hat{h}_i, \hat{C}_{ui}, \hat{a}_i \) and \( \hat{A} \) be a zigzag uncertain holding cost, production cost, storage area per unit and storage space represented by \( \hat{h}_i = Z(h_{i1}, h_{i2}, h_{i3}), \hat{C}_{ui} = Z(C_{ui1}, C_{ui2}, C_{ui3}), \hat{a}_i = Z(a_{i1}, a_{i2}, a_{i3}) \) and \( \hat{A} = Z(A_1, A_2, A_3), i = 1, 2, \ldots, n \) respectively. Using Lemmas 2, 3 and Theorem 2, the above problem can be expressed as

Minimize \( E[\hat{J}] \) \hspace{1cm} (27)

subject to (17), (22) and

\[ \sum_{i=1}^n \left( (2 - 2\alpha)a_{i2} + (2\alpha - 1)a_{i3} \right) x_i(t) \leq (2\alpha - 1)A_1 + (2 - 2\alpha)A_2. \] \hspace{1cm} (28)
In this model, we assume that the rate of production is a function of time

\[ U_i(t) = u_{i0} + u_{i1}t + \cdots + u_{im}t^m. \]  

(29)

where \( u_{i0}, u_{i1}, \ldots, u_{im} \) (constants) are unknown control variables determined for minimum total cost. Also demand function

\[ D_i(t) = d_{i0} + d_{i1}t + \cdots + d_{ir}t^r. \]  

(30)

where \( d_{i0}, d_{i1}, \ldots, d_{ir} \) are known constants.

Using Eqs. (29) and (30), from Eq. (17),

\[ x_i(t) = f(u_{ij}, t), \quad j = 0, 1, 2 \ldots m; \quad i = 1, 2 \ldots n. \]  

(31)

Putting the value of \( x_i(t) \) from (31) in (27) and (28), the above problem becomes to the form

\[
\text{Minimize } E[\tilde{J}] = \left[ \left| \int_0^T \sum_{i=1}^n \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) f(u_{ij}, t) dt - I_0 \right|^p \right]^{\frac{1}{p}} + \left[ \left| \int_0^T \sum_{i=1}^n \left( \frac{C_{ui1} + 2C_{ui2} + C_{ui3}}{4} \right) U_i(t) dt - p_0 \right|^p \right]^{\frac{1}{p}} \right.
\]

\[
- \lambda \left[ \int_0^T \left( (2\alpha - 1)A_1 + (2 - 2\alpha)A_2 \right.ight.
\]

\[
- \sum_{i=1}^n \left( (2 - 2\alpha)a_{i2} + (2\alpha - 1)a_{i3} \right) f(u_{ij}, t) \right] dt
\]

(32)

where \( \lambda \) is the Lagrange multiplier.

6.2.2. Model with fuzzy parameters \( \tilde{h}_i, \tilde{C}_{ui}, \tilde{a}_i, \tilde{A} \)

According to Liu,\(^{10}\) the above production inventory problems (20), (21) in fuzzy environment can be converted to crisp model as:

\[
\text{Minimize } E[\tilde{J}] = E \left[ \left| \int_0^T \sum_{i=1}^n \tilde{h}_i x_i dt - I_0 \right|^p \right]^{\frac{1}{p}} + \left[ \left| \int_0^T \sum_{i=1}^n \tilde{C}_{ui} U_i(t) dt - p_0 \right|^p \right]^{\frac{1}{p}} \right. \]  

(33)

subject to (17), (22) and Poss \( \left\{ \sum_{i=1}^n \tilde{a}_i x_i \leq \tilde{A} \right\} \geq \alpha \)  

(34)

or Nec \( \left\{ \sum_{i=1}^n \tilde{a}_i x_i \leq \tilde{A} \right\} \geq \alpha \).  

(35)

Let \( \tilde{h}_i \), fuzzy holding cost be a triangular number represented by \( \tilde{h}_i = (h_{i1}, h_{i2}, h_{i3}) \), \( \tilde{C}_{ui} \) be a triangular fuzzy production cost represented by \( \tilde{C}_{ui} = (C_{ui1}, C_{ui2}, C_{ui3}) \), \( \tilde{a}_i \) be a triangular fuzzy storage area per unit represented by \( \tilde{a}_i = (a_{i1}, a_{i2}, a_{i3}) \),
i = 1, 2, ..., n, and $\tilde{A}$ be a triangular fuzzy storage space represented by $\tilde{A} = (A_1, A_2, A_3)$. Using Lemmas 5 and 7, the above problem can be expressed as

Minimize $E[\tilde{J}] = \left[ \left| \int_0^T \sum_{i=1}^n \left( \frac{h_{1i} + 2h_{i2} + h_{i3}}{4} \right) f(u_{ij}, t) dt - I_0 \right|^p \right]^\frac{1}{p}$

subject to (17), (22) and

$$\sum_{i=1}^n \left\{ (1 - \alpha)a_{i1} + \alpha a_{i2} \right\} x_i(t) \leq \alpha A_2 + (1 - \alpha)A_3 \quad (37)$$

or

$$\sum_{i=1}^n \left\{ (1 - \alpha)a_{i2} + \alpha a_{i1} \right\} x_i(t) \leq \alpha A_3 + (1 - \alpha)A_2 \quad (38)$$

It may be noted that the objective function $E[\tilde{J}]$ (i.e. Eq. (36)) is same as (25).

