Abstract: The set of eigenvalues of a graph $G$ together with their multiplicities is called the spectrum of $G$. The knowledge of spectrum can be used to obtain various topological properties of graphs like connectedness, toughness and many more. In this paper we use MATLAB to completely describe the spectrum of Sierpiński graphs and Sierpiński triangles, thus adding to the classes of graphs whose spectrum is known.

Keywords: sierpiński triangle $S_n$, spectrum

I. INTRODUCTION

Algebraic graph theory is the branch of mathematics that studies graphs by using algebraic properties of associated matrices. More in particular, spectral graph theory studies the relation between graph properties and the spectrum of the adjacency matrix or Laplace matrix. And the theory of association schemes and coherent configurations studies the algebra generated by associated matrices.

Spectral graph theory is a useful subject. The founders of Google computed the Perron-Frobenius eigenvector of the web graph and became billionaires. The second largest eigenvalues of a graph gives information about expansion and randomness properties. The smallest eigenvalues gives information about independence number and chromatic number. The fact that the eigenvalues multiplicities must be integral provides strong restrictions. And the spectrum provides useful invariant.

Let $G$ be a simple graph with vertex set $V = \{v_1, v_2 \ldots v_n\}$. The adjacency matrix of $G$ is an $n \times n$ matrix, in $A = A(G) = [a_{ij}]$ which $a_{ij} = 1$ if $v_i$ is adjacent to $v_j$ and $a_{ij} = 0$, otherwise. It follows directly from the definition that $A$ is a real symmetric matrix, and that the trace of $A$ is zero. Since the rows and columns of $A$ correspond to an arbitrary labelling of the vertices of $G$, we shall be interested primarily in those properties of the adjacency matrix which are invariant under permutations of the rows and columns. Foremost among such properties are the spectral properties of $A$. The eigenvalues of the adjacency matrix $A(G)$ are the eigenvalues of $G$. The set of eigenvalues of $G$ together with their multiplicities is called the spectrum of $G$. The graph spectrum is an important tool one can use to find information about the physical properties of a network, such as robustness, diameter, connectivity [1]. Let $d(v)$ denote the degree of $v \in V(G)$ and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$. The matrix $L = D - A$ and $Q = D + A$ are called the Laplacian and Signless Laplacian matrix respectively [2].

In [3] the spectra and energy of several classes of graphs containing a linear polyene fragment are obtained. In [4] the energy of iterated line graphs of regular graphs and in [5] the spectrum and energy of some self complimentary graphs are discussed. The energy of regular graphs is discussed in [6]. In this paper we use MATLAB to completely describe the spectrum of Sierpiński triangles $S_n$ and of Sierpiński graphs $S(n,3)$.

II. SIERPIŃSKI TRIANGLES

The Sierpiński triangle, also called the Sierpiński gasket or the Sierpiński Sieve, is a fractal and an attractive fixed set named after the Polish Mathematician Waclaw Sierpiński who described it in 1915 [7]. Originally constructed as a curve, this is one of the basic examples of self-similar sets, in the sense that it is a mathematically generated pattern that can be reproducible at any magnification or reduction. Start with any triangle in a plane (any closed, bounded region in the plane will actually work). The canonical Sierpiński triangle uses an equilateral triangle with a base parallel to the horizontal axis (Figure 1(i)). Shrink the triangle to $1/2$ height and $1/2$ width, make three copies, and position the three shrunken triangles so that each triangle touches the two other triangles at a corner (Figure 1(ii)). Note the emergence of the central hole - because the three shrunken triangles can between them cover only $3/4$ of the area of the original. Holes are an important feature of Sierpiński triangle. This step is repeated with each of the smaller triangles (Figure 1(iii) and so on). Note that this infinite process is not dependent upon the starting shape being a triangle - it is just clearer that way. Michael Barnsley used an image of a fish to illustrate this in his paper "V-variable fractals and superfractals" [8].

We denote by $S_n$ the Sierpiński triangle obtained at the $n^{th}$ stage of the iterative process. It is immediate that $S_n$ consists of three attached copies of $S_{n-1}$ which will be referred as the top, bottom left and bottom right components of $S_n$, denoted by $S_{n,t}$, $S_{n,l}$, and $S_{n,r}$ respectively. See Figure 2.
Structural Properties of $S_n$

1. $S_n$ has $((3^n + 3)/2)$ vertices and $3^n$ edges; its diameter is $2^{n-1}$.

