Orderings of uniquely colorable hypergraphs

Csilla Bujtás*, Zsolt Tuza1

Department of Computer Science, University of Pannonia, H-8200 Veszprém, Egyetem u. 10, Hungary

Received 13 March 2005; received in revised form 15 January 2007; accepted 18 February 2007
Available online 3 March 2007

Abstract

For a mixed hypergraph \( H = (X, C, D) \), where \( C \) and \( D \) are set systems over the vertex set \( X \), a coloring is a partition of \( X \) into ‘color classes’ such that every \( C \in C \) meets some class in more than one vertex, and every \( D \in D \) has a nonempty intersection with at least two classes. A vertex-order \( x_1, x_2, \ldots, x_n \) on \( X \) (\( n = |X| \)) is uniquely colorable if the subhypergraph induced by \( \{x_j : 1 \leq j \leq i\} \) has precisely one coloring, for each \( i \) (\( 1 \leq i \leq n \)). We prove that it is NP-complete to decide whether a mixed hypergraph admits a uniquely colorable vertex-order, even if the input is restricted to have just one coloring. On the other hand, via a characterization theorem it can be decided in linear time whether a given color-sequence belongs to a mixed hypergraph in which the uniquely colorable vertex-order is unique.

© 2007 Elsevier B.V. All rights reserved.

Keywords: Algorithmic complexity; Mixed hypergraph; Uniquely colourable; Vertex-order

1. Introduction

Using the terminology of [9], a mixed hypergraph \( H = (X, C, D) \), where \( X \) is the vertex set, and \( C \) and \( D \) are families of subsets of \( X \). It will be assumed throughout that \( |H| \geq 2 \) holds for all \( H \in C \cup D \). For simplicity, we shall also use the shorthand ‘M-graph’ for ‘mixed hypergraph’.

An element of \( C \) is called \( C \)-edge and an element of \( D \) is a \( D \)-edge. This distinction gets meaning when we color a mixed hypergraph. A (proper) \( k \)-coloring of \( H \) is a mapping from the vertex set \( X \) into a set of \( k \) colors, where each \( C \)-edge has at least two vertices with a common color and each \( D \)-edge has at least two vertices with distinct colors.

We shall denote the colors by the positive integers \( 1, 2, \ldots, k \).

The motivation of introducing mixed hypergraphs [7,8] originates from theoretical biology. Chapter 12 of [9] discusses a number of further problems that can be modeled by M-graphs. Moreover, the related concept of ‘rich coloring’ [1] was studied for its applicability in information theory.

There exist mixed hypergraphs—termed uncolorable—which have no proper colorings at all (the simplest one consists of a 2-element \( C \)-edge and a 2-element \( D \)-edge on the same vertex pair), while others have many colorings. We can consider a coloring as a partition of \( X \) into color classes. In this case we disregard renumberings of colors; i.e.,...
two colorings are considered different only if there are two vertices having the same color in one of them and different colors in the other one.

In this sense a mixed hypergraph is uniquely colorable—UC-graph, or UC, for short—if it has precisely one feasible partition into color classes. Such M-graphs are on the boundary between colorable and uncolorable systems. It was shown in [6] that UC-graphs have a rather unrestricted structure. More precisely, every colorable M-graph can be embedded into some UC-graph as an induced subhypergraph. (For a given M-graph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, the subhypergraph induced by a vertex subset $X' \subseteq X$ is defined as the M-graph with vertex set $X'$ and having all those $\mathcal{C}$- and $\mathcal{D}$-edges of $\mathcal{H}$ which are contained wholly in $X'$.) In the same paper the algorithmic intractability of deciding whether a given M-graph is UC was proved, too.

**Definition 1.** A mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is **UC-orderable** if there exists a vertex-order $x_1, x_2, \ldots, x_n$ on the vertex set $X$ with the following property: for each $1 \leq i \leq n$, the subhypergraph $\mathcal{H}_i$ induced by $\{x_1, \ldots, x_i\}$ is uniquely colorable. Such a vertex-order on $X$ will be termed a **UC-order**.

Note, that if we color the vertices of a UC-orderable $\mathcal{H}$ in the given order $x_1, x_2, \ldots, x_n$ one by one, considering only the subhypergraph induced by $\{x_1, \ldots, x_i\}$, we have just one possible color for $x_i$ in each step.

Unique colorability and UC-orderability on mixed hypertrees mean the same [4]; but in general the two properties are not equivalent. The smallest example demonstrating their difference consists of two disjoint 2-element $\mathcal{C}$-edges and a $\mathcal{D}$-edge containing all the four vertices. This M-graph admits the unique color partition into two classes, but it has no UC-order.

Trivially, every UC-orderable M-graph is UC (apply the definition to $i = n$). One might expect that the converse is simple, too: UC-orderability seems to be such a special property that it might be easy to decide whether a UC-graph has it or not. This intuition, however, is far from being correct; one of our main results, Theorem 1, states that this problem is NP-complete. Along the way, an auxiliary result—may be of interest in itself, too—is proved (Corollary 1), namely that it is NP-complete to decide whether a 3-uniform hypergraph contains a vertex subset that meets every edge in precisely one vertex.

