Direct and inverse results in Hölder norms

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Received 19 November 2004; accepted 19 October 2005
Communicated by Vilmos Totik
Available online 15 December 2005

Abstract

We present a general approach to obtain direct and inverse results for approximation in Hölder norms. This approach is used to obtain a collection of new results related with estimates of the best polynomial approximation and with the approximation by linear operators of non-periodic functions in Hölder norms. © 2005 Elsevier Inc. All rights reserved.

Keywords: Best approximation; Approximation by linear operators; Approximation in Hölder norm

1. Introduction

In last years, there have been some interest in studying the rate of convergence of different approximation processes in Hölder (Lipschitz) norms. The first one, due to A.I. Kalandiya [10], was motivated by applications in the theory of differential equations. Some improvements were obtained by N.I. Ioakimidis [9]. D. Elliot [8] gave other direct estimates. Later other papers were devoted to analyze approximation of periodic functions. For more historical comments on this subject we refer to [3].

The main subject of this paper is to present direct and converse results related with the best approximation and with approximation by linear operators of non-periodic functions in Hölder norms. This will be accomplished in the last section with the help of weighted moduli of smoothness associated to the so-called Ditzian–Totik moduli of smoothness.

In Section 2, we develop a general approach to show how to construct Hölder spaces $E_{\omega,\theta}$ associated to a given modulus of smoothness $\omega$ on a Banach space $E$. Then, we introduce a modulus of smoothness $\theta_{\omega,\theta}$ in this new space and characterize it in terms of an appropriated
In Section 3, we show how theorems concerned with approximation in the basic space $E$ can be used to derive similar ones in the Hölder spaces $E_{\omega,\alpha}$. We remark that we are interested in applications of the abstract approach more than in a general theory in Banach spaces. Of course other results can be derived from our approach, we only include some important ones. This paper can be compared with [2] where approximation in Hölder norms is studied in the periodical case. We remark that the results of [2] can be deduced from the approach given here.

In what follows the letter $E$ will denote a real Banach space which norm $\| \cdot \|_E$ and $W$ a linear subspace of $E$ with a seminorm $| \cdot |_W$.

2. Generalized Hölder spaces

There are different approaches to present generalized Hölder spaces. One of them assumes that we have in hand a certain modulus of smoothness. This last notion can be replaced by a $K$-functional when we are working with an abstract Banach space. In concrete examples one pass from a $K$-functional to a modulus of smoothness by means of a theorem which asserts that both notions are equivalent. There is a standard way to define what a $K$-functional is, but we cannot say the same for the notion of a modulus of smoothness of a given order. Thus, we begin this section by presenting a definition (convenient for our purposes) of a modulus of smoothness on a Banach space.

**Definition 1.** A modulus of smoothness on $E$ is a function $\omega : E \times [0, +\infty) \to \mathbb{R}^+$ such that: (a) For each fixed $t \in (0, +\infty)$, the function $\omega(\cdot, t)$ is a seminorm on $E$ and for all $f \in E$, $\omega(f, 0) = 0$; (b) For each fixed $f \in E$, the function $\omega(f, \cdot)$ is increasing on $[0, +\infty)$ and continuous at 0; (c) There exists a constant $C > 0$ such that for each $(f, t) \in E \times [0, +\infty)$, one has

$$\omega(f, t) \leq C \| f \|.$$

Given a real $r > 0$, we say that the modulus $\omega$ is of order $r$ if $N(E, \omega, r) \neq \text{Ker}(\omega)$ and $N(E, \omega, s) = \text{Ker}(\omega)$ for all $s > r$, where

$$\text{Ker}(\omega) = \left\{ g \in E : \sup_{t \geq 0} \omega(g, t) = 0 \right\}$$

and

$$N(E, \omega, r) = \left\{ f \in E : \sup_{t > 0} \frac{\omega(f, t)}{t^r} < \infty \right\}.$$

To each modulus of smoothness $\omega$ on $E$ we associate some (generalized) Hölder spaces as follows.

**Definition 2.** Given a modulus of smoothness $\omega$ on $E$ and a real $\alpha > 0$, we denote $\theta_{\omega,\alpha}(f, 0) = 0$,

$$\theta_{\omega,\alpha}(f, t) = \sup_{0 < s \leq t} \frac{\omega(f, s)}{s^\alpha}$$

and

$$\| f \|_{\omega,\alpha} = \| f \|_E + \sup_{t > 0} \theta_{\omega,\alpha}(f, t).$$
The Hölder space $E_{0,\alpha}$ is formed by those $f \in E$ such that $\|f\|_{0,\alpha} < \infty$ with the norm $\|f\|_{0,\alpha}$.