2. All the vertices of the $S_n$ have degree 4 except the three corner vertices which have degree 2.

3. $S_n$ is Hamiltonian and is pancyclic for each $n$. [9]

A Labelling Algorithm

Our aim in this paper is to determine completely the spectrum of Sierpiński triangle $S_n$. This requires that we need to find first the adjacency matrix of $S_n$. But writing down the adjacency matrix using the definition is not feasible. So we propose a labelling algorithm called Sier Labelling to label the vertices of Sierpiński triangle $S_n$ and use it in a MATLAB program to express the adjacency matrix of $S_n$ in terms of the adjacency matrix of $S_{n-1}$. For convenience of notation we refer to a vertex of the graph by the label it gets in the algorithm.

Algorithm Sier Labelling

**Input:** Sierpiński triangle $S_n$ with three copies of $S_{n-1}$ denoted by $S_{n,T}$, $S_{n,L}$ and $S_{n,R}$. Let $x = V(S_{n,T}) \cap V(S_{n,L})$, $y = V(S_{n,T}) \cap V(S_{n,R})$, $z = V(S_{n,L}) \cap V(S_{n,R})$.

**Step 1:** Label $S_2$ as in Figure 3(a).

**Step 2:** At stage $k$, let $y$ be the largest label of $S_{k,T}$, $S_{k,L}$ and $S_{k,R}$. Let $x = V(S_{k,T}) \cap V(S_{k,L})$, $y = V(S_{k,T}) \cap V(S_{k,R})$, $z = V(S_{k,L}) \cap V(S_{k,R})$.

**Step 3:** Stop when step 2 cannot be implemented further.

**Output:** Sier Labelling of $S_n$.

We observe that the above algorithm is proper since each vertex in $S_n$ receives a unique label, as at every stage of the algorithm only the unlabelled vertices are labelled.

B. Spectrum of Sierpiński Triangles

The following MATLAB program called sierspectrum generates the adjacency matrix of the Sierpiński triangle $S_n$ for any $n$, with Sier Labelling and determines its spectrum.

```matlab
function A=sierspectrum(n)
x=0;
A=[0 1 1 0 0 0;1 0 1 1 0 1;1 1 0 1 0 1;1 1 1 0 1 1;1 1 0 1 0 0;1 0 0 1 0 1];
switch n
    case 1
        A=[0 1 1 1 0 1;1 0 1 0 1 0;1 1 0 1 0 1;0 1 0 1 0 1];
    case 2
        A;
```

Figure 2: Sierpiński triangle $S_n$ with $S_{n,T}$, $S_{n,L}$, $S_{n,R}$

Figure 3: Labelling of $S_2$ and $S_3$ using Sier Labelling

Figure 4: Labelling of $S_4$ using Sier Labelling
Assume that the result is true for the dimension \(1.6180, -1.2361, 0.6180, 0.6180, 3.2361\). Consider the Sierpiński triangle \(S_2\). Let \(A\) be the adjacency matrix of \(S_2\), obtained at stage \(k - 1\). In view of the isomorphism between \(S_{k,R} - \{1, y\}\) and the unlabelled vertices of \(S_{k,L} - \{z\}\), the adjacency matrix \(A_1\) of \(S_{k,L} - \{x, z\}\) is obtained by deleting the first row and column and the last row and column from \(A\). Similarly, the adjacency matrix \(A_2\) of \(S_{k,R} - \{y\}\) is obtained by deleting the first row and column of \(A\). When these matrices are concatenated using MATLAB function \texttt{blkdiag}, we obtain a matrix of the form

\[
\begin{pmatrix}
A & * \\
* & A_1 & * \\
* & * & A_2
\end{pmatrix}
\]

where \(*\) denotes the entries to be accounted for.

Remark 1: From the adjacency matrix \(A\), the Laplace matrix and signless Laplace matrix are calculated using \(L = D - A\) and \(Q = D + A\) respectively where \(D\) is the diagonal matrix indexed by the vertices of \(G\) such that \(D_{xx}\) is the degree of \(x\). Thus we get the Laplace spectrum and signless Laplace spectrum as well. Laplace spectrum can be used to calculate the exact number of spanning trees [10].

III. CONCLUSION

In this paper we have devised a method to compute the spectrum of Sierpiński triangles \(S_n\) and that of Sierpiński graphs \(S(n, 3)\) for any \(n\). This is of great relevance in studying the structural properties of Sierpiński triangles in a new perspective. The spectrum of Sierpiński related networks and other important interconnection networks are under investigation.

REFERENCES

Appendix

**Calculation of Spectrum of Sierpiński Triangles**

In this paper, the program `sierspectrum` has been proved to be true for Sierpiński of any dimension. Simulation for Sierpiński triangle when $n=3$ is given below.

**Input:** `sierspectrum(3)`

**Output:**

$A =$

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

`spectrum =`

\[-2.0000 -2.0000 -1.5529 -1.5529 -1.4893 -1.0000 -0.2577 -0.2577 0.7108 0.8418 0.8418 2.9688 2.9688 3.7785\]