A simple definition for the universal Grassmannian order

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Abstract

In this paper, we provide a combinatorial definition of the Universal Grassmannian order (or the Grassmannian Bruhat order) of Bergeron and Sottile. This defines the order in terms of inversions, and thus the order can be viewed as a generalization of the weak order for Coxeter groups. Finally, we use this understanding of the order to analyze the generating function of the number of elements at rank $n$ in this order.

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In [BeS], Bergeron and Sottile defined a new partial order on the symmetric group, which they called the universal Grassmannian order. They used this order to examine the coefficients of the Littlewood–Richardson polynomials, obtaining some new and interesting results. They continued their study of this order in [BeS2, BeS3], where they noted a strange correspondence between the right-sided universal Grassmannian order and the left-sided weak Bruhat order.

Unfortunately, this order is quite complicated to define, involving the $k$-Bruhat order (related to Grassmannian permutations with a single descent at $k$), making it difficult to examine the combinatorial properties of the Grassmannian order. In this paper, we present an equivalent definition of this order based on permutation statistics together with an application of this characterization. This new definition also explains the correspondence between the right universal Grassmannian order and the left weak Bruhat order.

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In Section 1 of the paper, we state our definitions and main results; in Section 2, we examine the properties of these orders; in Section 3 we prove our main theorems; and then in Section 4 we apply our characterization of the order to discuss the rank generating function of the order.

This paper does not assume a great deal of background on the part of the reader. Definitions not stated in the paper can be found in [H], and much of the motivation behind studying these partial orders appears in [F].

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1. Definitions of the orders

In this section we define the BeSo order and the universal Grassmannian order. Let $S_n$ denote the symmetric group on the numbers $1, \ldots, n$, with permutations acting on the left so that multiplication is read from right to left. As we will need to distinguish between ordinary parentheses, cycle notation, and operations, for $\sigma \in S_n$ and $i \in \{1, \ldots, n\}$, we use $\sigma[i]$ to denote $\sigma$ applied to $i$. For permutations, we shall use both the cycle notation (where we will not use commas) and line notation, where $a_1; \ldots; a_n$ denotes the permutation $\sigma$ with $\sigma[i] = a_i$. We define an inversion of the permutation $\sigma$ as an ordered pair $(i, j)$ with $i < j$ and $\sigma[i] \sigma[j]$, and Inv$(\sigma)$ denotes the set of all inversions of $\sigma$. An output inversion is the pair $(\sigma[i], \sigma[j])$, where $(i, j)$ is an inversion of $\sigma$. (Note that in [S], Stanley uses inversion for our output inversions. Our notation, however, is consistent with that of [BeS].) Finally, if $\prec$ is a partial order on a set $S$, we say that $\beta$ is a cover of $\alpha$ if $\alpha \prec \beta$ and $\alpha \preceq \gamma \prec \beta$ implies $\alpha = \gamma$ or $\beta = \gamma$. We denote this covering relation by $\alpha \prec \beta$. (This notation is different from that of [BeS], but is more standard in the literature.)

Define $\sigma \preceq_{\text{weak}} \alpha$ in the left weak Bruhat order on $S_n$ if and only if $\text{Inv}(\sigma) \subseteq \text{Inv}(\alpha)$. Writing $\ell(\sigma) = |\text{Inv}(\sigma)|$ for the length of the permutation $\sigma$, we have $\alpha$ covers $\sigma$ in the left weak Bruhat order if and only if $\alpha = (i \ i + 1) \sigma$ and $\ell(\alpha) = \ell(\sigma) + 1$, explaining the use of the adjective left in this definition. (The right weak Bruhat order is given by containment on output inversions.) The rank of $\sigma$ in the left weak Bruhat order is given by the length of $\sigma$ (see [S]). Throughout this paper, we shall use the term weak order to denote the left weak Bruhat order.

For a fixed $k$, the (right) $k$-Bruhat order $(\leq_k)$ on $S_n$ is generated by the covering relation $\sigma \leq_k \sigma(i \ j)$, where $i \leq k < j$ and $\ell(\sigma) + 1 = \ell(\sigma(i \ j))$. We can now define the Grassmannian order for $S_n$ by defining $\alpha \leq \beta$ in the Grassmannian order if and only if there exists a $k$ together with a $\gamma \in S_n$ such that $\gamma \leq_k \alpha \leq_k \gamma \beta$. In [BeS], it is shown that the Grassmannian order forms a partial order on $S_n$.

This definition is useful when using the Grassmannian order for understanding relationships associated to the generic $k$-Bruhat orders, but it is difficult to use to examine the combinatorial aspects of the order.
Given a permutation \( \sigma \), we make the following definitions:

\[
\sigma^+ = \{ i \mid \sigma[i] < i \}
\]

\[
\sigma^- = \{ i \mid \sigma[i] > i \}
\]

and

\[
\sigma_\perp = \{ i \mid \sigma[i] \leq i \}.
\]

We refer to these as the \textit{ups} and \textit{downs} of \( \sigma \). The inversions of \( \sigma \) can then be broken into classes as follows:

\[
\sigma^{++} = \{ (i,j) \in \text{Inv}(\sigma) \mid i,j \in \sigma^+ \}
\]

\[
\sigma^{+-} = \{ (i,j) \in \text{Inv}(\sigma) \mid i \in \sigma^+, j \in \sigma^- \}
\]

and

\[
\sigma^-- = \{ (i,j) \in \text{Inv}(\sigma) \mid i,j \in \sigma^- \}.
\]

We, respectively, term these as the \textit{up–up}, the \textit{up–down}, and the \textit{down–down} inversions of \( \sigma \). (Note that there cannot be a down–up inversion of \( \sigma \).) We now define a permutation statistic on \( S_n \):

\[
\lambda(\sigma) = |\sigma^{++}| - |\sigma^{+-}| - |\sigma^{--}|
\]

Although it is not immediately clear, it is the case that \( \lambda(\sigma) \geq 0 \) for all \( \sigma \), as we shall see later.

For \( \alpha, \beta \in S_n \), we define \( \alpha \leq \beta \) in the (left) BeSo (for Bergeron and Sottile) order if and only if \( \alpha \leq \mathcal{w} \beta \) and \( \lambda(\beta) = \lambda(\alpha) + \lambda(\mathcal{w}^{-1} \beta) \). We shall prove later that \( \lambda \) is in fact the rank function of the BeSo order.

\textbf{Theorem 1.} If \( \alpha \) and \( \beta \) are two permutations of \( S_n \), then \( \alpha \preceq \mathcal{G} \beta \) if and only if \( \alpha \preceq \beta \).

As a corollary to this we obtain:

\textbf{Corollary 2.} The function \( \lambda : S_n \rightarrow \mathbb{Z}_{\geq 0} \) is the rank function for the Grassmannian order.

Thus, the BeSo order and the Grassmannian order are equivalent. One advantage of this equivalence is that it gives the BeSo order a natural combinatorial definition based on permutation statistics. In particular, the BeSo order naturally arises from the rank order and the weak order as is seen in our second theorem:

\textbf{Theorem 3.} The BeSo order for \( S_n \) is the strongest order that is weaker than both the (Grassmannian-) rank order and the left weak order on \( S_n \).
That is, in the partially ordered set of orders on $S_n$, the BeSo order is the meet of the left weak order and the rank order.

2. Analysis of the orders

To begin with, we shall review some of the work on the Grassmannian order. In [BeS], Bergeron and Sottile prove

**Proposition 4.** Let $\alpha, \beta \in S_n$. Then $\alpha \leq_{G} \beta$ if and only if the following conditions are satisfied:

1. For all $a \in \{1, \ldots, n\}$, the sequence $a, \alpha[a], \beta[a]$ is either increasing or decreasing (not necessarily strictly).
2. For all $a, b \in \{1, \ldots, n\}$ with $a < b$, and $\beta[a] < \beta[b]$, if $a \in \beta_{\downarrow}$ or $a, b \in \beta_{\perp}$ then $\alpha[a] < \alpha[b]$.