Moreover we denote

$$E_{0,\alpha}^0 = \left\{ f \in E_{0,\alpha} : \lim_{t \to 0} \theta_{0,\alpha}(f, t) = 0 \right\}.$$ 

Later we will prove that $\theta_{0,\alpha}$ is a modulus of smoothness of order $r - \alpha$ on $E_{0,\alpha}^0$ provided that $\omega$ is of order $r$. For the moment notice that $\text{Ker}(\theta_{0,\alpha}) = \text{Ker}(\omega)$. For completeness we recall the notion of $K$-functional.

**Definition 3.** If $E$ and $W$ are given as above, the $K$-functional $K^W$ on $E$ is defined for $f \in E$ and $t \geq 0$ by,

$$K^W(f, t) = \inf \{ \| f - g \|_E + t|g|_W : g \in W \}.$$ 

If $\omega$ is a modulus of smoothness of order $r$ on $E$, we say that $\omega$ and the $K$-functional $K^W$ are equivalent if there are positive constants $C_1$, $C_2$ and $t_0$ such that for $f \in E$ and $t \in (0, t_0)$, we have

$$C_1 \omega(f, t) \leq K^W(f, t^r) \leq C_2 \omega(f, t). \tag{2}$$

Now we can state one of the main problems to be considered in this section. Given a linear space $E$, a real $r > 0$, $\alpha \in (0, r)$ and a modulus of smoothness $\omega$ of order $r$ on $E$, characterize (1) in terms of a $K$-functional.

Since our approach will be used in concrete situations, it can be assumed that we have some additional information about $\omega$. In many cases the proof of (2) is obtained as follows. It is shown that there exist positive constant $C$ and $t_0$ such that for any $g \in W$ and $t \in (0, t_0]$,

$$\omega(g, t) \leq Ct^r|g|_W. \tag{3}$$

Moreover for each $t \in (0, t_0]$, there exists a function $L_t : E \to W$, such that for all $f \in E$,

$$\| f - L_t f \|_E \leq C \omega(f, t) \quad \text{and} \quad t^r|L_t f|_W \leq C \omega(f, t). \tag{4}$$

Notice that if (3) holds, then $W \subset E_{0,\alpha}^0 (\alpha \in (0, r))$. This fact will be used below.

In what follows we write $(E, W, L_t, \omega, r, \alpha, t_0)$ to assume that we have a Banach space $E$, a linear subspace $W$ of $E$ (with a seminorm $\| \cdot \|_W \neq 0$), a modulus of smoothness $\omega$ of order $r$ on $E$, and a family of functions $\{L_t\}$ such that conditions (3) and (4) hold and $\alpha \in (0, r)$.

**Theorem 4.** If $(E, W, L_t, \omega, r, \alpha, t_0)$ is given as above, then $\theta_{0,\alpha}$ is a modulus of smoothness of order $r - \alpha$ on $E_{0,\alpha}^0$. Moreover if $g \in W$ and $t > 0$, then

$$\theta_{0,\alpha}(f - g, t) \leq \| f - g \|_{0,\alpha} \quad \text{and} \quad \theta_{0,\alpha}(g, t) \leq Ct^{r-\alpha}|g|_W$$

(where $C$ is the constant given in (4)) and there exist positive constants $D_1$ and $D_2$ such that for $f \in E_{0,\alpha}^0$ and $t \in (0, t_0]$,

$$D_1 \theta_{0,\alpha}(f, t) \leq K_{0,\alpha}(f, t^{r-\alpha}) \leq D_2 \theta_{0,\alpha}(f, t), \tag{5}$$

where

$$K_{0,\alpha}(f, t) = \inf \{ \| f - g \|_{0,\alpha} + t|g|_W : g \in W \}.$$
Thus, since \( \theta_{o,\omega}(f + g, t) \leq \theta_{o,\omega}(f, t) + \theta_{o,\omega}(g, t) \) and \( \theta_{o,\omega}(af, t) = |a| \omega(f, t) \). Therefore for \( s \in \mathbb{R} \) we have

\[
\omega(f + g, t) \leq \omega(f, t) + \omega(g, t), \quad \omega(af, t) = |a| \omega(f, t).
\]

Therefore \( \theta_{o,\omega}(f + g, t) \leq \theta_{o,\omega}(f, t) + \theta_{o,\omega}(g, t) \) and \( \theta_{o,\omega}(af, t) = |a| \theta_{o,\omega}(f, t) \). On the other hand

\[
\theta_{o,\omega}(f - g, t) = \sup_{0 < s \leq t} \frac{\omega(f - g, s)}{s^\alpha} \leq \sup_{s > 0} \frac{\omega(f - g, s)}{s^\alpha} \leq \|f - g\|_{o,\omega}.
\]

Assume now that \( g \in W \). Taking into account (3) we obtain

\[
\theta_{o,\omega}(g, t) = \sup_{0 < s \leq t} \frac{\omega(g, s)}{s^\alpha} \leq C \sup_{0 < s \leq t} s^{r - \alpha} |g|_W = C t^{r - \alpha} |g|_W.
\]