We consider a more restricted class of UC-graphs, too. Note first that if $x_1, x_2, x_3, \ldots, x_n$ is a UC-order, then so is $x_2, x_1, x_3, \ldots, x_n$ as well, obtained by the transposition of $x_1$ and $x_2$. The M-graphs with no more UC-orders were introduced in [6] (cf. also [9, Problem 3 p. 76]).

**Definition 2.** A UC-graph is called **uniquely UC-orderable**—UUC-graph, or UUC, for short—if it has just one UC-order apart from the transposition of the first two vertices. (The smallest UC-orderable non-UUC-graph consists of three vertices mutually joined by 2-element $\mathcal{D}$-edges, that is the simple graph $K_3$.)

We study the color-orders belonging to the (unique) UC-orders of UUC-graphs, and completely characterize them in Theorem 2. This result shows some analogy with the paper [2] where the size distributions of coloring partitions are characterized for the *uniform UC-graphs with $\mathcal{C} = \mathcal{D}$*. (A hypergraph is $r$-uniform if each of its edges contains exactly $r$ vertices.) Both in [2] and in our theorem, the structure of M-graphs themselves is not well-described, but necessary and sufficient conditions are given for their characteristics on a higher level.

We close this introduction with a brief summary of complexity results on M-graphs.

**Complexity of some M-graph coloring problems.**

- It is NP-complete to decide whether a given M-graph is colorable [6].
- Given $\mathcal{H}$ together with a feasible coloring, it is co-NP-complete to decide whether $\mathcal{H}$ is UC [6]. (Equivalently, deciding whether $\mathcal{H}$ admits at least one further proper coloring is NP-complete.)
- Given a UC-graph $\mathcal{H}$, it is NP-complete to decide whether $\mathcal{H}$ is UC-orderable (our Theorem 1).
- It can be decided in linear time whether a given vertex-order of $\mathcal{H}$ is a UC-order (our Proposition 1).
- Given an integer $r \geq 3$ and a sequence $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$, it can be decided in linear time whether there exists an $r$-uniform UC-graph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ such that $\mathcal{C} = \mathcal{D}$ and in the unique coloring of $\mathcal{H}$ the color classes have respective cardinalities $n_1, \ldots, n_k$ (from the characterization in [2]).
- Given a color-order $c_1, c_2, \ldots, c_n$, it can be decided in linear time whether there exists a UUC-graph whose unique UC-order generates the given color-order (from our characterization Theorem 2).
Open problems on complexity.

Determine the time complexity of recognizing UUC-graphs, under the condition that the input is restricted to

1. colorable,
2. uniquely colorable,
3. UC-orderable

mixed hypergraphs. Moreover, study the analogous problem where also (1) a proper coloring, (2) the unique color partition, or (3) a UC-order is given in the input.

2. NP-completeness of UC-orderability

The main goal of this section is to prove the following result:

Theorem 1. Given a uniquely colorable mixed hypergraph $H$ with its coloring as an input, it is NP-complete to decide whether $H$ has a UC-ordering.

Before the details of the proof, let us verify first the membership of UC-orderability in NP. As a matter of fact, a polynomial-time (more precisely, quadratic) test for any fixed vertex-order can be read out from the combination of ideas presented in [5,6]. Here we prove a stronger (best possible) time bound, as follows.

Proposition 1. For any $H = (X, C, D)$, it is decidable in linear time whether a given vertex-order $x_1, \ldots, x_n$ on $X$ is a UC-order.

Proof. Let us note first that the expression

$$|X| + \sum_{C \in \mathcal{C}} |C| + \sum_{D \in \mathcal{D}} |D|$$

is a lower bound on the input size. We are going to present an algorithm whose running time is proportional to this sum.

Let $X_i := \{x_1, \ldots, x_i\}$, for $i = 1, 2, \ldots, n$. Assuming that $X_{i-1}$ has been colored, the possible colors for $x_i$ are determined by precisely those edges of $H$ that are induced by $X_i$, contain $x_i$ and, moreover, are of one of the following two kinds:

- $C \in \mathcal{C}$, and all colors in $C \cap X_{i-1}$ are distinct.
- $D \in \mathcal{D}$, and all colors in $D \cap X_{i-1}$ are the same.