From this result, we deduce the following easy result implying that the Grassmanian order is weaker than the weak order.

**Proposition 5.** Let $\alpha, \beta \in S_n$ be given. If $\alpha \leq_{G} \beta$ then $\text{Inv}(\alpha) \subseteq \text{Inv}(\beta)$. Consequently, if $\alpha \leq_{G} \beta$ then $\alpha \leq_{w} \beta$.

**Proof.** It suffices to show that $\text{Inv}(\alpha) \subseteq \text{Inv}(\beta)$ since this implies that $\alpha \leq_{w} \beta$. To see that this condition holds, let $(a, b) \in \text{Inv}(\alpha)$ with $a < b$. If $(a, b) \in \beta_{\downarrow}$ or $(a, b) \in \beta_{\perp}$, condition (2) of Proposition 4 implies $(a, b) \in \text{Inv}(\beta)$. Conversely, if $a \in \beta_{\downarrow}$ and $b \in \beta_{\perp}$, it follows that $a < \beta[a]$, so that condition (1) above implies that $a \leq \alpha[a] < \beta[a]$, while $b \geq \alpha[a] > \beta[a]$. As $(a, b)$ is an inversion, $\alpha[a] > \alpha[b]$ so that $\beta[a] > \beta[b]$ implying $(a, b) \in \text{Inv}(\beta)$. The last case is where either $a = \beta[a]$ or $b = \beta[b]$. If $a = \beta[a]$, then condition (1) implies $a = \alpha[a]$ also. As $(a, b) \in \text{Inv}(\alpha)$, we have $\alpha[b] < a < b$. Condition (1) now tells us that $\beta[b] \leq \alpha[b]$ so that $(a, b) \in \text{Inv}(\beta)$. A similar argument handles the case where $\beta[b] = b$. Thus $\text{Inv}(\alpha) \subseteq \text{Inv}(\beta)$. $\square$

We further note that condition (1) of Proposition 4 together with the above proposition implies

$\alpha_{\downarrow \downarrow} \subseteq \beta_{\downarrow \downarrow}$,

$\alpha_{\uparrow \downarrow} \subseteq \beta_{\uparrow \downarrow}$

and

$\alpha_{\downarrow \uparrow} \subseteq \beta_{\downarrow \uparrow}$.

To show that the Grassmannian order is weaker than the BeSo order, we will need a couple of technical lemmas allowing us to give several equivalent forms for the
rank function \( \lambda \). To this end, given subsets \( A \) and \( B \) of \( \{1, \ldots, n\} \), define

\[
\sigma_{\uparrow A} = \{(i,j) \in \text{Inv}(\sigma) \mid i \in \sigma_{\uparrow}, \ j \in A, \ \sigma[j] = j\}
\]

\[
\sigma_{\downarrow A} = \{(i,j) \in \text{Inv}(\sigma) \mid i \in A, \ \sigma[i] = i, \ j \in \sigma_{\downarrow}\},
\]

\[
\sigma_{\uparrow B} = \{(i,j) \in \text{Inv}(\sigma) \mid i \in \sigma_{\uparrow}, \ \sigma[j] = j, \ j \in B\},
\]

\[
\sigma_{\downarrow B} = \{(i,j) \in \text{Inv}(\sigma) \mid i \in \sigma_{\downarrow}, \ \sigma[j] = j, \ j \in B\}
\]

and

\[
\sigma_{\uparrow AB} = \sigma_{\uparrow A} \cup \sigma_{\uparrow B}.
\]

Note that the meaning behind the notation is that \( A \) represents the points that when fixed will count as ups and \( B \) represents the points that when fixed points will count as downs. For \( \sigma \in S_n \) and these sets we define

\[
\lambda_{A,B}(\sigma) = \lambda(\sigma) + |\sigma_{\uparrow A}| - |\sigma_{\uparrow B}| - |\sigma_{\downarrow A}| - |\sigma_{\downarrow B}|.
\]

\[
= \lambda(\sigma) + |\sigma_{\uparrow A}| - |\sigma_{\uparrow B}| - |\sigma_{\uparrow AB}|.
\]

**Lemma 6.** For all subsets \( A \) and \( B \) of \( \{1, \ldots, n\} \) and \( \sigma \in S_n \), we have \( \lambda_{A,B}(\sigma) = \lambda(\sigma) \).

**Proof.** It suffices to show that \( |\sigma_{\uparrow A}| = |\sigma_{\uparrow A}| \) and \( |\sigma_{\uparrow B}| = |\sigma_{\uparrow B}| \). For the former, note that using complements and inverses, we have the following chain of equalities:

\[
|\sigma_{\uparrow A}| = |\{j > i \mid \sigma(j) < \sigma[i] = i, \ i \in A\}|
\]

\[
= |\{k < i \mid \sigma^{-1}[k] > i = \sigma^{-1}[i], \ i \in A\}|
\]

\[
= (i - 1) - |\{k < i \mid \sigma^{-1}[k] < i = \sigma^{-1}[i], \ i \in A\}|
\]

\[
= (i - 1) - |\{j < i \mid \sigma[j] < \sigma[i] = i, \ i \in A\}|
\]

\[
= |\{j < i \mid \sigma[j] > \sigma[i] = i, \ i \in A\}|
\]

\[
= |\sigma_{\uparrow A}|.
\]

The proof that \( |\sigma_{\uparrow B}| = |\sigma_{\downarrow B}| \) is similar. \( \square \)

Given permutations \( \alpha \) and \( \beta \) with \( \alpha \leq_G \beta \), define

\[
A_{\alpha,\beta} = \{i \in \beta_{\uparrow} \mid \alpha[i] = i\}
\]

and

\[
B_{\alpha,\beta} = \{i \in \beta_{\downarrow} \mid \alpha[i] = i\}.
\]

We then have the following lemma:
Lemma 7. Let $\pi, \beta \in S_n$ be permutations such that $\pi \leq_G \beta$, and let $A = A_{\pi, \beta}$ and $B = B_{\pi, \beta}$ be as above. Then

(a) $(i, j) \in \pi \uparrow \downarrow A$ implies $(i, j) \in \beta \uparrow \downarrow$,
(b) $(i, j) \in \pi \downarrow \downarrow B$ implies $(i, j) \in \beta \downarrow \downarrow$, and
(c) $(i, j) \in \pi \uparrow \downarrow AB$ implies $(i, j) \in \beta \uparrow \downarrow$.

Proof. Let $\pi$ and $\beta$ be as given and suppose $(i, j) \in \pi \uparrow \downarrow A$. Proposition 5 implies $\pi \leq_w \beta$. Consequently, $(i, j)$ is an inversion of $\beta$. If $j = \pi[j]$, then $(i, j) \in \pi \uparrow \downarrow A$ implies $j \in A$ and hence $j \in \beta$, so that $\beta[j] > j$. On the other hand, Proposition 4(2) tells us that if $j < \pi[j]$ then $j < \pi[j] < \beta[j]$ as $\pi \leq_G \beta$. By the same proposition, $i < \pi[i]$ implies $i < \pi[i] < \beta[i]$. Consequently, $(i, j) \in \pi \uparrow \downarrow A$ implies $(i, j) \in \beta \uparrow \downarrow$ as desired. Similar arguments work for parts (b) and (c).

Our next step is to show that the conditions on $\pi$ and $\beta$ of Proposition 4 imply that $\pi \leq_{B\beta}$.

First, we introduce the following notation for $\sigma \in S_n$ given sets $A$ and $B$. Let

\[
\begin{align*}
\sigma_{\uparrow \downarrow A} &= \sigma_{\downarrow \uparrow A} \cup \sigma_{\uparrow \uparrow}, \\
\sigma_{\downarrow \uparrow A} &= \sigma_{\downarrow \downarrow A} \cup \sigma_{\uparrow \downarrow}, \\
\sigma_{\downarrow \downarrow B} &= \sigma_{\downarrow \downarrow B} \cup \sigma_{\downarrow \downarrow}, \\
\sigma_{\uparrow \downarrow B} &= \sigma_{\downarrow \downarrow B} \cup \sigma_{\uparrow \downarrow}, \\
\sigma_{\uparrow \downarrow AB} &= \sigma_{\downarrow \downarrow AB} \cup \sigma_{\uparrow \downarrow}. 
\end{align*}
\]

Proposition 8. Let $\pi, \beta \in S_n$. If $\pi \leq_G \beta$ then $\pi \leq_{B\beta}$.