Fix \( s > r - \alpha \). If \( f \in N(E_{o,\omega}, \theta_{o,\omega}, s) \), then \( \theta_{o,\omega}(f, t) \leq C t^{r} \). Thus \( \omega(f, t) \leq C t^{r + \alpha} \). This says that \( f \in N(E, \omega, s) = \text{Ker}(\omega) = \text{Ker}(\theta_{o,\omega}) \). Hence \( N(E_{o,\omega}, \theta_{o,\omega}, s) = \text{Ker}(\theta_{o,\omega}) \). Finally, if \( f \in N(E, \omega, r) \setminus \text{Ker}(\omega) \), then \( f \in N(E_{o,\omega}, \theta_{o,\omega}, r - \alpha) \setminus \text{Ker}(\theta_{o,\omega}) \). We have proved that \( \theta_{o,\omega} \) is a modulus of smoothness of order \( r - \alpha \) on \( E_{o,\omega} \).

Fix \( f \in E_{o,\omega} \). For each \( g \in W \),

\[
\theta_{o,\omega}(f, t) \leq \theta_{o,\omega}(f - g, t) + \theta_{o,\omega}(g, t) \leq C_1 \left\{ \|f - g\|_{o,\omega} + \theta_{o,\omega}(g, t) \right\} \]

\[
\leq C_1 \left\{ \|f - g\|_{o,\omega} + t^{r - \alpha} |g|_W \right\}.
\]

Thus

\[
\frac{1}{C_1} \theta_{o,\omega}(f, t) \leq \inf \left\{ \|f - g\|_{o,\omega} + t^{r - \alpha} |g|_W : g \in W \right\} = K_{r,\omega}(f, t^{r - \alpha}).
\]

For the second inequality in (5) for each \( t \in (0, t_0] \) we fix a function \( L_t : E \to W \) which satisfies (4). For \( s > t \) we obtain the estimates

\[
\omega(f - L_t f, s) \leq C_2 \|f - L_t f\|_E \leq C_3 \omega(f, t) \leq C_3 s^{\alpha} \theta_{o,\omega}(f, s).
\]

Let us find a similar estimate for \( s \leq t \). Recall that for \( f \in E_{o,\omega} \) and \( t \in (0, t_0] \), \( L_t f \in W \). Therefore for \( s \in (0, t] \), we deduce from (3) and (4) that

\[
\omega(L_t f, s) \leq C_4 s^{r} |L_t f|_W = C_4 \left( \frac{s}{t} \right)^r t^{r} |L_t f|_W \leq C_5 \left( \frac{s}{t} \right)^r \omega(f, t).
\]

Thus, since \( K^W \) is a concave function and \( s \leq t \leq t_0 \)

\[
\omega(L_t f, s) \leq C_6 s^{r} \frac{K^W(f, t^r)}{t^r} \leq C_6 s^{r} \frac{K^W(f, s^r)}{s^r} \leq C_7 \omega(f, s).
\]

Now

\[
\omega(f - L_t f, s) \leq \omega(L_t f, s) + \omega(f, s) \leq C_8 \omega(f, s) \leq C_8 s^{\alpha} \theta_{o,\omega}(f, s).
\]

Therefore

\[
\sup_{s > 0} \frac{\omega(f - L_t f, s)}{s^\alpha} \leq C_8 \theta_{o,\omega}(f, t).
\]
From the last inequality and (4) we infer that
\[ \|f - L_t f\|_{\omega, \alpha} \leq C_9 \theta_{\omega, \alpha}(f, t) \]
and
\[ t^{r-\alpha} |L_t|_W \leq C_{10} \frac{\omega(f, t)}{t^{\alpha}} \leq C_{10} \theta_{\omega, \alpha}(f, t), \]
respectively. From this two last inequalities and the definition of a \( K_{\omega, \alpha} \) we have
\[ K_{\omega, \alpha}(f, t^{r-\alpha}) \leq \|f - L_t f\|_{\omega, \alpha} + t^{r-\frac{\alpha}{\alpha+1}} |L_t f| \leq C_{11} \theta_{\omega, \alpha}(f, t). \]
\[ \square \]

3. Best approximation and linear approximation in Hölder spaces

In this section, we assume that there is a sequence \( \{A_n\}_{n=0}^\infty \) of linear subspaces of \( E \) such that \( A_n \subset A_{n+1} \), \( \dim(A_n) = n \) and \( \cup_{n=0}^\infty A_n \) is dense in \( E \).