Such edges are called *influencing* $\mathcal{C}$-edges and *influencing* $\mathcal{D}$-edges for $x_i$, respectively. Denoting in general by $\mathcal{P}(Y)$ the set of colors occurring on the vertices in a set $Y \subseteq X$, the *forcing set* $FS(i) = \bigcap \{\mathcal{P}(C \setminus \{x_i\}) : C$ is an influencing $\mathcal{C}$-edge for $x_i\}$ lists the colors from which $x_i$ has to get one, while the *veto set* $VS(i) = \bigcup \{\mathcal{P}(D \setminus \{x_i\}) : D$ is an influencing $\mathcal{D}$-edge for $x_i\}$ contains the colors excluded from $x_i$. It is readily seen that $x_i$ is a uniquely colorable vertex if and only if either $|FS(i) \setminus VS(i)| = 1$ or there is no influencing $\mathcal{C}$-edge for $x_i$ and $VS(i) = \mathcal{P}(X_{i-1})$ (depending on whether the uniquely determined color of $x_i$ has appeared already in $X_{i-1}$ or not). Hence, the heart of the matter is to generate the sets $FS(i), VS(i)$ in linear time.

The algorithm runs in two phases. First, in reverse order $x_n, x_{n-1}, \ldots, x_1$ it determines the collections $\mathcal{C}_i, \mathcal{D}_i$ of those edges whose vertex of largest subscript is $x_i$. Having them at hand for all $i$, the second phase scans $X$ in the original order $x_1, x_2, \ldots$ as long as it is unique. We shall also store $|\mathcal{P}(X_{i-1})|$, that equals the number of vertices $x_j$ ($1 \leq j \leq i$) without influencing $\mathcal{C}$-edges. In each step $i > 1$, first the influencing edges are selected from $\mathcal{C}_i$ and $\mathcal{D}_i$ for $x_i$, and then it is tested whether $|FS(i) \setminus VS(i)| = 1$, or $|VS(i)| = |\mathcal{P}(X_{i-1})|$.
and there is no influencing $C$-edge for $x_i$. If none of these holds for some $i \leq n$, then the algorithm terminates with concluding that $x_1, \ldots, x_n$ is not a UC-order.

One way to proceed with this in linear time—assuming adjacency list representation, that is easily constructed from another input format if necessary—is to duplicate $C$ and $D$ as $C'$ and $D'$, the set systems that will consist of the edges actually available. While at $x_i$, it is checked for each edge in the list of $x_i$ whether the edge still occurs in $C'$ or $D'$. If so, then the edge in question is moved from there into $C_i$ or $D_i$. This phase is obviously fast.

For the second phase, it is convenient to create ‘dual lists’, i.e., listing for each edge the vertices contained in it. It will then take just $O(|H|)$ steps for any $H \in C_i \cup D_i$ to test whether $H$ is influencing for $x_i$; and if so, then the corresponding colors will be inserted into $FS(i)$ or $VS(i)$. After that, the color of $x_i$ is easily determined, always taking for new color the smallest positive integer still available. (The colors assigned are conveniently stored in a block of size $n$.) □

There are other alternatives, too, for a linear-time test, but we do not find it important to describe here more than the solution above.

2.1. Structure of the NP-hardness proof

We now turn to the substantial part of Theorem 1, that is the hardness of deciding whether an input M-graph admits a UC-order. The complexity of this problem will be traced back to the classical problem of hypergraph 2-coloring, more precisely to the 2-colorability of 3-uniform hypergraphs. For the latter, the input is a hypergraph (i.e., $D$-hypergraph in the terminology of M-graphs, having $C = \emptyset$) in which each hyperedge contains precisely three vertices, and the question is whether there exists a vertex partition into two classes, none of them containing any hyperedge. This problem is well-known to be NP-complete [3].

The reduction will be carried out in two steps. First we make a reduction from hypergraph colorability to a new type of hypergraph covering problem (still no involved), and then go on to UC-orderings. The proof of the latter will be postponed to the next subsection. We begin with introducing the following concept.

**Definition 3.** Let $F$ be a set system over an underlying set $X$. A set $B \subset X$ is a strong blocking set (SBS, for short) if it contains precisely one element from each member of $F$; that is, $|B \cap F| = 1$ holds for all $F \in F$. (This term is borrowed from design theory, where ‘blocking set’ means a set $B$ that meets all the $F \in F$ but does not contain any of them.)

We shall prove the following lemmas. The technical conditions included in first one will play a role in the proof of the main result later.

**Lemma 1.** For any given 3-uniform hypergraph $\mathcal{E}$, a 3-uniform hypergraph $F$ can be constructed in polynomial time, with the following properties: $\mathcal{E}$ is colorable with two colors if and only if $F$ has a SBS, moreover

(i) $F$ has no blocking vertex (that is, $\bigcap_{F \in \mathcal{F}} F = \emptyset$),
(ii) $F$ contains two vertices not belonging to a common hyperedge.

**Lemma 2.** Let $F$ be any hypergraph with at least two vertices not belonging to a common hyperedge. Then a uniquely colorable mixed hypergraph $H$ (whose coloring is known) can be constructed in polynomial time, such that $H$ has a UC-order if and only if $F$ has a SBS with more than one element.

From these two assertions, the main result of the section can easily be deduced.