Proof. Let $\pi, \beta \in S_n$ and suppose $\pi \leq_G \beta$. By Proposition 5, it suffices to check the second condition in the definition of the BeSo order, namely that $\lambda(\beta^{-1}) = \lambda(\beta) - \lambda(\pi)$. At this point we shall look at the inversions of $\pi$, $\beta$, and $\beta^{-1}$ as we claim that $|(\beta^{-1})_{\downarrow \downarrow A}| = |\beta_{\downarrow \downarrow} - |\pi_{\downarrow \downarrow A}|$ and $|(\beta^{-1})_{\downarrow \downarrow B}| = |\beta_{\downarrow \downarrow} - |\pi_{\downarrow \downarrow B}|$.

Suppose $(i, j) \in \beta \uparrow \downarrow$, but it is not an inversion of $\pi$. Then

\[\beta^{-1}[\pi[i]] = \beta[i] > \beta[j] = \beta^{-1}[\pi[j]]\]

and

\[\pi[i] < \pi[j].\]

Thus $(\pi[i], \pi[j])$ is an inversion of $\beta^{-1}$. As $i < \beta[i]$ and $j < \beta[j]$, Proposition 4(1) implies $i \leq \pi[i] \leq \beta[i]$ and $j \leq \pi[j] \leq \beta[j]$. Hence $(\pi[i], \pi[j]) \in (\beta^{-1})_{\downarrow \downarrow A}$ (note that $\beta[j] = \pi[j]$ is
possible). This condition tells us that
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \geq |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha] | = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|.
\]
Similarly, \((i,j) \in \beta_{\mathcal{T}[\alpha]} \setminus \text{Inv}(\alpha)\) implies \((\alpha[i], \alpha[j]) \in (\beta x^{-1})_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]\), and \((i,j) \in \beta_{\mathcal{T}[\alpha]} \setminus \text{Inv}(\alpha)\) implies \((\alpha[i], \alpha[j]) \in (\beta x^{-1})_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]\). These conditions similarly imply
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \geq |\mathcal{T}[\alpha] \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|,
\]
and
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \geq |\mathcal{T}[\alpha] \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|.
\]
For the reverse inequalities, suppose \((\alpha[i], \alpha[j]) \in \text{Inv}(\beta x^{-1})\). If \((i,j) \in \text{Inv}(\alpha)\) by Proposition 5, so that \((\alpha[j], \alpha[i]) \notin \text{Inv}(\beta x^{-1})\). Therefore, we may assume \((i,j) \notin \text{Inv}(\alpha)\).

If \((\alpha[i], \alpha[j]) \in (\beta x^{-1})_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]\), then \(\beta[i] > \alpha[i]\) and \(\beta[j] > \alpha[j]\). If \(\beta[j] = \alpha[j]\), then \(j \in A\). But this implies \(\beta[j] > j\), contradicting \(\alpha[j] = j\). Hence \(\beta[j] > j\). Therefore, by Proposition 4(1), we conclude \(i \leq \alpha[i] < \beta[i]\) and \(j \leq \alpha[j] < \beta[j]\). Consequently, \((i,j) \in \beta_{\mathcal{T}[\alpha]}\). It thus follows that
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \leq |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|.
\]
Similarly, we conclude
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \leq |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]|.
\]
and
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} \leq |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]|.
\]
But we now have the following equalities:
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} = |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|,
\]
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} = |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|,
\]
and
\[
|\beta x^{-1} |_{\mathcal{T}[\alpha]} = |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]|.
\]
Thus
\[
\lambda_{\mathcal{T}[\alpha]}(\beta x^{-1}) = |\beta x^{-1} |_{\mathcal{T}[\alpha]} - |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| - |\beta_{\mathcal{T}[\alpha]} \setminus \mathcal{T}[\alpha]| = |\beta_{\mathcal{T}[\alpha]} | - |\mathcal{T}[\alpha]| - |\mathcal{T}[\alpha]| - |\mathcal{T}[\alpha]| = \lambda(\beta) - \lambda_{AB}(\alpha).
\]
By Lemma 6, it follows that
\[ \lambda(\beta x^{-1}) = \lambda(\beta) - \lambda(x) \]
desired. Therefore \( x \leq_B \beta \).  \( \square \)

We now embark on the more complicated process of showing \( x \leq_B \beta \) implies \( x \leq_G \beta \). This direction appears to be harder as we are unable to find a proof of this that avoids using covers. Consequently, our next step is to classify the covers in the BeSo order. We first show that if \( \gamma x \) is a cover of \( x \) in the BeSo order, then \( \gamma \) is a transposition. Then we characterize the properties of transpositions \( t \) with the property that \( tx \) covers \( x \). To begin the process, we show for every permutation \( x \in S_n \), there exists a sequence of covers
\[ e <_B t_1 <_B t_2 t_1 <_B \cdots <_B t_k t_{k-1} \cdots t_1 = x, \]
such that \( t_i \) is a transposition for all \( i \) and \( k = \lambda(x) \). As a consequence we obtain that every cover of the identity is a transposition. Thereafter, we shall show if \( x \leq_B \beta \), then \( x \leq_B \gamma \leq_B \beta \) if and only if \( e \leq_B \gamma x^{-1} \leq_B \beta x^{-1} \). This latter implies the interval \([x, \beta]_B\) in the BeSo order is isomorphic to the interval \([e, \beta x^{-1}]_B\). We next show that if \( x \) is a cover of \( \beta \), then \( \beta x^{-1} \) is a cover of the identity, implying \( \beta x^{-1} \) is a transposition as desired. The final consequence of this proof will be that the BeSo order is ranked with rank function \( \lambda \). This proof takes a slightly different tack from that of [E], where the proof proceeds inductively using rank conditions.

To begin this process, we establish a condition on the BeSo order noticed by Bergeron and Sottile for the Grassmannian order.

**Proposition 9.** Let \( x \in S_n \) with \( x \neq e \). Then there exist \( i, j \in \{1, \ldots, n\} \) with \( i \) minimal such that \( x[i] > i \) and \( x[j] \) maximal such that \( j \geq x[i] \) and \( j \geq i \). Moreover, if \( t \) is the transposition switching \( x[i] \) and \( x[j] \), then \( tx \leq_B \alpha \) and \( \lambda(tx) = \lambda(x) - 1 \).

**Proof.** If \( x[i] \leq i \) for all \( i \in \{1, \ldots, n\} \), then \( x[i] = i \) for all \( i \). Thus, \( x \neq e \) implies that there exists a minimal \( i \) such that \( x[i] > i \). Moreover, \( x[k] = k \) for all \( k < i \). Let
\[ T = \{ j \mid j \geq x[i] \} \]
and
\[ U = \{ j \mid x[j] > x[i] \} \]
and note that \( |T| = n - x[i] + 1 \), while \( |U| = n - x[i] \). Thus, the set \( S = T \setminus U \) is not empty. Moreover, \( j \geq x[i] \geq x[j] \) if and only if \( j \in S \). As \( x[j] \neq j \), it follows that \( x[j] > i \). Choose \( j \in S \) so that \( x[j] \) is maximal.

At this point, we need to show that if \( t = (x[i], x[j]) \), then \( tx \leq_B x \) and \( \lambda(tx) = \lambda(x) - 1 \). Starting with the latter, we count the changes in the inversions. Let \( A = \{ i \} \) and \( B = \{ j \} \), and we will prove \( \lambda_{AB}(tx) = \lambda(x) - 1 \) giving the result by Lemma 6.