Recall that for \( f \in E \) the best approximation of \( f \) by \( A_n \) is defined by
\[ E_n(f) = \text{dist}(f, A_n) = \inf \{\|f - h\| : h \in A_n\}. \]

**Theorem 5.** Let \((E, W, L_t, \omega, r, \alpha, t_0)\) be given as in the previous section and suppose that, for each \( n \), \( A_n \subset W \). For \( f \in E_{\omega, \alpha}^0 \) let \( E_{n, \alpha}(f) \) be the best approximation of \( f \) (in \( E_{\omega, \alpha} \)) by \( A_n \). If there exists a constant \( C_1 \) such that for each \( n \), every \( g \in W \) and each \( h \in A_n \) one has
\[ E_{n, \alpha}(g) \leq C_1 \frac{1}{n^{r-\alpha}} |g|_W \quad \text{and} \quad |h|_W \leq C_1 n^{r-\frac{\alpha}{\alpha+1}} \|h\|_E, \]
then there exist positive constants \( C_2 \) and \( C_3 \) such that for \( f \in E_{\omega, \alpha}^0 \) and each \( n \) one has
\[ C_2 E_{n, \alpha}(f) \leq \theta_{\omega, \alpha} \left( f, \frac{1}{n} \right) \leq C_3 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_{k, \alpha}(f). \]

**Proof.** From the main results in [4] we know that there exist positive constants \( C_4 \) and \( C_5 \) such that for every \( f \in E_{w, \alpha}^0 \) and every \( n \),
\[ C_4 E_{n, \alpha}(f) \leq K_{\omega, \alpha} \left( f, \frac{1}{n^{r-\alpha}} \right) \leq C_5 \frac{1}{n^{r-\alpha}} \sum_{k=1}^{n} k^{r-\alpha-1} E_{k, \alpha}(f). \]
Therefore the result follows from Eq. (5). \( \square \)

When a good approximation on \( E \) is obtained by means of an operator with a shape preserving property, then we can derive a direct-type result without using the first inequality in (6).

**Theorem 6.** Let \((E, W, L_t, \omega, r, \alpha, t_0)\) be given as in the previous section and suppose that, for each \( n \), \( A_n \subset W \). If there exists a constant \( D \) and a sequence \( \{H_n\} \) of functions, \( H_n : E \to A_n \) such that, for each \( f \in E \),
\[ \|f - H_n f\| \leq D \omega \left( f, \frac{1}{n} \right) \quad \text{and} \quad \omega(H_n f, t) \leq D \omega(f, t) \quad (t > 0), \]
then for \( h \in E_{\omega, \alpha}^0 \) the first inequality in (7) holds.
Proof. If \( f \in E_{0, \omega, \alpha} \), then for each \( n \)
\[
\| f - H_n f \|_E \leq C_1 \omega \left( f, \frac{1}{n} \right) \leq C_1 \frac{1}{n^2} \theta_{\omega, \alpha} \left( f, \frac{1}{n} \right).
\]

On the other hand, for \( t \geq 1/n \)
\[
\frac{\omega(f - H_n f, t)}{t^{\alpha}} \leq C_2 \frac{1}{t^{\alpha}} \| f - H_n f \|_E \leq C_3 \frac{1}{t^{\alpha}} \omega \left( f, \frac{1}{n} \right) \leq C_3 \theta_{\omega, \alpha} \left( f, \frac{1}{n} \right)
\]
and, for \( t \in (0, 1/n) \),
\[
\frac{\omega(f - H_n f, t)}{t^{\alpha}} \leq \frac{\omega(f, t)}{t^{\alpha}} + \frac{\omega(H_n f, t)}{t^{\alpha}} \leq C_4 \theta_{\omega, \alpha} \left( f, \frac{1}{n} \right).
\]

Therefore \( E_{n, \alpha}(f) \leq \| f - H_n f \|_{\omega, \alpha} \leq D_4 \theta_{\omega, \alpha} \left( f, \frac{1}{n} \right) \). □

Let us discuss some problems of approximation by linear operators in Hölder spaces. For the inverse estimate we need a result analogous to a lemma of Berens and Lorentz in [1]. Since the proof can be obtained with a modification of the one presented in [5, p. 312–313], we omit it.

Lemma 7. If \( 0 < \alpha < 2, \alpha \in (0, 1) \) and \( \phi \) is an increasing positive function on \([0, a]\) with \( \phi(0) = 0 \), then for \( \beta \in (0, 2 - \alpha) \) the inequalities \( \phi(a) \leq M_0 a^\beta \) and \( \phi(x) \leq M_0 \left( y^\beta + (x/y)^{2-\alpha} \right) \) \( (0 \leq x \leq y \leq a) \) imply for some \( C = C(\alpha, \beta) \)
\[
\phi(x) \leq C M_0 x^\beta, \quad 0 \leq x \leq a.
\]

Theorem 8. Let \( (E, W, L, \omega, r, \alpha, t_0) \) be given as in the previous section and suppose that, for each \( n, A_n \subset W \). Let \( \{F_n\} \) be a bounded sequence of linear operators for which there exist a constant \( C \) such that for each \( f \in E \), every \( g \in W \) and all \( n \), one has \( F_n f \in A_n \) and \( |F_n g|_W \leq C |g|_W \). If for each \( f \in E \) and every \( n \), one has \( \|f - F_n f\| \leq D\omega(f, \psi(n)) \), where \( \{|\psi(n)\} \) is a decreasing sequence which converges to zero, then there exists a constant \( D_1 \) such that, for every \( h \in E_{0, \omega, \alpha} \), and each \( n \)
\[
\|h - F_n h\|_{\omega, \alpha} \leq D_1 \theta_{\omega, \alpha}(h, \psi(n)).
\]