**Proof of Theorem 1.** A 1-element SBS in a hypergraph $F$ would be a blocking vertex of $F$. Therefore, combining Lemmas 1 and 2, we obtain that for each 3-uniform hypergraph $\mathcal{E}$, a uniquely colorable mixed hypergraph $H$ (with its known coloring) can be constructed in polynomial time, such that $\mathcal{E}$ is 2-colorable if and only if $H$ has a uniquely colorable ordering. Since the former property is NP-complete to decide [3], it follows that the latter is intractable, too. □
Stopping at half way, from Lemma 1 we obtain:

**Corollary 1.** It is NP-complete to decide whether a 3-uniform hypergraph has a SBS.

**Proof of Lemma 1.** Let $\mathcal{E}$ be a 3-uniform hypergraph. To construct $\mathcal{F}$, we keep the (‘old’) vertices of $\mathcal{E}$ and supplement them with nine new vertices for each edge. If $E = \{x, y, z\}$ is an edge of $\mathcal{E}$ and the new $E$-vertices in $\mathcal{F}$ are $f_1, f_2, \ldots, f_9$, then the edges of $\mathcal{F}$ are the triples $\{x, f_1, f_2\}, \{y, f_3, f_4\}, \{z, f_5, f_6\}, \{f_2, f_4, f_6\}, \{f_1, f_3, f_7\}, \{f_3, f_5, f_8\}, \{f_1, f_5, f_9\}$ (see Fig. 1 for illustration). Note that the ‘old’ edge $E$ is not included anymore. Then, $\mathcal{F}$ is constructed from such gadgets that meet only at the ‘old’ vertices and are mutually vertex-disjoint outside.

First, we assume that $\mathcal{E}$ has a proper coloring with two colors. Let us consider the (old) vertices of the first color in $\mathcal{E}$. If the edge $E \in \mathcal{E}$ has two vertices with this color, say $x$ and $y$, then, in $\mathcal{F}$ we put the vertices $x, y, f_6, f_7, f_8, f_9$ into the SBS to be constructed. If $E$ has only one vertex with this color, say $x$, we can choose $x, f_3, f_6, f_9$ from this gadget into the SBS. We apply this method for all the edges of $\mathcal{E}$ and finally obtain a SBS containing exactly one vertex from each edge of $\mathcal{F}$.

Second, we prove that there exists a 2-coloring of $\mathcal{E}$ whenever $\mathcal{F}$ has a SBS (denoted $B$). For any gadget in $\mathcal{F}$, it is impossible that all the three of its ‘old’ vertices $x, y, z$ belong to $B$ at the same time. Indeed, otherwise none of the connected $f_2, f_4, f_6$ could belong to $B$, and it would be in contradiction to the assumption that $B$ meets the edge $\{f_2, f_4, f_6\}$. Similarly, it cannot be the case that $B$ does not contain any of the ‘old’ vertices of the gadget. Therefore, we can find just one or two ‘old’ vertices belonging to $B$ in each gadget. Let these vertices be colored in $\mathcal{E}$ with the first color, and the other vertices with the second color. This is a proper 2-coloring since every $\mathcal{E}$-edge has two vertices with distinct colors.

Clearly, the construction can be carried out in linear time and $\mathcal{F}$ complies with the restrictions (i) and (ii). □

**Remark.** If the hypergraph $\mathcal{F}$ constructed in the proof has some SBS $T$, then $|T| \geq 3|\mathcal{E}|$, because in each gadget, the four edges disjoint from $\{x, y, z\}$ cannot be covered with fewer than three vertices. (A more careful analysis yields the lower bound $3|\mathcal{E}| + \tau(\mathcal{E})$, where $\tau(\mathcal{E})$ is the smallest number of vertices in a set that meets all edges of $\mathcal{E}$.) Later on we shall only use the fact that $\tau(\mathcal{F}) > 1$.

### 2.2. Strong blocking sets vs. UC-orders

Here we prove Lemma 2. We shall need the following definition.

**Definition 4.** For a hypergraph $\mathcal{H}$ with vertex set $X$ and with edges $H_1, H_2, \ldots, H_k$, an *edge-crossing set* is a subset $B \subset X$ such that $|B \cap H_i| \leq 1$ for each $i, 1 \leq i \leq k$. By definition, a set is a SBS if and only if it is edge-crossing and also meets all edges.