Since \( x[k] = k \) for \( k < i \), there are no inversions of \( x \) or \( tx \) of the form \( (k, i) \) or \( (k, j) \) for \( k < i \). Next, suppose \( i < k < j \). If \( x[j] > x[k] \), then \( (i, k) \) is an inversion of both \( x \) and \( tx \) of the same type, while \( (k, j) \) is not an inversion of either \( x \) or \( tx \). Thus, in this case,
the contributions for $k$ to $\lambda(\pi)$ and to $\lambda_{AB}(\pi\tau)$ are the same. Similarly, if $\pi[i] > \pi[k] > \pi[j]$, then $(i,k)$ and $(k,j)$ are both inversions of $\pi$, and since $i \in \pi_\pi$ and $j \in \pi_\pi$, the inversions of $i$ and $j$ with $k$ contribute nothing to the value of $\lambda(\pi)$, but as neither are inversions of $\pi\tau$, the joint contribution to $\lambda_{AB}(\pi\tau)$ is also zero. If $\pi[k] > \pi[i]$, then $(i,k)$ is not an inversion of either $\pi$ or $\pi\tau$, while $(k,j) \in \pi_\pi$ implies $(k,j) \in (\pi\tau)\pi$. Consequently, inversions with $k$ again contribute equally to both $\lambda(\pi)$ and $\lambda_{AB}(\pi\tau)$.

If $k > j$ and $\pi[j] > \pi[k]$, then $(i,k)$ and $(j,k)$ are both inversions of $\pi$ and $\pi\tau$, and again they contribute equally to both $\lambda(\pi)$ and $\lambda_{AB}(\pi\tau)$.

To complete the proof of the proposition, we need to show

$$\ell(\pi\tau) = \ell(\pi) - \ell(\pi).$$

For this, it is sufficient to note that we need only see that $\pi\tau$ has $\ell(\pi)$ fewer inversions than $\pi$. A simple calculation shows that $\ell(\pi) = 2(\pi[j] - \pi[i]) - 1$. Moreover, the number of inversions, $m$, of $\pi$ that are not inversion of $\pi\tau$ is given by

$$m = 1 + 2 \cdot |\{k \mid i < k < j, \pi[j] < \pi[k] < \pi[i]\}|.$$

Since $\pi[j]$ is maximal with respect to $j \geq \pi[i] > \pi[j] \geq i$, if $\pi[i] > \pi[k] > \pi[j]$, it follows that $k < \pi[i] < j$. As $k < i$ implies $\pi[k] = k < i < \pi[i]$, if $\pi[i] > \pi[k] > \pi[j]$ then $i < k < j$. Consequently,

$$m = 1 + 2(\pi[i] - \pi[j] - 1) = 2(\pi[i] - \pi[j]) - 1 = \ell(\pi)$$

as desired. Therefore $\pi\tau \leq_w \pi$, and hence we have shown that $\pi\tau \leq_B \pi$, establishing the proposition. \[\square\]

**Corollary 10.** Let $\pi \in S_n$. If $e < \pi$ is a cover, then $\pi$ is a transposition.

**Proof.** By Proposition 9, there exists a transposition $\pi$ such that $\pi\tau < \pi\pi$. As it is trivially the case that for all $\sigma \in S_n$, $\pi < \sigma$, it follows that $\pi < \sigma\pi$. Since $\pi$ is a cover, it follows that $\pi = e$. Consequently, $\pi = \pi^{-1} = \pi$, and $\pi$ is a transposition. \[\square\]

Proposition 9 has two other important corollaries:

**Corollary 11.** If $\pi \in S_n$ is not the identity, then $\lambda(\pi) > 0$.

**Proof.** We proceed by induction on $|\text{Inv}(\pi)|$. By Proposition 9, there exists a transposition $\pi$ such that $\pi\tau < \pi\pi$ and $\lambda(\pi\tau) = \lambda(\pi) - 1$. Consequently, $\pi\tau$ has fewer inversions than $\pi$. Therefore, continuing inductively, we can find inversions $t_1, \ldots, t_k$
such that $t_k \ldots t_1 a$ has no inversions and $\lambda(t_k \ldots t_1 a) = \lambda(a) - k$. But $t_k \ldots t_1 a = e$ since it has no inversions. Therefore $\lambda(a) - k = 0$, and hence $\lambda(a) = k > 0$. □

**Corollary 12.** Let $a \in S_n$ be given. If $\lambda(a) = k$, then there exist transpositions $t_1, \ldots, t_k$ such that $a = t_k \ldots t_1$.

At this point, we could prove that all covers correspond to multiplying on the left by transpositions in two ways. First, we could generalize the proof of Proposition 9. This is roughly what is done by Bergeron and Sottile [BeS]. As we are starting with the BeSo order definition, however, it is preferable to work differently. We begin by examining what we can say about the rank $\lambda(tz)$ in terms of $\lambda(x)$, given that $t$ is a transposition such that $\ell(tz) = \ell(t) + \ell(x)$. This characterization will give us the key to proving that $\alpha \leq_\beta \beta$ implies $\alpha \leq_\alpha \beta$. In addition, in [BBB], this idea is used for a more general result on arbitrary Coxeter groups.

**Proposition 13.** Let $a \in S_n$ and $t$ be a transposition. Suppose $\ell(tz) = \ell(t) + \ell(x)$. Then $\lambda(tz) \leq 1 + \lambda(x)$. Moreover, we have equality if and only if $t = (a[i] a[j])$ with $i < j$, $a[i] < a[j]$, $i \in a^+_1$, and $j \in a^-_1$.

**Proof.** Let $t$ denote the transposition $(a[i] a[j])$. The condition that $\ell(tz) = \ell(t) + \ell(x)$ implies first that $a[i] < a[j]$, and second that if $a[i] < a[k] < a[j]$, then $i < k < j$. Let

$$I_1 = \{(k, l) \mid (k, l) \in \text{Inv}(a) \text{ and } \{i, j\} \cap \{k, l\} = \emptyset\},$$

and let $I_2$ denote the inversions of $a$ that involve either $i$ or $j$. For $i = 1, 2$ we define $\lambda(I_i)$ by

$$\lambda(I_i) = |a^+_1 \cap I_i| - |a^-_1 \cap I_i| - |a^+_1 \cap I_i| = I_1 - I_2.$$

Similarly, we define $I'_1$ and $I'_2$ as the set of inversions of $tz$ involving neither $i$ nor $j$, and involving $i$ or $j$, respectively. Clearly,

$$\lambda(tz) - \lambda(x) = |I'_2| - |I_2|.$$
\[T_{(1,ij,<<)} = \{ k \mid k \in \mathcal{A}, \; i < k < j, \; \text{and} \; \mathcal{A}[k] < \mathcal{A}[j]\},\]

\[T_{(1,ij,<<)} = \{ k \mid k \in \mathcal{A}, \; i < k < j, \; \text{and} \; \mathcal{A}[i] < \mathcal{A}[k] < \mathcal{A}[j]\},\]

\[T_{(1,ij,>>)} = \{ k \mid k \in \mathcal{A}, \; i < k < j, \; \text{and} \; \mathcal{A}[k] > \mathcal{A}[j]\},\]

\[T_{(1,jn,<<)} = \{ k \mid k \in \mathcal{A}, \; k > j, \; \text{and} \; \mathcal{A}[k] < \mathcal{A}[i]\},\]

\[T_{(1,jn,>>)} = \{ k \mid k \in \mathcal{A}, \; k > j, \; \text{and} \; \mathcal{A}[i] < \mathcal{A}[k] < \mathcal{A}[j]\},\]

\[T_{(1,jn,>>)} = \{ k \mid k \in \mathcal{A}, \; k > j, \; \text{and} \; \mathcal{A}[k] > \mathcal{A}[j]\},\]

and we similarly define sets \(T_{(\ldots)}\) (where \(k \in \mathcal{A}\) now). Since \(\mathcal{A} \leq u_{tx}\) implies that if \(\mathcal{A}[k]\) is between \(\mathcal{A}[i]\) and \(\mathcal{A}[j]\), then \(i < k < j\), it follows that \(T_{(1,ii,<<)}\), \(T_{(1,jn,<<)}\), \(T_{(1,ii,<<)}\), and \(T_{(1,jn,<<)}\) are all empty sets and need not be considered. Consequently, only fourteen sets need to be examined. At this point, we make a chart noting the contribution to \(\lambda(tx) - \lambda(x)\) for each set, depending on the relationships between \(i, j, \mathcal{A}[i]\), and \(\mathcal{A}[j]\). Since \(i < j\) and \(\mathcal{A}[i] < \mathcal{A}[j]\), we have only the following cases to consider:

1. \(i \in \mathcal{A}, \; j \in \mathcal{A}\),
2. \(i \in \mathcal{A}, \; j \in \mathcal{A}, \; j > \mathcal{A}[i]\),
3. \(i \in \mathcal{A}, \; j \in \mathcal{A}, \; j < \mathcal{A}[i]\),
4. \(i \in \mathcal{A}, \; j \in \mathcal{A}, \; j = \mathcal{A}[i]\),
5. \(i \in \mathcal{A}, \; j \in \mathcal{A}, \; \mathcal{A}[i] < \mathcal{A}[j]\),
6. \(i \in \mathcal{A}, \; j \in \mathcal{A}\).