6.3. Multi-item models with different production and demand functions

6.3.1. Quadratic production and quadratic demand function (Model-1)

For $m = 2 = r$, i.e. quadratic production and demand functions are assumed, let $U_i(t) = u_{i0} + u_{i1}t + u_{i2}t^2$ and $D_i(t) = d_{i0} + d_{i1}t + d_{i2}t^2$. From (17), the optimal stock,

$$x_i(t) = \left\{ (q_iu_{i0} - d_{i0}) + (q_iu_{i1} - d_{i1})t + (q_iu_{i2} - d_{i2})t^2 \right\} \frac{1}{\alpha_i}$$

$$- \left\{ (q_iu_{i1} - d_{i1}) + 2(q_iu_{i2} - d_{i2})t \right\} \frac{1}{\alpha_i^2} + \left\{ (q_iu_{i2} - d_{i2}) \right\} \frac{2}{\alpha_i^3} - A_ie^{-\alpha_i t}$$

where $A_i = \left\{ (q_iu_{i0} - d_{i0}) \right\} \frac{1}{\alpha_i} - \left\{ (q_iu_{i1} - d_{i1}) \right\} \frac{1}{\alpha_i^2} + \left\{ (q_iu_{i2} - d_{i2}) \right\} \frac{2}{\alpha_i^3}$.

From Kuhn-Tucker conditions and Theorem 4, we have, $\frac{\partial E[\tilde{J}(u_i)]]}{\partial u_{i0}} = \frac{\partial E[\tilde{J}(u_i)]]}{\partial u_{i1}} = \frac{\partial E[\tilde{J}(u_i)]]}{\partial u_{i2}} = 0$. Using above relations, we get

$$(H_0 - I_0) \left\{ \frac{h_{1i} + 2h_{i2} + h_{i3}}{4} q_i \right\} + (G_0 - p_0) \alpha_i \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) = 0 \quad (39)$$

$$(H_0 - I_0) \left\{ \frac{h_{1i} + 2h_{i2} + h_{i3}}{4} q_i (T\alpha_i - 2) \right\}$$

$$+ (G_0 - p_0) \frac{T^2}{2} \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) T\alpha_i^2 = 0 \quad (40)$$
For the optimum value of minimum cost function is given by

\[ (H_0 - I_0) \left\{ \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) q_i (\alpha_i^2 T^2 - 3\alpha_i T + 6) \right\} + (G_0 - p_0) \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) \alpha_i^3 T^2 = 0 \]  

(41)

where

\[ H_0 = \int_0^T \sum_{i=1}^n \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) f(u_{ij}, t) dt \]

\[ = \sum_{i=1}^n \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) \left\{ \left( q_i u_{i0} - d_{i0} \right) T + \left( q_i u_{i1} - d_{i1} \right) \frac{T^2}{2} + \left( q_i u_{i2} - d_{i2} \right) \frac{T^3}{3} \right\} \frac{1}{\alpha_i} \]

\[ - \left\{ \left( q_i u_{i1} - d_{i1} \right) T + \left( q_i u_{i2} - d_{i2} \right) T^2 \right\} \frac{1}{\alpha_i} + \left\{ \left( q_i u_{i2} - d_{i2} \right) \frac{2T}{3} - A_i (1 - e^{-\alpha_i T}) \right\} \]

\[ G_0 = \int_0^T \sum_{i=1}^n \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) U_i(t) dt \]

\[ = \sum_{i=1}^n \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) \left\{ u_{i0} T + u_{i1} \frac{T^2}{2} + u_{i2} \frac{T^3}{3} \right\} . \]

The optimum value of minimum cost function is given by

\[ E[\tilde{J}] = \int_0^T \sum_{i=1}^n \left[ \left( \frac{h_{i1} + 2h_{i2} + h_{i3}}{4} \right) x_i(t) + \left( \frac{C_{u_{i1}} + 2C_{u_{i2}} + C_{u_{i3}}}{4} \right) U_i(t) \right] dt \]

\[ = H_0 + G_0 \]  

(42)

Solving the Eqs. (39)–(41), with the help of GRG technique we get, \( u_{i0}, u_{i1} \) and \( u_{i2} \) and then find the optimal rate of production, \( (U_i(t)) \) and then corresponding stock level \( (x_i(t)) \).