Proof. To obtain (8) we only need to verify that \( \sup_{t>0} \|f - F_n f\|_E \leq C_1 \theta_{\omega, \alpha}(h, \psi(n)) \). If \( t > \psi(n) \), then
\[
\omega(h - F_n h, t) \leq C_1 \|h - F_n h\|_E \leq C_2 \omega(f, \psi(n))
\leq C_2 \psi(n)^2 \theta_{\omega, \alpha}(f, \psi(n)) \leq C_3 t^2 \theta_{\omega, \alpha}(f, t).
\]

If \( t \in (0, \psi(n)) \), then \( \omega(h - F_n h, t) \leq C_4 \omega(h, t) + \omega(F_n h, t) \). Thus, it is sufficient to prove that \( \omega(F_n h, t) \leq C_5 \omega(f, t) \). But
\[
\omega(F_n h, t) \leq C_6 \inf \left\{ \|F_n h - g\|_E + t' |g|_W : g \in W \right\}
\leq C_6 \inf \left\{ \|F_n h - L_n g\|_E + t' |L_n g|_W : g \in W \right\}
\leq C_7 \inf \left\{ \|h - g\|_W + t' |g|_W : g \in W \right\} \leq C_8 \omega(h, t). \quad □
\]
Theorem 9. Assume the conditions given in theorem 8 with \( r = 2 \). If there exists a constant \( C \) such that for each \( f \in E \),
\[
|F_n f|_W \leq C n^2 \|f\|_E \quad \text{and} \quad |F_n g|_W \leq C \|g\|_W ,
\]
then there exists a constant \( D_1 \) such that for each couple of positive integers \( n \) and \( k \) and \( f \in E_{o,\alpha}^0 \), one has
\[
\theta_{o,\alpha} \left( f, \frac{1}{n} \right) \leq D_1 \left\{ \|f - F_k f\|_{o,\alpha} + \left( \frac{k}{n} \right)^{2-\alpha} \theta_{o,\alpha} \left( f, \frac{1}{k} \right) \right\} .
\]
Moreover, if for \( \beta \in (0, 2 - \alpha) \) and \( f \in E_{o,\alpha}^0 \) there exists a constant \( C_f \) such that,
\[
\|f - F_n f\|_{o,\alpha} \leq C_f \frac{1}{n^{\beta/2}}
\]
for each positive integer \( n \), then there exists a constant \( D_f \) such that
\[
\theta_{o,\alpha}(f, t) \leq D_f t^\beta .
\]

Proof. Fix \( g \in W \) and integers \( n \) and \( k \). From the definition of \( K_{o,\alpha} \) and considering that \( F_k f \in W \subset E_{o,\alpha} \) and the inequality (5) we obtain that there exists a positive constant \( C_1 \) such that
\[
C_1 \theta_{o,\alpha} \left( f, n^{-1} \right) \leq K_{o,\alpha} \left( f, n^{2-\alpha} \right) \leq \|f - F_k f\|_{o,\alpha} + n^{2-\alpha} |F_k f|_W
\]
\[
\leq \|f - F_k f\|_{o,\alpha} + n^{2-\alpha} (|F_k(f - g)|_W + |F_k g|_W)
\]
\[
\leq \|f - F_k f\|_{o,\alpha} + n^{2-\alpha} k^2 \left( \|f - g\|_E + k^{-1} \|g\|_W \right) .
\]

We consider that \( g \in W \) is arbitrary and use again (5), to infer that there exists a constant \( C_2 \) such that
\[
C_1 \theta_{o,\alpha} \left( f, n^{-1} \right) \leq \|f - F_k f\|_{o,\alpha} + (k/n)^{2-\alpha} k^x \frac{K_f (f, k^{-2})}{K_W (f, k^{-2})}
\]
\[
\leq \|f - F_k\|_{o,\alpha} + C_2 (k/n)^{2-\alpha} \theta_{o,\alpha} \left( f, 1/k \right) .
\]

This proves (10).

The estimate (12) is obtained from Lemma (7) and Eq. (10). \( \square \)

4. Approximation of non-periodic functions

In this section, we realize the abstract approach presented above in the case of continuous or integrable functions defined on an interval of the real line. As before \( r \) is a fixed integer.