**Construction of $\mathcal{H}$ Lemma 2.** Let the vertex-set of $\mathcal{H}$ be the disjoint union of the following sets:

- $X = \{x_1, \ldots, x_m\}$: it has $m = |\mathcal{F}|$ elements, one for each edge $F_i \in \mathcal{F}$;
- $Y = \{y_1, \ldots, y_n\}$ and $Y^* = \{y^*_1, \ldots, y^*_n\}$: these are two copies of the vertex set $\{v_1, \ldots, v_n\}$ of $\mathcal{F}$;
- $\{w; w^*\}$: two further vertices.
All the $\mathcal{D}$-edges of $\mathcal{H}$ will have just two vertices. Vaguely speaking, their induced subgraphs on $X \cup Y \cup \{w\}$ will be nearly complete (but if $v_{i}$ and $v_{j}$ are contained in a common edge in $\mathcal{F}$ then the $\mathcal{D}$-edge $\{y_{i}, y_{j}\}$ is missing in $\mathcal{H}$), and complete-bipartite minus a perfect matching between $Y$ and $Y^{*}$. Formally,

$$\mathcal{D}(\mathcal{H}) = \{\{x_{i}, x_{j}\} : 1 \leq i < j \leq m\} \cup \{\{x_{i}, y_{j}\} : x_{i} \in X \land y_{j} \in Y\} \cup \{\{w, z\} : z \in X \cup Y\} \cup \{\{y_{i}, y_{j}\} : y_{i}, y_{j} \in Y \land \exists F_{k} \in \mathcal{F} : (\{v_{i}, v_{j}\} \subset F_{k}\} \cup \{\{w^{*}, y_{i}^{*}\} : y_{i}^{*} \in Y^{*}\} \cup \{\{y_{i}, y_{j}^{*}\} : y_{i} \in Y \land y_{j}^{*} \in Y^{*} \land i \neq j\}.$$

The $\mathcal{G}$-edges of $\mathcal{H}$ are of two types:

- $\mathcal{G}$-Type-1 edges: $C_{i}^{1} = \{w^{*}, y_{i}^{*}\}$, for every $1 \leq i \leq n$.
- $\mathcal{G}$-Type-2 edges: $C_{i}^{2} = \{w, w^{*}, y_{i}\} \cup \{x_{j} \in X : v_{j} \notin F_{j}\}$ for every $1 \leq i \leq n$, where $x_{j}$ corresponds to the edge $F_{j}$ of $\mathcal{F}$ and $y_{i}$ is the copy of the vertex $v_{i}$ of $\mathcal{F}$. So, $y_{i}$ and $x_{j}$ belong to a common $\mathcal{G}$-edge in $\mathcal{H}$ if and only if the $i$th vertex is not an element of the $j$th edge in $\mathcal{F}$.

$$\mathcal{G}(\mathcal{H}) = \{C_{i}^{1}, C_{i}^{2} : 1 \leq i \leq n\}.$$

It is clear that the construction of $\mathcal{H}$ from $\mathcal{F}$ can be carried out in polynomial time.

**Unique coloring.** First, let us observe that the M-graph $\mathcal{H}$ is colorable. Obviously, the partition where $\{y_{1}, y_{1}^{*}\}, \ldots, \{y_{n}, y_{n}^{*}\}$, and $\{w, w^{*}\}$ are 2-element classes and all the other classes are singletons, is a proper coloring of $\mathcal{H}$. It will be shown that for any $\mathcal{F}$ satisfying the condition of Lemma 2, the M-graph $\mathcal{H}$ constructed from $\mathcal{F}$ in the way described above, is the only suitable coloring of $\mathcal{H}$; that is, $\mathcal{H}$ is UC in any case. Moreover, we shall prove that $\mathcal{H}$ is UC-orderable if and only if $\mathcal{F}$ has a SBS.

Let us consider an edge-crossing set $B$ in $\mathcal{F}$, which contains at least two vertices. (Due to the assumptions of Lemma 2, such an edge-crossing set exists.) Passing on to the M-graph $\mathcal{H}$, let $B'$ be the subset of $Y$ with the elements corresponding to the vertices of $B$. Since $B$ is edge-crossing, no edge of $\mathcal{F}$ can contain more than one vertex of $B$; hence, any two elements of $B'$ are surely joined by a $\mathcal{D}$-edge in $\mathcal{H}$.

First, we prove that $\mathcal{H}$ is uniquely colorable in any construction, and then search for a UC-ordering of $\mathcal{H}$ if the above $B$ is a SBS.

**Step 1:** $X \cup \{w\} \cup B'$ is complete in $\mathcal{D}$-edges, so its vertices all have different colors. Their coloring is unique in any order.

**Step 2:** To color $w^{*}$, let us consider all the $\mathcal{G}$-Type-2 edges $C_{i}^{2}$ belonging to the members of $B'$. All elements of their union, except $w^{*}$, have been colored in Step 1 with mutually distinct colors. Therefore, these $\mathcal{G}$-edges can be colored properly only if $w^{*}$ gets the color of some common vertex. This cannot be from $Y$ because there are at least two elements in $B'$, and hence none of the corresponding vertices of $Y$ belong to the intersection of their $\mathcal{G}$-Type-2 edges. Suppose that $w^{*}$ gets the color of some $x_{k} \in \bigcap_{y_{i} \in B'} C_{i}^{2}$. Choosing a vertex $v_{\ell} \in F_{k}$, we have $x_{k} \notin C_{\ell}^{2}$, thus all vertices of $C_{\ell}^{2}$ would have different colors, what is forbidden. Thereby in every $\mathcal{H}$ constructed from any $\mathcal{F}$, the color of $w$ and $w^{*}$ must be the same.