6.3.2. Linear production and quadratic demand function (Model-2)

For \( m = 1, r = 2 \) i.e. linear production and quadratic demand function, let \( U_i(t) = u_{i0} + u_{i1} t \) and \( D_i(t) = d_{i0} + d_{i1} t + d_{i2} t^2 \). Putting \( u_{i2} = 0 \) and solving the Eqs. (39)–(41) for \( k = 1, 2 \), we get, \( u_{i0}, u_{i1} \) and then find the optimal rate of linear production function and the corresponding stock level.

6.3.3. Constant production and demand function (Model-3)

For \( m = 0 = r \), i.e. constant production, i.e., \( U_i(t) = u_{i0} \) and constant demand function \( D_i(t) = d_{i0} \). As before, putting \( u_{i2} = u_{i1} = 0, d_{i2} = d_{i1} = 0 \), and solving the Eqs. (39)–(41) for \( k = 1 \), we get, \( u_{i0}, d_{i0} \) and then find the optimal value of production and hence optimum stock level.
6.4. Single item model with different production and demand function

Taking \( n = 1 \) in Models-1, -2, -3 and proceeding in the same way, we get the single item models with quadratic production and demand functions (Model-4), linear production and quadratic demand functions (Model-5), constant production and demand functions (Model-6) respectively.

6.5. Particular case

If the imperfectness is removed from the system i.e. \( \delta_1 = 0 = \delta_2 \), then the system considered here is same with that of Maity and Maiti, but the derivation and solution methodology are quite different for the present models.

7. Numerical Experiments

Input data

We take two items i.e. \( n = 2 \) and \( d_{10} = 2.5; \ d_{20} = 2.0; \ d_{11} = 1.0; \ d_{21} = 1.3; \ d_{12} = 2.1; \ d_{22} = 1.9; x_1(0) = 0; \ x_2(0) = 0; \ p_0 = 60; \ \alpha_1 = 0.04; \ \alpha_2 = 0.06; \ \delta_1 = 0.01; \ \delta_2 = 0.02; \ I_0 = 10; T = 1; \) and total space is triangular uncertain/fuzzy number \( \bar{A} = (120, 140, 160) \) and remaining uncertain/fuzzy parameters are given in Table 1. Here, the demands are \( D_1(t) = 2.5 + 1.0t + 2.1t^2 \) and \( D_2(t) = 2.0 + 1.3t + 1.9t^2 \). Solving (39), (40) and (41) equations and then evaluating (42) for the Model-1 and and its corresponding forms for models 2–6 using LINGO-10.0, we get optimum results for multi items and single item models which are presented in Table 2.

Table 1. Input data for different uncertain/fuzzy cost.

<table>
<thead>
<tr>
<th>Item</th>
<th>Holding cost</th>
<th>Production cost</th>
<th>Storage per unit area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \bar{h}_1 ) = (1.5, 1.8, 2.4)</td>
<td>( \bar{C}_{u1} ) = (4.6, 6.3, 7.5)</td>
<td>( \bar{a}_1 ) = (0.7, 1.1, 1.4)</td>
</tr>
<tr>
<td>2</td>
<td>( \bar{h}_2 ) = (1.8, 2.1, 2.4)</td>
<td>( \bar{C}_{u2} ) = (3.6, 4.9, 5.8)</td>
<td>( \bar{a}_2 ) = (1.4, 1.8, 2.2)</td>
</tr>
</tbody>
</table>

Table 2. Results obtained for different models.

<table>
<thead>
<tr>
<th>Model</th>
<th>Item</th>
<th>( U_i(t) ) in uncertain system</th>
<th>( \lambda )</th>
<th>( J ) in uncertain</th>
<th>( J ) in fuzzy Poss</th>
<th>( J ) in fuzzy Nec</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>59.23 + 9.93( t^2 )</td>
<td>–27.12</td>
<td>4320.12</td>
<td>4426.14</td>
<td>4227.98</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>56.25 + 10.32( t^2 )</td>
<td>–27.12</td>
<td>4320.12</td>
<td>4426.14</td>
<td>4227.98</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>58.18 + 10.5t</td>
<td>–31.21</td>
<td>4325.65</td>
<td>4427.54</td>
<td>4228.21</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>51.19 + 7.09t</td>
<td>–31.21</td>
<td>4325.65</td>
<td>4427.54</td>
<td>4228.21</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>58.18</td>
<td>–33.87</td>
<td>4326.07</td>
<td>4428.54</td>
<td>4229.21</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>51.19</td>
<td>–33.87</td>
<td>4326.07</td>
<td>4428.54</td>
<td>4229.21</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>58.18 + 8.853( t^2 )</td>
<td>–</td>
<td>2230.67</td>
<td>2231.12</td>
<td>2232.15</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>58.18 + 10.45t</td>
<td>–</td>
<td>2231.51</td>
<td>2232.12</td>
<td>2233.25</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>58.18</td>
<td>–</td>
<td>2232.52</td>
<td>2233.09</td>
<td>2232.89</td>
</tr>
</tbody>
</table>
Numerical Approach to an Optimal Multi-Item Imperfect Production Control