Here the letter \( I \) will always denote an interval of the real line and \( \varphi \) an admissible function in the sense of Ditzian–Totik (see [7, p. 8]). Recall that the function \( \varphi(x) = \sqrt{x(1-x)} \), \( \sqrt{x} \), \( \sqrt{x(1+x)} \) is admissible for the interval \( (0, 1) ((0, +\infty)) \). For \( p \in [1, +\infty) \), let \( L_p(I) \) we denote the usual Lebesgue space with the norm \( \|f\|_p = (\int_I |f(x)|^p \, dx)^{1/p} \). For \( f \in L_p(I) \) and \( t > 0 \),
Theorem 10. Fix an admissible function \( \varphi \) the weighted (Ditzian–Totik) modulus of smoothness of order \( r \) is defined by

\[
\omega_r^\varphi (f, t)_p := \sup_{h \in (0, t]} \| \Delta^r h f \|_p.
\]

Let \( W_{\varphi}^{p,r}(I) \) denote the space of all \( g \in L_p(I) \) such that, \( g \) is \( r-1 \) times differentiable, \( g^{(r-1)} \) is absolutely continuous on each compact subinterval of \( I \) and \( \| \varphi^r g^{(r)} \|_p < \infty \). In \( W_{\varphi}^{p,r}(I) \) we consider the seminorm \( \| g \|_{p,r} := \| \varphi^r g^{(r)} \|_p \). These notations are related to the ones considered in the previous section as follows: \( L_p(I) = E, \omega_r^\varphi(f, t)_p = \omega(f, t) \) and \( W_{\varphi}^{p,r}(I) = W (K_{r,\varphi}(f, t)_p = K^W(f, t)) \).

It is easy to verify that \( \omega_r^\varphi(f, t)_p \) is a modulus of smoothness of order \( r \) in the sense we have considered before. Thus for \( \alpha \in (0, r) \) the Hölder space is well defined and we set \( \text{lip}^{\alpha,r}_p(I) = E^{\alpha,r}_\varphi, \| \circ \|_{p,r,\alpha} = \| \circ \|_{\alpha,\varphi}, \theta^{\alpha,r}_\varphi(f, t)_p = \theta_{\alpha,\varphi}(f, t) \) and \( K_{r,\varphi,\alpha}(f, t)_p = K_{\alpha,\varphi}(f, t) \).

For the space \( C(I) \) of bounded continuous functions we obtain similar definitions by changing the \( L_p \) norm by the sup norm. In this case, we use the last notations with \( p = \infty \). In particular \( L_\infty(I) = C(I) \).

From the proof of Theorem 2.1.1 in [7] we have

**Theorem 10.** Fix \( 1 \leq p \leq \infty \) and an admissible function \( \varphi \) for \( I \). There exist constants \( C \) and \( t_0 \) and, for each \( t \in (0, t_0] \) a function \( L_t : L_p(I) \to W_{\varphi}^{p,r}(I) \) such that for \( f \in L_p(I), g \in W_{\varphi}^{p,r}(I) \) and \( h > 0 \),

\[
\| \Delta^r h g \|_p \leq Ch^r \| \varphi^r g^{(r)} \|_p, \quad \| f - L_t f \|_p \leq C \omega_r^\varphi(f, t)_p \tag{13}
\]

and

\[
t^r \| \varphi^r (L_t f)^{(r)} \|_p \leq C \omega_r^\varphi(f, t)_p \tag{14}
\]

Moreover, there exist constant \( C_1 \) and \( C_2 \) such that for \( t \in (0, t_0] \) and \( f \in L_p(I) \)

\[
C_1 \omega_r^\varphi(f, t)_p \leq K_{r,\varphi}(f, t^r)_p \leq C_1 \omega_r^\varphi(f, t)_p. \tag{15}
\]

Now we can state a similar theorem for spaces of Hölder functions. We remark that for the first inequality in (15) the restriction \( t \leq t_0 \) is not needed.

**Theorem 11.** Fix \( \alpha \in (0, r) \). Under the conditions of Theorem 10 there exist positive constants \( D_1, D_2 \) and \( t_0 \) such that for every \( f \in \text{lip}^{\alpha,r}_p(I) \) and \( t \in (0, t_0] \)

\[
D_1 \theta_{\alpha,\varphi}^{\alpha}(f, t)_p \leq K_{r,\varphi,\alpha}(f, t^{r-\alpha})_p \leq D_2 \theta_{\alpha,\varphi}^{\alpha}(f, t)_p. \tag{16}
\]

**Proof.** We use Theorem 4. From (13) and (14) we know that conditions (3) and (4) hold. Then (16) follows from (5). \( \square \)
Let $\Pi_n$ denote the family of all algebraic polynomials of degree no greater than $n$. In order to use the results of Section 3, we set $\Pi_n = A_n$, $E_n(f)_p = E_n(f)$ and $E_{n,2}(f)_p = E_{n,2}(f)$. We first give a proof of the shape-preserving property needed in Theorem 5 and of the Bernstein-type inequality needed in Theorem 6. We remark that the result of Theorem 12 is seen to be known. Since it is important for us we include a proof.