**Step 3:** Let $B^{*}$ be the subset of $Y^{*}$ corresponding to the elements of $B$. For every $y_{i}^{*} \in B^{*}$ we have edges $\mathcal{G}$-Type-1: $C_{i}^{1} = \{y_{i}, y_{i}^{*}, w^{*}\}$, $\mathcal{D}$-edge: $\{y_{i}^{*}, w^{*}\}$.

Since $w^{*}$ has got a color different from $y_{i}$ in Step 2, the only chance to color properly the edge $C_{i}^{1}$ is that $y_{i}^{*}$ gets a common color with $y_{i}$.

**Step 4:** In this step we color the vertices of $Y \setminus B'$ and $Y^{*} \setminus B^{*}$. First, we take a $y_{1} \in Y \setminus B'$. It has to be colored differently from $w$, from the elements of $X$, and from the colored elements of $Y^{*}$ (which are colored like the corresponding elements
of $Y$). Therefore, we can assign only a totally new color to $y_1$. Then, looking at the influencing edge $C_1$, the vertex $y_1^*$ must be colored like $y_l$. Thus, taking the pairs $y_l, y_1^*$ one by one, each of them turns out to be monochromatic in a color different from all preceding colors.

**UC-order from SBS.** Suppose that $B$ is a SBS in $\mathcal{F}$ with at least two elements; say, $B = \{v_1, v_2, \ldots, v_k\}$. Then we can construct the following UC-order of $\mathcal{H}$:

$$x_1, \ldots, x_m, y_1, \ldots, y_k, w, w^*, y_1^*, \ldots, y_k^*, y_{k+1}, y_{k+1}^*, \ldots, y_n, y_n^*.$$ 

Until $w$, we obtain a heterochromatic color-order, by Step 1. The crucial point is that now Step 2 applies to $w^*$, even if we disregard the later vertices of $\mathcal{H}$, because $B$ meets all edges of $\mathcal{F}$—implying that the intersection of the corresponding $\mathcal{C}$-edges is empty inside $X$—and therefore $w^*$ cannot get any color from $X$. Thus, $w$ and $w^*$ must have a common color, and after that the vertex-order remains UC, by Steps 3 and 4.

This argument already indicates the substantial difference between an edge-crossing set and a SBS with respect to UC-orders. At the moment when both $w$ and $w^*$ are present in the subsequence (whichever comes later), it should be verified that they must get a common color. For this purpose, an edge-crossing set is insufficient if it fails to be a SBS.

**SBS from UC-order.** We have already seen that every $\mathcal{H}$ obtained by the $\mathcal{F} \rightarrow \mathcal{H}$ construction is uniquely colorable, with well-defined monochromatic pairs of vertices. Suppose that $\mathcal{H}$ is not only UC but also admits a UC-order. We concentrate on the subsequence where the very first monochromatic pair appears. By what has been said, only the following possibilities may occur:

1. $w$ repeats the color of $w^*$;
2. some $y_i$ repeats the color of $y_i^*$;
3. some $y_i^*$ repeats the color of $y_i$;
4. $w^*$ repeats the color of $w$.

We are going to prove that the first three of these cannot be the case in a UC-order; and if the fourth one does, then it also results in a SBS of $\mathcal{F}$. Note that any UC-order (if it exists) has to satisfy the following requirement:

$(\ast)$ The vertices preceding the occurrence of the first repeated color must induce a complete $D$-graph. In particular, up to that point there are no colored pairs $(y_i, y_i^*)$, and at most one $y_i^*$ may occur.

1. If the first repeated color is at $w$, then $w^*$ has been colored before coloring $w$. Since all $D$-edges incident with $w^*$ have their other endpoint in $Y^*$, $(\ast)$ implies that $w$ is preceded either by $w^*$ alone or by $w^*$ and just one $y_i^*$. Hence, the subsequence ending with $w$ does not induce any $\mathcal{C}$-edges, therefore nothing can force $w$ to get a common color with $w^*$.
2. The unique coloring of $y_i$ with the first repeated color requires some $\mathcal{C}$-edge containing $y_i$. Since every $\mathcal{C}$-edge involves $w^*$, this case can occur only if $w^*$ and $y_i^*$ have been colored before $y_i$. But with the presence of $w^*$ we obtain the same situation as in Case 1: Only $y_i^*$ and $w^*$ are colored before $y_i$, so $y_i$ may have a common color with $w^*$ instead of $y_i^*$.
3. Assuming, that the first color repetition is at $y_i^*$, the vertices $y_i$ and $w^*$ of the influencing $\mathcal{C}$-edge must be previously colored. But this is in contradiction to the requirement $(\ast)$ since $y_i$ and $w^*$ are not joined by a $D$-edge.
4. This is the only possible case: we have the first repeated color at $w^*$. Then $w$ already appeared, and $(\ast)$ implies that the vertices colored before $w^*$ induce a complete $D$-subgraph inside $X \cup Y \cup \{w\}$.