Table 3. Values of $X_i(t)$, $U_i(t)$ and $D_i(t)$ for $\delta_1 = 0, \delta_2 = 0$.

<table>
<thead>
<tr>
<th></th>
<th>Model-1</th>
<th>Model-2</th>
<th>Model-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0.00 0.50 0.075 0.010</td>
<td>0.00 0.50 0.75 0.010</td>
<td>0.00 0.50 0.75 0.010</td>
</tr>
<tr>
<td>$X_i(t)$</td>
<td>0.00 28.09 42.68 57.54</td>
<td>0.00 28.21 42.60 57.52</td>
<td>0.00 29.38 43.59 56.19</td>
</tr>
<tr>
<td>$U_i(t)$</td>
<td>58.25 62.39 65.12 67.85</td>
<td>58.18 63.4366 65 68.68</td>
<td>58.18</td>
</tr>
<tr>
<td>$D_i(t)$</td>
<td>02.50 03.52 04.43 05.6</td>
<td>same as Model-1</td>
<td>same as Model-1</td>
</tr>
</tbody>
</table>

Values of $X_i(t)$, $U_i(t)$, $D_i(t)$ for different multi-item models in uncertain environment are calculated using GRG technique and presented in the following Table 3. These functions are plotted in Fig. 4 for different $t$’s.

7.1. Sensitivity analysis

Here, a study has been made on the objective function due to the percentage change of rate of deterioration and defective. Table 3 gives the values of the objective function $J$ (of Model-1) in uncertain environment for different values of $\alpha_i$ and $\delta_i$, $i = 1, 2$. The percentage change of these values is shown with respect to the values used in the previous example with $\alpha_1, \delta_1, \alpha_2, \delta_2$ and the minimum objective value $J = 4320.12$. Table 4 shows that if $\delta_i$ and $\delta_i$ are increased/decreased by $+5\%$, $+10\%$, $-5\%$ and $-10\%$, the values of the objective function change by $3.19\%$ and $3.87\%$, $4.45\%$ and $6.15\%$, $-1.25\%$ and $-1.98\%$ and $-2.23\%$ and $-2.97\%$ respectively.

8. Discussion

From Table 2, it is observed that amongst all models, the model formulated in uncertain environment gives minimum cost. In this case production rate at $t = T$ is
Table 4. Change of $J$ due to change of $\alpha_i$ and $\delta_i$ for the Model-1.

<table>
<thead>
<tr>
<th>Change in $\alpha_i$ (%)</th>
<th>Change in $\delta_i$ (%)</th>
<th>Change of $J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>4.45</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>6.15</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>3.19</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>3.87</td>
</tr>
<tr>
<td>−5</td>
<td>0</td>
<td>−1.25</td>
</tr>
<tr>
<td>0</td>
<td>−5</td>
<td>−1.98</td>
</tr>
<tr>
<td>−10</td>
<td>0</td>
<td>−2.23</td>
</tr>
<tr>
<td>0</td>
<td>−10</td>
<td>−2.97</td>
</tr>
</tbody>
</table>

lowest but the amount of stock at the end is highest. Due to the first fact, the total cost is lowest and for the second incident, the available space is used as maximum as possible. Again, among the possibility and necessity formulations, production is more in possibility and for that cost in necessity formulation. But the stock label in both cases are almost same and hence the available space utilization does not differ much.

9. Conclusion

A realistic optimal control problem with uncertain/fuzzy production, holding cost and constraint in inventory control system has been formulated. These Uncertain/fuzzy objective and constraint are changed to crisp equivalent by uncertain/possibility/necessity measure theories and then solved by using global criteria method, Kuhn-Tucker conditions and GRG technique. The numerical illustrations has been discussed for multi-items and single item. A numerical example along with one graphical illustration is considered and its sensitivity analysis is provided to test feasibility of the model. The present technique and solution procedures can be extended for future research work to other uncertain inventory control problem with different uncertain parameters.

Acknowledgements

We would like to thank the anonymous referees for their valuable comments and suggestions for improving this paper. This work was supported by the Minor Research Project (PSW-092, 12/13) under UGC, Government of India.

References