Note that there must be \(\mathcal{A}[j] - \mathcal{A}[i] + 1\) numbers appearing between \(i\) and \(j\) since \(\mathcal{A}[i] < \mathcal{A}[k] < \mathcal{A}[j]\) occurs only if \(i < k < j\). Condition 6 then implies that \(\mathcal{A}[j] - \mathcal{A}[i] > j - i\), yielding a contradiction. Thus Case 6 is vacuous and may be omitted.

The next step is to calculate the change that each element of one of the \(T\) sets contributes to \(\lambda(tx) - \lambda(x)\). We summarize this in Table 1 (where \(\emptyset\) implies the case is vacuous).

Unfortunately, the only method we have for checking Table 1 requires a tedious case-by-case analysis. For the interested reader, we give a few guideposts for checking it. To begin with, the four sets \(T_{(1,ii,<<)}, T_{(1,jn,>>)}, T_{(1,ii,<<)}, T_{(1,jn,>>)}\) never contribute to any inversions of \(\mathcal{A}\) or \(tx\) involving \(i\) or \(j\), so their contribution to \(\lambda(tx) - \lambda(x)\) must be 0. Moreover, if \(\omega\) is the longest element of \(S_n\) (the element sending \(x\) to \(n - x + 1\)), conjugating permutations by \(\omega\) will interchange columns 2 and 4 as well as columns 3 and 5, while inverting the rows. Thus, reading up column 2 must be the same as reading down column 4 and similarly for columns 3 and 5. Of course, such an inversion leaves column 1 intact, so for column 1 it suffices to simply check the first half of the entries. Next, in the third column, since \(i,j \in \mathcal{A} \cap (tx)\), the value of \(\lambda\) with respect to \(k\) will only change if an inversion involving \(k\) has been created. This immediately tells us that almost all of the entries in this column are 0,
Table 1

\[
\begin{array}{|c|c|c|c|c|}
\hline
i \in \mathcal{A}_1 & j \in \mathcal{A}_1 & i \in \mathcal{A}_1 & j \in \mathcal{A}_1 & j \in \mathcal{A}_1 \\
\hline
i > x[i] & j > x[i] & j \leq x[i] & i < x[j] & i \geq x[j] \\
\hline
T(\uparrow, l_i < \downarrow) & 1, \downarrow \mapsto \uparrow, \downarrow & 1, \uparrow \mapsto \uparrow, \downarrow & 1, \uparrow \mapsto \uparrow, \downarrow & 1, \downarrow \mapsto \downarrow, \downarrow \\
\hline
\end{array}
\]

and similarly for the fifth column. After this, however, it appears to be necessary to complete the rest of the table with a case-by-case analysis.

Using Table 1, we now check the value of \(\lambda(tx) - \lambda(x)\) is for each of the five cases corresponding to each column.

**Case 1** \((i \in \mathcal{A}_1 \text{ and } j \in \mathcal{A}_1)\): In this case, \(\lambda(tx) - \lambda(x) = 1\), since Table 1 tells us the change in the value of \(\lambda\) is due entirely to the new inversion \((i,j)\). As this is an up–down inversion, the change in value is 1. Note that in this case, \(tx \geq_B x\).

**Case 2** \((i \in \mathcal{A}_1, \ j \in \mathcal{A}_1, \text{ and } j > x[i])\): In this case we have

\[
\lambda(tx) - \lambda(x) = 1 + 2|T(\uparrow, l_i > \downarrow)| + 2|T(\downarrow, l_j > \downarrow)| - 2|T(\downarrow, jn, < \downarrow)|. \tag{1}
\]

Notice that

\[
|T(\downarrow, jn, > \downarrow)| + |T(\uparrow, l_i > \downarrow)| + |T(\downarrow, l_j > \downarrow)| = n - \alpha[j]
\]
as these are all the cases where \(\alpha[k] > \alpha[j]\). Then

\[
2 \cdot |T(\uparrow, l_i > \downarrow)| + 2 \cdot |T(\downarrow, l_j > \downarrow)| = 2(n - \alpha[j] - 2 \cdot |T(\downarrow, jn, < \downarrow)|). \tag{2}
\]

Also,

\[
|T(\downarrow, jn, > \downarrow)| + |T(\uparrow, jn, < \downarrow)| = n - j
\]
since for \(j > k\), we must have \(\alpha[j] < \alpha[i]\) or \(\alpha[j] < \alpha[k]\) (see the second line of the proof). Thus,

\[
2 \cdot |T(\uparrow, jn, < \downarrow)| = 2(n - j) - 2 \cdot |T(\downarrow, jn, > \downarrow)|. \tag{3}
\]
Combining Eqs. (1)–(3), we obtain
\[ \lambda(tx) - \lambda(z) = 1 + 2n - 2x[j] - 2|T_{(\uparrow, jn, >)}| - 2(n - j) + 2|T_{(\uparrow, jn, >)}| 
= 1 - 2(x[j] - j). \]
Since \( j \in x_\uparrow \), however, \( x[j] - j \geq 1 \), implying that \( \lambda(tx) - \lambda(z) < 0 \). Consequently, \( tx \not\geq_B x \) in this case.

**Case 3** \((i \in x_\uparrow, j \in x_\downarrow, \text{ and } j \leq x[i])\): In this case
\[ \lambda(tx) - \lambda(z) = -1 - 2|T_{(\uparrow, ij, <)}| < 0 \]
since the \((i, j)\) inversion created by \( t \) is an up–up inversion. Again, \( tx \not\geq_B x \).

**Case 4** \((i \in x_\downarrow, j \in x_\downarrow, \text{ and } j \leq x[j])\): This is similar to Case 2 (after conjugation by the longest element of \( S_n \)).

**Case 5** \((i \in x_\downarrow, j \in x_\downarrow, \text{ and } i > x[j])\): This is similar to Case 3.

These five cases complete the proof of the proposition. \( \square \)

As an important corollary to Proposition 13 we have:

**Corollary 14.** If \( t_1, \ldots, t_k \in S_n \) are all transpositions such that \( \ell(t_k \ldots t_1) = \ell(t_1) + \cdots + \ell(t_k) \), then \( \lambda(t_k \ldots t_1) \leq k \). Moreover, if \( \lambda(t_k \ldots t_1) = k \) then \( \lambda(t_i \ldots t_1) = i \) for all \( i = 1, \ldots,k \).

**Proof.** The case \( k = 1 \) is trivial. Proceeding by induction, suppose
\[ \lambda(t_{k-1} \ldots t_1) \leq k - 1. \]
Since \( \ell(t_k \ldots t_1) = \ell(t_k) + \cdots + \ell(t_1) \), recalling that the length of a permutation is also the length of the shortest way to write the permutation as a product of elementary transpositions, we have that \( \ell(t_i \ldots t_1) = \ell(t_i) + \cdots + \ell(t_1) \) for all \( i \) so that \( \ell(t_i \ldots t_1) = \ell(t_i) + \ell(t_{i-1} \ldots t_1) \). Proposition 13 now implies \( \lambda(t_k \ldots t_1) \leq k \). Moreover, if we have equality, then \( \lambda(t_k \ldots t_1) \) must equal \( k - 1 \) so that the second part follows by induction. \( \square \)

We now state the key lemma in classifying covers.