Let $B'$ be the set of elements in $Y$ that were colored before $w^*$. As they are joined by $D$-edges in $\mathcal{H}$, no $F_j \in \mathcal{F}$ contains more than one of them; that is, the corresponding $B$ in $\mathcal{F}$ is an edge-crossing set. Furthermore, the color of $w^*$ is uniquely determined only if $|B'| \geq 2$ and the $\mathcal{C}$-Type-2 edges belonging to the $y_j \in B'$ do not contain any common element $x_j \in X$. (Otherwise at this point of the sequence $w^*$ could get the color of $w$ or $x_j$ as well.) So, under the assumption that we have a UC-order, for every $x_j$ there exists a $y_j \in B'$ such that $x_j$ does not belong to the $\mathcal{C}$-Type-2 edge of $y_j$. Passing over to the hypergraph $\mathcal{F}$, for every edge $F_i$ there is a vertex $v_j \in B$, which is contained in $F_i$. Thus, the edge-crossing set $B$ meets all edges of $\mathcal{F}$, so that it is a SBS. \[ \square \]
3. Uniquely UC-orderable hypergraphs

In this section we will investigate the structure of mixed hypergraphs that have exactly one UC-order, disregarding the transposition of the first two vertices. It may be noted in general that if an edge is a subset of another edge of the same type (both are $\mathcal{E}$-edges or both are $\mathcal{D}$-edges), then the larger edge is redundant with respect to coloring, because it does not impose any new condition: any proper coloring for the smaller edge properly colors the larger one, too.

If the number $n$ of vertices is at most 2, then the properties UC, UC-orderable, and UUC are equivalent. Hence, in order to avoid the few trivial exceptions, we shall assume $n \geq 3$ throughout this section. The first assertion is immediate by definition.

Proposition 2. If $x_1, x_2, \ldots, x_n$ is the UC-order of a UUC-graph $\mathcal{H}$, then the subhypergraph of $\mathcal{H}$ induced by $\{x_j : 1 \leq j \leq i\}$ is UUC for every $i \leq n$.

Proposition 3. If $\mathcal{H}$ is a UUC-graph with UC-order $x_1, \ldots, x_n$ on $n \geq 3$ vertices, then $\{x_1, x_2\}$ is a $\mathcal{E}$-edge and $\{x_1, x_2, x_3\}$ is a $\mathcal{D}$-edge. So, the subhypergraph induced by the first three vertices of the UC-order is the same in every UUC-graph without redundant edges.

Proof. By definition, $x_1, x_2, x_3$ is a UC-order if and only if the subhypergraphs induced by $\{x_1, x_2\}$ and by $\{x_1, x_2, x_3\}$ are uniquely colorable. Because of the uniqueness of this UC-order neither $\{x_1, x_3\}$ nor $\{x_2, x_3\}$ can be UC. Consequently, there exists an edge $\{x_1, x_2\}$, but no other 2-element edge inside $\{x_1, x_2, x_3\}$.

Color $x_3$, we need an influencing edge for it. If $\{x_1, x_2\} \in \mathcal{D}$, the influencing edge could be $\{x_1, x_2, x_3\} \in \mathcal{E}$, but this—without a 2-element $\mathcal{D}$-edge containing $x_3$—does not determine the color of $x_3$ uniquely. In the other case: If $\{x_1, x_2\} \in \mathcal{D}$, the influencing edge for $x_3$ surely is the $\mathcal{D}$-edge $\{x_1, x_2, x_3\}$. This is the only structure without redundant edges that yields a UUC subhypergraph. Note, that permitting the presence of redundant edges, this M-graph can be supplemented with the $\mathcal{E}$-edge containing all the three vertices. \(\square\)

We distinguish three types of vertices in a UC-order, depending on their colors:

- $x_i$ has a continuing color if it is the same as the color of the preceding vertex $x_{i-1}$.
- $x_i$ has a returning color if it is not continuing but this color has already occurred at some $x_j$ ($j < i - 1$).
- $x_i$ has a new color if this color has not occurred up to this point, at any $x_j$ with $j < i$.

Accordingly, a vertex will be called continuing/returning/new if so is its color.

Proposition 4. If $n \geq 3$, then there are no two consecutive new vertices in a UC-order.

Proof. Due to Proposition 3, $x_2$ is a continuing vertex, so that the assertion is valid within $\{x_1, x_2, x_3\}$. Assuming that both $x_i$ and $x_{i+1}$ have new colors, for some $i \geq 3$, their positions could be switched, because of the following facts.