**Theorem 12.** Fix $1 \leq p \leq \infty$, a positive integer $r$ and set $\varphi(x) = \sqrt{1-x^2}$ and $I = [-1, 1]$. For each $n$ let $M_n : L_p(I) \to \Pi_n$ be a (non-linear) operator such that for each $f \in L_p(I)$, $\| f - M_n f \| = E_n(f)$. Then there exists a constant $C$ such that for each $f \in L_p(I)$ and every $n > r$,

$$\omega_r^\varphi(M_n f, t)_p \leq C \omega_r^\varphi(f, t)_p, \quad t \in (0, 1/r].$$

**Proof.** From [7, p. 79, 84] we know that there exists a constant $C_1$ such that $(n > r)$

$$E_n(f)_p \leq C_1 \omega_r^\varphi(f, 1/n)_p \quad \text{and} \quad \| \varphi_r (M_n f)^{(r)} \|_p \leq C_1 n^r \omega_r^\varphi \left( f, \frac{1}{n} \right)_p. \quad (17)$$

Recall that there exist constant $D_1$, $D_2$ and $t_0$ such that for $f \in L_p(I)$ and $t \in (0, t_0]$, Eq. (15) holds.

Fix a positive integer $n$, $f \in L_p(I)$ and $t > 0$. If $t > 1/n$, then

$$\omega_r^\varphi(M_n f, t)_p \leq \omega_r^\varphi(f - M_n f, t)_p + \omega_r^\varphi(f, t)_p \leq C_2 \left\| f - M_n f \right\|_p + \omega_r^\varphi(f, t)_p \leq C_3 \omega_r^\varphi(f, t)_p.$$

On the other hand, if $t \leq \min \{1/n, t_0\}$, then using (15) and (17) we obtain

$$\omega_r^\varphi(M_n f, t)_p \leq C_4 K_{r,\varphi}(M_n f, t^r)_p \leq C_4 t^r \left\| \varphi_r (M_n f)^{(r)} \right\|_p \leq C_5 t^r n^r \omega_r^\varphi(f, 1/n)_p \leq C_5 t^r n^r K_{r,\varphi}(f, n^{-r})_p \leq C_6 K_{r,\varphi}(f, t^r)_p \leq C_7 \omega_r^\varphi(f, t)_p,$$

where we have used the fact that $K_{r,\varphi}(f, t)_p$ is a concave function. From this we have the proof for the case $t \leq t_0$ ($t \leq 1/n$). If $t > t_0$ ($t \leq 1/n$), then using (15) we have

$$\omega_r^\varphi(M_n f, t)_p \leq C_8 K_{r,\varphi}(M_n f, t^r)_p \leq C_8 t^r \frac{1}{t_0^r} K_{r,\varphi}(M_n f, t_0^r)_p \leq C_8 \frac{1}{r^r t_0^r} K_{r,\varphi}(M_n f, t_0^r)_p \leq C_9 \omega_r^\varphi(f, t_0)_p. \quad \square$$

**Theorem 13.** Set $I = [-1, 1]$ and $\varphi(x) = \sqrt{1-x^2}$. Fix $0 \leq p \leq + \infty$, a positive integer $r$ and $\alpha \in (0, r)$. There exists a constant $C$ such that, for any positive integer $n$ and every $P \in \Pi_n$

$$\| \varphi_r P^{(r)} \|_p \leq C n^{r-\alpha} \| P \|_{p,\alpha}.$$

**Proof.** We present a proof for $p < \infty$. For $p = \infty$ similar arguments can be used. If $P$ is a polynomial of degree $n$, then $\operatorname{dist} (P, \Pi_n) = 0$. Thus from the second inequality in (17) it
follows that
\[ \| q^r P^{(r)} \|_p \leq C_1 n^r \omega_{q^r}(P, \frac{1}{n})_p \leq C_1 n^{r-\alpha} \theta_{q^r}(P, \frac{1}{n})_p \leq C_2 n^{r-\alpha} \| P \|_{p, \alpha}, \]
where we have considered Theorem 4. \( \square \)

**Theorem 14.** Set \( I = [-1, 1] \) and \( \varphi(x) = \sqrt{1 - x^2} \). Fix \( 0 \leq p \leq +\infty \), a positive integer \( r > 0 \), \( \alpha \in (0, r) \). Then there exist positive constants \( C_1 \) and \( C_2 \), such that, for every \( f \in \text{lip}_{p, \alpha}(I) \) and all \( n > r \)
\[ C_1 E_n,\alpha(f)_p \leq C_1 \| f \|_{p, \alpha} \leq \sum_{k=1}^{n} k^{r-\alpha-1} E_{k,\alpha}(f)_p. \]

**Proof.** The first inequality follows from Theorem 6, Eq. (17) and Theorem 12. The inverse inequality follows from Theorem 5, since we have verified the Bernstein-type inequality in Theorem 13. \( \square \)

Recall that for a real function \( f \) on \([0, 1]\) the Bernstein polynomial is given by
\[ B_n(f, x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) \binom{n}{k} x^k (1-x)^{n-k}. \]

For these operators we consider the weight function \( \varphi(x) = \sqrt{x(1-x)} \) and set \( E = C[0, 1] \) and \( F = \text{lip}_{p, \alpha} [0, 1] \).