**Lemma 15.** Suppose \( \alpha, \beta \in S_n \) and \( \tau \) is a transposition, such that
1. \( \ell(\tau \beta \alpha) = \ell(\tau \beta) + \ell(\alpha) = \ell(\tau) + \ell(\beta) + \ell(\alpha) \), and
2. \( \lambda(\tau \beta \alpha) = \lambda(\tau \beta) + \lambda(\alpha) = \lambda(\tau) + \lambda(\beta) + \lambda(\alpha) \) (note that \( \lambda(\tau) = 1 \)).

Then \( \ell(\tau \beta \alpha) = \ell(\tau) + \ell(\beta \alpha) \) and \( \lambda(\tau \beta \alpha) = \lambda(\tau) + \lambda(\beta \alpha) \). That is \( \beta \alpha <_B \tau \beta \alpha \).

**Proof.** Again as the length of a permutation counts any shortest way to write it as a product of elementary transpositions, the first condition tells us that one shortest way to write \( \tau \beta \alpha \) as such a product involves simply concatenating such products for \( \tau, \beta, \) and \( \alpha \). Consequently, concatenating the products for \( \tau \) and for \( \beta \alpha \) must also give a shortest product, implying the first conclusion. Thus, we are left to show that
\[ \lambda(\tau \beta z) = 1 + \lambda(\beta z). \] Given that \[ \lambda(\tau \beta) = \lambda(\tau) + \lambda(\beta) \] together with the condition on the length function, we have \[ \beta <_{B} \tau \beta. \] Consequently, there exists transpositions \[ t_1, \ldots, t_k = \tau \] such that \[ t_k t_{k-1} \ldots t_1 = \tau \beta \] with \[ \lambda(\tau \beta) = k, \] and \[ \ell(t_k \ldots t_1) = \ell(t_k) + \cdots + \ell(t_1). \] Let \[ s_1, \ldots, s_1 \] be transpositions such that \[ z = s_l \ldots s_l, \lambda(z) = l, \] and \[ \ell(z) = \ell(s_l) + \cdots + \ell(s_l). \] By our hypothesis, it follows that

\[ \ell(t_k \ldots t_1 s_l \ldots s_l) = \ell(t_k) + \cdots + \ell(s_l) + \cdots + \ell(s_l), \]

\[ \lambda(t_k \ldots t_1 s_l \ldots s_l) = \lambda(t_k) + \cdots + \lambda(s_l) + \cdots + \lambda(s_l). \]

Consequently, by Corollary 14, \[ \lambda(\beta z) = \lambda(t_{k-1} \ldots t_1 s_l \ldots s_l) = k + l - 1. \] Consequently, \[ \lambda(\tau \beta z) = 1 + \lambda(\beta z) = \lambda(\tau) + \lambda(\beta z) \] as desired. \[ \square \]

We can now prove one of the key facts about the BeSo order that we have yet to need, but have been tacitly assuming without proof. Namely, that it is indeed a partial order.

**Theorem 16.** The BeSo order is a partial order on \( S_n. \) Moreover, if \( z <_{B} \beta z <_{B} \gamma \beta z, \) then \( \beta <_{B} \gamma \beta. \)

**Proof.** To check the first portion, we need to check the conditions that \( <_{B} \) is a transitive anti-symmetric relation. Anti-symmetry follows from the anti-symmetry of the weak order, so it remains to check that \( z <_{B} \beta z \) and \( \beta z <_{B} \gamma \beta z \) imply \( z <_{B} \gamma \beta z. \) Let \( \lambda(z) = k, \lambda(\beta) = l, \) and \( \lambda(\gamma) = m. \) Let \( t_1, \ldots, t_k \) and \( r_1, \ldots, r_l \) and \( s_1, \ldots, s_m \) be transpositions such that

\[ z = t_k \ldots t_1, \]

\[ \beta = r_l \ldots r_1 \] and

\[ \gamma = s_m \ldots s_1, \]

where

\[ \ell(z) = \ell(t_k) + \cdots + \ell(t_1), \]

\[ \ell(\beta) = \ell(r_l) + \cdots + \ell(r_1) \] and

\[ \ell(\gamma) = \ell(s_m) + \cdots + \ell(s_1). \]

Since the BeSo order is a weaker order than the weak order, it follows that

\[ \ell(\gamma \beta z) = \ell(s_m) + \cdots + \ell(s_1) + \ell(r_l) + \cdots + \ell(r_1) + \ell(t_k) + \cdots + \ell(t_1). \] (4)

Moreover, we know that \( \lambda(\beta z) = \lambda(\beta) + \lambda(z), \) and \( \lambda(\gamma \beta z) = \lambda(\gamma) + \lambda(\beta z) = \lambda(\gamma) + \lambda(\beta) + \lambda(z). \)

Consequently, it suffices to show that \( \lambda(\gamma \beta) = \lambda(\gamma) + \lambda(\beta). \) To do this, we shall prove inequality in both directions. To see that \( \lambda(\gamma \beta) \leqslant \lambda(\gamma) + \lambda(\beta), \) note that Eq. (4) implies that \( \ell(\gamma \beta) = \ell(s_m) + \cdots + \ell(s_1) + \ell(r_l) + \cdots + \ell(r_1). \) Consequently, by
Proposition 13 and induction, $\lambda(\gamma \beta) \leq m + l = \lambda(\gamma) + \lambda(\beta)$. For the reverse inequality, suppose that $\lambda(\gamma \beta) = p$. Then there exist transpositions $u_p, \ldots, u_1$ such that $\gamma \beta = u_p \ldots u_1$ and $\ell(\gamma \beta) = \ell(u_p) + \cdots + \ell(u_1)$. Since $\ell(\gamma \beta \alpha) = \ell(\gamma \beta) + \ell(\alpha)$, it follows that

$$\ell(\gamma \beta \alpha) = \ell(u_p) + \cdots + \ell(u_1) + \ell(\alpha).$$

Using Proposition 13 again, it follows that

$$k + l + m = \lambda(\gamma \beta \alpha) \leq p + \lambda(\alpha) = p + k.$$

Hence $l + m \leq p$, implying $\lambda(\gamma) + \lambda(\beta) \leq \lambda(\gamma \beta)$. Thus $\lambda(\gamma) + \lambda(\beta) = \lambda(\gamma \beta)$ as desired implying transitivity. This also shows that $\beta \prec_B \beta$ as desired. $\Box$

In addition to showing that the BeSo order is a partial order, Theorem 16 has the following important corollary:

**Corollary 17.** If $\alpha \prec_B \beta \alpha$ is a cover in the BeSo order, then $\beta$ is a transposition.

**Proof.** Simply consider the condition that $e \prec_B \beta$ and use Corollary 10. $\Box$

We now prove the equivalence of the two orders.

**Theorem 18.** Let $\alpha, \beta \in S_n$. Then $\alpha \leq_B \beta$ if and only if $\alpha \leq_G \beta$, and consequently the Grassmannian order is equivalent to the BeSo order.

**Proof.** By Proposition 8, we already know that $\alpha \leq_G \beta$ implies $\alpha \leq_B \beta$. Consequently, it remains to prove the reverse. Since we are dealing with partial orders, it suffices to prove this condition for covers (i.e., to prove $\alpha \prec_B \beta$ if and only if $\alpha \prec_B \beta$). Suppose $\alpha \prec_B \beta \alpha$. By the preceding Corollary, we know that $\tau$ is a transposition. By Proposition 13, this implies that $\tau = (\alpha[i], \alpha[j])$ where $i \in \alpha, j \in \alpha$, and $\alpha[i] < \alpha[j]$. We will now show that $\alpha$ and $\tau \alpha$ satisfy the conditions of Proposition 4, namely that

1. For all $a \in \{1, \ldots, n\}$, the sequence $a, \alpha[a], \tau \alpha[a]$ is either increasing or decreasing (not necessarily strictly); and
2. For all $a, b \in \{1, \ldots, n\}$ with $a < b$, and $\tau \alpha[a] < \tau \alpha[b]$ if $a, b \in (\tau \alpha) \downarrow$, or $a, b \in (\tau \alpha) \uparrow$ then $\alpha[a] < \alpha[b]$.

To show (1), note for $k \in \{1, \ldots, n\}$, we have either $k \notin \{i, j\}$, in which case $\alpha[k] = \tau \alpha[k]$, or $k \in \{i, j\}$, in which case for $k = i$ we have $i < \alpha[i] < \alpha[j] = \tau \alpha[i]$, and for $k = j$, we have $j > \alpha[j] > \alpha[i] = \tau \alpha[j]$. Thus (1) holds.

For the second condition, note that $\alpha \leq_B \tau \alpha$ implies that $\alpha \leq \alpha \tau \alpha$ in the weak order. However, this implies that any inversion of $\alpha$ is an inversion of $\tau \alpha$. Restated, however, the second condition states that if $(i, j)$ is a non-inversion of $\tau \alpha$ such that either $i, j \in (\tau \alpha) \downarrow$ or $i, j \in (\tau \alpha) \uparrow$, then $(i, j)$ is a non-inversion of $\alpha$. Since this is true for all non-inversions, the condition is satisfied. Consequently $\alpha \leq_G \beta \alpha$ as desired, and the two orders are equivalent. $\Box$
In [BeS] Bergeron and Sottile showed that the Grassmannian order satisfied the property that if \( \alpha \leq_G \beta \), then the interval from \( \alpha \) to \( \beta \) in the Grassmannian order (written [\( \alpha, \beta \)]\( \leq_G \)) is isomorphic to the interval from the identity to \( \beta \alpha^{-1} ([e, \beta \alpha^{-1}]_G) \).

We have already proven half of this claim (for the equivalent BeSo order) in Theorem 16, as there we showed that \( \alpha \leq_B \gamma \leq_B \beta \) implied \( \gamma \alpha^{-1} \leq_B \beta \alpha^{-1} \). We complete this proof here.

**Theorem 19.** If \( \alpha \leq_B \beta \), then the intervals \([\alpha, \beta]_B \) and \([e, \beta \alpha^{-1}]_B \) are isomorphic.

**Proof.** As mentioned above, it suffices to check that \( \gamma \alpha^{-1} \leq_B \beta \alpha^{-1} \) implies \( \alpha \leq_B \gamma \leq_B \beta \). By transitivity, we may restrict our attention to the case where \( \beta \) is a cover of \( \gamma \). That is, we may assume \( \beta = t \gamma \) with \( \lambda(t \gamma \alpha^{-1}) = 1 + \lambda(\gamma \alpha^{-1}) \) and \( \ell(t \gamma \alpha^{-1}) = \ell(t) + \ell(\gamma \alpha^{-1}) \).

By assumption that \( \alpha \leq_B \beta = t \gamma \), we have both \( \ell(t) = \ell(t \gamma \alpha^{-1}) + \ell(\alpha) = \ell(t) + \ell(\gamma \alpha^{-1}) + \ell(\alpha) \) and \( \lambda(t \gamma \alpha^{-1}) = \lambda(t \gamma) - \lambda(\alpha) \). Consequently, it follows that \( \ell(\gamma) = \ell(\gamma \alpha^{-1}) + \ell(\alpha) \).

Thus to see that \( \alpha \leq_B \gamma \), it remains to show that \( \lambda(\gamma) = \lambda(\gamma \alpha^{-1}) + \lambda(\alpha) \).

By Proposition 13 together with the condition on \( \ell(\gamma) \), it follows that \( \lambda(\gamma) \leq \lambda(\gamma \alpha^{-1}) + \lambda(\alpha) \).

On the other hand, by the same reasoning,

\[
\lambda(t \gamma) \leq 1 + \lambda(\gamma).
\]

Therefore

\[
\lambda(\gamma) \geq \lambda(t \gamma) - 1 = \lambda(t \gamma \alpha^{-1}) + \lambda(\alpha) - 1 = 1 + \lambda(\gamma \alpha^{-1}) + \lambda(\alpha) - 1 = \lambda(\gamma \alpha^{-1}) + \lambda(\alpha) \geq \lambda(\gamma).
\]

Thus we have equality throughout, implying that \( \lambda(\gamma) = \lambda(\gamma \alpha^{-1}) + \lambda(\alpha) \) implying in turn that \( \alpha \leq_B \gamma \). This self-same string of equalities, however, implies that \( \lambda(t \gamma) = 1 + \lambda(\gamma) \), so that \( \gamma \leq B \beta \) as required. The theorem now follows. \( \square \)

### 3. An application

In this final section, we shall give an application of the BeSo order definition of the Grassmannian order to investigate the generating function for the number of elements at given rank in the Grassmannian order. In a pre-print version [BeS4] of their paper [BeS3], Bergeron and Sottile give these generating functions \( P_n(x) \) for \( S_n \).
with \( n \leq 10 \). Factoring these polynomials shows that \((1 + x)^{\lfloor n/2 \rfloor}\) divides \( P_n(x) \) in all known cases. Our goal is to prove this result in general. While we can prove this directly, for expositional purposes, it seems helpful to define another partial order on the symmetric group that might also prove to be of general interest. The correspondence between this order and the Grassmannian order, is analogous to the correspondence between the Bruhat order and the weak order.

**Definition.** The two-sided BeSo order is generated by the covering relation \( \alpha \prec_{BS} \beta \), where \( \alpha, \beta \in S_n \) are such that

1. \( \lambda(\beta) = 1 + \lambda(\alpha) \), and
2. \( \beta \alpha^{-1} \) is a transposition.

We note that this order is two-sided since \( \beta \alpha^{-1} \) is a transposition if and only if \( \alpha^{-1} \beta \) is a transposition.

**Theorem 20.** Let \( P_n(x) \) denote the generating function for the number of elements at given rank \( k \) in the BeSo order on \( S_n \). Then \((1 + x)^{\lfloor n/2 \rfloor}\) divides \( P_n(x) \).

We will prove this theorem via the following two lemmas. While we could state both lemmas in their left-sided version, the notation is much easier if we use the right-sided version.

**Lemma 21.** Let \( \tau = (i \ i + 1) \) be an elementary transposition of \( S_n \) and \( \alpha \in S_n \) be any element. Then either \( \alpha \leq_{BS} \alpha \tau \) or \( \alpha \tau \leq_{BS} \alpha \).

**Proof.** Set

\[
A = \begin{cases} 
{i + 1} & \text{if } i = \alpha[i + 1], \\
0 & \text{otherwise,}
\end{cases}
\]

\[
B = \begin{cases} 
i & \text{if } i + 1 = \alpha[i], \\
0 & \text{otherwise,}
\end{cases}
\]

\[
A' = \begin{cases} 
i + 1 & \text{if } i + 1 = \alpha[i + 1], \\
0 & \text{otherwise,}
\end{cases}
\]

\[
B' = \begin{cases} 
i & \text{if } i = \alpha[i], \\
0 & \text{otherwise.}
\end{cases}
\]

Next define

\[\alpha'_{i\uparrow i', \uparrow} = \alpha_{i\uparrow i'} \setminus \{(i, i + 1)\},\]

\[\alpha'_{i\downarrow i', \downarrow} = \alpha_{i\downarrow i'} \setminus \{(i, i + 1)\},\]
We note that \((i, i + 1)\) lies in exactly one of the six sets on the right, so that in the other five cases the right-hand side is identical to the left-hand set. Moreover, in the sixth case, they differ by exactly one element. Consequently, if we can show

\[
|\alpha'_{\downarrow 1,\downarrow} - |(\alpha \tau)^{\downarrow 1,\downarrow} |
\]

\[
|\alpha'_{\downarrow 1,\downarrow'}| = |(\alpha \tau)^{\downarrow 1,\downarrow'} |
\]

\[
|\alpha'_{\downarrow 1,\downarrow'}| = |(\alpha \tau)^{\downarrow 1,\downarrow} |
\]

then we would have that \(\lambda_{AB}(\alpha \tau) - \lambda_{A'B'}(\alpha) = \pm 1\). By Lemma 6, we would then have \(\lambda(\alpha \tau) - \lambda(\alpha) = \pm 1\), which would imply the result.