For \( f \in L_1 [0, 1] \) and a positive integer \( n \) the Kantarovich polynomial are defined by
\[ K_n(f, x) = (n+1) \sum_{k=0}^{n} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(s) \, ds \right) \binom{n}{k} x^k (1-x)^{n-k}. \]

For these operator we consider the weight function \( \varphi(x) = \sqrt{x(1-x)} \) and set \( E = L_p [0, 1] \) and \( F = \text{lip}_{p, \alpha} [0, 1] \).

For \( f \in C_\infty [0, +\infty) \) and a positive integer \( n \), the Szasz–Mirakyan operator is given by
\[ S_n(f, x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right). \]

For these operators we consider the weight function \( \varphi(x) = \sqrt{x} \) and set \( E = C_\infty [0, \infty) \) and \( F = \text{lip}_{p, \alpha} [0, \infty) \).

For \( f \in L_p [0, +\infty) \) the operators of Szasz–Kantarovich are defined as
\[ S_n^*(f, x) = e^{-nx} \sum_{k=0}^{\infty} \left( \int_{k/(n+1)}^{(k+1)/(n+1)} f(s) \, ds \right) \frac{(nx)^k}{k!}. \]

In this case we consider the weight \( \varphi(x) = \sqrt{x} \) and the spaces \( E = L_p [0, \infty) \) and \( F = \text{lip}_{p, \alpha} [0, \infty) \).
For \( f \in C_\infty [0, +\infty) \), the Baskakov operators are defined by
\[
V_n(f, x) = \sum_{k=0}^{\infty} f \left( \frac{k}{n} \right) \left( \frac{n + k - 1}{k} \right) x^k (1 + x)^{-n-k}.
\]
In this case we consider the weight \( \varphi(x) = \sqrt{x(1+x)} \) and set \( E = C_p [0, \infty) \) and \( F = \text{lip}^{\varphi,2}_{p,\varphi} [0, \infty) \).

The Baskakov–Kantarovich polynomials are defined analogously. In this case we consider the weight \( \varphi(x) = \sqrt{x(1+x)} \) and set \( E = L_p [0, \infty) \) and \( F = \text{lip}^{\varphi,2}_{p,\varphi} [0, \infty) \).

**Theorem 15.** Let \( \{F_n\} \) be the sequence of Bernstein (Kantorovich, Szasz–Mirakyan, Szasz–Kantarovich, Baskakov) operators with the weight function \( \varphi \) and the associated space \( E \) and \( F \) be given as above where \( \alpha \in (0, 2) \).

(i) There exist a constant \( C \) such that, for \( f \in F \) and each positive integer \( n \)
\[
\|f - F_n(f)\|_{w,\alpha} \leq C_\alpha \cdot (f, \frac{1}{\sqrt{n}})_p.
\]

(ii) For \( k \leq n \) one has
\[
\theta_{f,\alpha}^\alpha \left( f, \frac{1}{n} \right) \leq D_1 \left\{ \|f - F_k f\|_{p,2,\alpha} + \left( \frac{k}{n} \right)^{2-\alpha} \theta_{f,\alpha}^\alpha \left( f, \frac{1}{k} \right) \right\}.
\]

(iii) Fix \( \beta \in (0, 2 - \alpha) \) and \( f \in F \). There exists a constant \( C_f \) such that, for all \( n \),
\[
\|f - F_n f\|_{p,2,\alpha} \leq C_f \cdot \frac{1}{n^\beta/2}
\]
if and only if there exists a constant \( D_f \) such that
\[
\theta_{f,\alpha}^\alpha (f, t) \leq D_f t^\beta.
\]

**Proof.** It follows from Theorem 9.3.2 in [7, p. 117] that,
\[
\|f - F_n (f)\|_p \leq C \left\{ \frac{1}{n} \|f\|_p + \omega_2^\alpha \left( f, \frac{1}{\sqrt{n}} \right)_p \right\}.
\]
On the other hand, there exists a constant \( D \) such that, for any \( g \in W \),
\[
\|\varphi^2 F_n^{(2)} g\|_p \leq D_2 \|\varphi^2 g^{(2)}\|_p
\]
(see (9.3.7) in [7, p. 118]). Then the result follows from Theorem 8.

(ii) For the inverse result we only need to verify condition (9), that is the Bernstein type inequality
\[
\|\varphi^2 L_n^{(2)} f\|_p \leq C n^2 \|f\|_p.
\]
But this last inequality is known (see Eq. (9.3.5) in [7, p. 118]).

(iii) It is a consequence of (i) and (ii). \( \square \)

**References**