To establish the above equalities, we show that the transposition \(\tau\) induces bijections between the above pairs of sets. That is, we show that \((k, l) \in \mathcal{X}_{\downarrow 1,\downarrow}\) and \((\alpha \tau)^{\downarrow 1,\downarrow}\) if and only if \((\tau[k], \tau[l]) \in (\alpha \tau)^{\downarrow 1,\downarrow}\) and \((\tau[k], \tau[l]) \in (\alpha \tau)^{\downarrow 1,\downarrow}\). We will establish the first of these statements, and leave the rest to the reader as they are all similar. Suppose \((k, l) \in \mathcal{X}_{\downarrow 1,\downarrow}\). Since \(k < l\) and \((k, l) \neq (i, i + 1)\), it follows that \(\tau[k] < \tau[l]\). Consequently, \((\tau[k], \tau[l])\) is an inversion of \(\alpha \tau\). Since \((k, l) \in \mathcal{X}_{\downarrow 1,\downarrow}\), we have \(\alpha[k] > k\) and \(\alpha[l] > l\), with equality holding if and only if \(l \in A'\) (in which case \(l = i + 1\)). If \(k \neq i + 1\), then \(\tau[k] = k\), and \(\alpha[\tau[k]] = \alpha[k] > k = \tau[k]\), implying that \(\tau[k] \in (\alpha \tau)\). If \(k = i\), as \(l \neq i + 1\), we have \(\alpha[l] > k\) implies \(l > i + 1\) and \(l > k\). Thus \(\alpha[k] > \alpha[l] > k\), and \(\alpha[\tau[k]] = \alpha[k] > k + 1\). Hence, \(\alpha[\tau[k]] \in (\alpha \tau)\), implies \(\tau[k] \in (\alpha \tau)\). Similar arguments to those for \(k\), show that the only case in which \(\alpha[l] \) might not be an element of \((\tau \alpha)\) is when \(l = i\). As \(l \notin A'\), in this case, we have \(\alpha[l] = \alpha[i] > i = \tau[l] - 1\). Thus \(\alpha[\tau[l]] \in (\alpha \tau)\), with equality if and only if \(\alpha[i] = i + 1\). Finally, if \(l \in A'\) we have \(l = i + 1\) and \(\alpha[i + 1] = i + 1\). Clearly we have \(\alpha[\tau[l]] = i + 1 > \tau[l]\), so that \((k, l) \in \mathcal{X}_{\downarrow 1,\downarrow}\) implies \((\alpha \tau[l], \tau[l]) \in (\alpha \tau)^{\downarrow 1,\downarrow}\) and \(\mathcal{X}_{\downarrow 1,\downarrow} \leq |(\alpha \tau)^{\downarrow 1,\downarrow}|\).

Conversely, suppose \((\tau[k], \tau[l]) \in (\alpha \tau)^{\downarrow 1,\downarrow}\), so that \(\tau[k] < \tau[l]\), \(\alpha[k] > \tau[k]\), and \(\alpha[l] > \tau[l]\), with equality implying that \(\tau[l] \in A\) (and hence equal to \(i + 1\)), and \(\alpha[i] = i + 1\). We wish to show that \((k, l) \in \mathcal{X}_{\downarrow 1,\downarrow}\). Again, the result is immediate if \(\{k, l\} \cap \{i, i + 1\} = \emptyset\), and in any case, \((k, l)\) is an inversion of \(\alpha\), since \(\tau[k] < \tau[l]\) implies \(k < l\) (as \((k, l) \neq (i, i + 1)\)). If \(k = i\), then \(\alpha[k] > \tau[k] = k + 1 > k\), and if \(k = i + 1\), then \(\alpha[k] > \tau[l] > i + 1 = k\) (as \(\tau[l] > \tau[k] = i\) and \(l \neq i\)). Therefore, \(k \in \mathcal{X}_{\downarrow 1,\downarrow}\). If
l = i, then \( x[l] \geq \tau[l] \) = \( l + 1 \) > \( l \), while if \( l = i + 1 \), then \( x[l] > \tau[l] = i \), implying that \( x[l] \geq \tau[l] \) with equality if and only if \( x[i + 1] = i + 1 \). Hence \( (k, l) \in x_{\uparrow \downarrow} \). Thus we have shown \( |x_{\uparrow \downarrow}^{\uparrow \downarrow} | \geq |(x \tau)^{\uparrow \downarrow} \), establishing the equality of the two cardinalities.

One similarly shows that \( |x_{\uparrow \downarrow}^{\uparrow \downarrow} | = |(x \tau)^{\uparrow \downarrow} | \) and \( |x_{\downarrow \uparrow}^{\downarrow \uparrow} | = |(x \tau)^{\downarrow \uparrow} | \), establishing the lemma.

Fix \( n > 0 \), and let \( \left[ \frac{n}{3} \right] = k \). Let \( r_l = (2l - 1 \ 2l) \), and consider the subgroups \( H_j \) of \( S_n \) generated by \( r_1, \ldots, r_j \). Clearly \( H \cong (Z_2)^j \) as the \( r_i \)'s all commute.

**Lemma 22.** Fix \( 0 < j \leq \left[ \frac{n}{3} \right] \), and suppose \( x \in S_n \) is such that \( \lambda(x r_i) > \lambda(x) \) for all \( r_i \in H_j \).

Then

\[
\lambda(x r_{i_1} \ldots r_{i_t}) = \lambda(x) + t
\]

for all \( i_1 < i_2 < \ldots < i_t \leq j \).

**Proof.** We prove this by induction on \( j \), using the function \( \lambda \). Fix \( j \geq 1 \). If \( j = 1 \), then the result follows immediately from Lemma 21. Now, suppose we have the result for \( j - 1 \), and consider the element

\[
r_{i_1} \ldots r_{i_t} x,
\]

where \( i_1 < \ldots < i_t \leq j \). By the induction hypothesis, we know that

\[
\lambda(x r_{i_1} \ldots r_{i_{t-1}}) = \lambda(x) + t - 1.
\]

Let \( \beta = x r_{i_1} \ldots r_{i_{t-1}} \). Write \( r_{i_t} = (2x - 1 \ 2x) \), and note that since the \( r_i \)'s all commute, then \( \beta^{-1}[2x - 1] = x^{-1}[2x - 1] \) and \( \beta^{-1}[2x] = x^{-1}[2x] \). Hence

\[
(x^{-1}[2x - 1], x^{-1}[2x]) \in x_{\uparrow \downarrow}
\]

if and only if it is an element of \( \beta_{\uparrow \downarrow} \), and similarly for the other types of inversion sets. Consequently, \( \beta \leq x_{\uparrow \downarrow} \beta r_{i_t} \) if and only if \( x \leq x_{\uparrow \downarrow} r_{i_t} \). By the hypothesis, we therefore have \( \beta \leq x_{\uparrow \downarrow} \beta r_{i_t} \), and hence \( \lambda(\beta r_{i_t}) = \lambda(\beta) + 1 = \lambda(x) + t \).

At this point, we note that the above imply that the \( B_2 \)-induced order on each coset of \( H_j \) is therefore isomorphic to the subset lattice for a set with \( j \) elements. Consequently, as \( S_n \) can be written as a disjoint union of the \( H_j \) cosets, the corresponding generating function for the number of elements at rank \( k \) is divisible by \((1 + x)^j \). Since we can choose \( j = \left[ \frac{n}{3} \right] \), we have proved Theorem 20.

**References**


[E] A. Evani, Results on the Grassmannian Bruhat Order, Bowling Green State University, Bowling Green, OH, 2000, (Supervisor: Prof. C. D. Bennett).