There are $\mathcal{D}$-edges guaranteeing that the color of $x_{i+1}$ is different from each color used up to $x_{i-1}$. These influencing edges cannot contain $x_i$. Consequently, $x_{i+1}$ can be uniquely colored right after $x_{i-1}$, and then $\{x_1, \ldots, x_{i-1}, x_{i+1}, x_i\}$ also determines a UC-order of the induced subhypergraph. This cannot occur if $\mathcal{H}$ is a UUC-graph. \(\square\)

For a given UUC-graph $\mathcal{H}$, we consider its unique UC-order $x_1, x_2, \ldots, x_n$. The coloring of $\mathcal{H}$ is the function $c$ that assigns a positive integer to each $x_i$: $c(x_i) = c_i$. In order to associate precisely one sequence $c_1, \ldots, c_n$ of colors with the coloring $c$, we shall assume that the new colors $1, 2, \ldots$ appear in increasing order, without skipping any intermediate values. This $c_1, c_2, \ldots, c_n$ will be called the color-order of $\mathcal{H}$. If $\mathcal{H}$ is UUC, then it has one well-defined color-order determined by its unique (vertex) UC-order.

Let us summarize the necessary conditions obtained for the color-orders of UUC-graphs on $n \geq 3$ vertices.

C0: (By assumption). The natural numbers appear in increasing order and without gaps; i.e., $c_1 = 1$, and $1 \leq c_i \leq \max_{k<i} \{c_k\} + 1$ holds for each $2 \leq i \leq n$.

C1: (According to Proposition 3). $c_1 = c_2 = 1$ and $c_3 = 2$.

C2: (According to Proposition 4). If $c_i \neq c_{i+1}$, then at least one of them has occurred in the subsequence $c_1, c_2, \ldots, c_{i-1}$.
The main result of this section claims that these conditions are sufficient, too. In Section 4, we shall compare the construction given in the proof with all UUC-graphs of the same color sequence, and prove that it is minimal from several aspects. (The hypergraph $H$ constructed here will be denoted by $H^*$ in the last section.)

**Theorem 2.** A sequence $c_1, c_2, \ldots, c_n$ of positive integers is the color-order of some UUC-graph on $n \geq 3$ vertices if and only if it satisfies requirements $C_0, C_1,$ and $C_2$.

**Proof.** Necessity has already been shown. To prove sufficiency, we construct a suitable mixed hypergraph $H$ for any given color sequence satisfying $C_0, C_1, C_2$.

For an M-graph $H$ and the fixed vertex-order $x_1, \ldots, x_n$, we shall denote by $H_i$ its subhypergraph induced by $\{x_j : 1 \leq j \leq i\}$ ($i = 1, 2, \ldots, n$). Whenever $H$ is UUC, the colors and the edges in its $H_3$ have been determined above:

- **Colors:** $c(x_1) = c(x_2) = 1$; $c(x_3) = 2$.
- **Edges:** $\{x_1, x_2\} \in \mathcal{C}$; $\{x_1, x_2, x_3\} \in \mathcal{D}$.

For each $i \geq 3$ we extend $H_i$ with the vertex $x_{i+1}$ and with the following new edges:

- If $c_{i+1}$ is a continuing color, the only new edge is: $\{x_i, x_{i+1}\} \in \mathcal{C}$ (Fig. 2).
- If $c_{i+1}$ is a returning color and $x_j$ is the last vertex before $x_{i+1}$ with the same color as $x_{i+1}$ (i.e., $c_i = c_j, j < i$, and $c_k \neq c_{i+1}$ holds for every $j < k < i + 1$), the new edges are: $\{x_j, x_i, x_{i+1}\} \in \mathcal{C}$ and $\{x_i, x_{i+1}\} \in \mathcal{D}$ (Fig. 3).
- If $c_{i+1}$ is a new color, we need $\mathcal{D}$-edges to distinguish the color of $x_{i+1}$ from the previously used colors. For every color $d$ ($d < c_{i+1}$) let $x_d/i+1$ denote the latest vertex colored with $d$ before $x_{i+1}$. The new edges are: $\{x_d/i+1, x_{i+1}\} \in \mathcal{D}$, for $d = 1, \ldots, c_{i+1} - 1$ (Fig. 4).

It is clear that $x_1, \ldots, x_n$ is a UC-order of $H$ with the given color-order $c_1, \ldots, c_n$, because the newly inserted edges and condition $C_0$ force $x_{i+1}$ to get color $c_{i+1}$ in each step.
It remains to be proved that the constructed $\mathcal{H}$ has this unique UC-order only. We prove by induction on $i$ that every subhypergraph $\mathcal{H}_i$ (induced by $\{x_1, x_2, \ldots, x_i\}$) is UUC.

It was shown that $\mathcal{H}_3$ is UUC. Suppose that $\mathcal{H}_i$ is UUC, with $x_1, x_2, \ldots, x_i$ as its only UC-order. The proof that the same holds true for $\mathcal{H}_{i+1}$ consists of two parts.

1. Deleting $x_{i+1}$ from any UC-order of $\mathcal{H}_{i+1}$, the (unique) UC-order of $\mathcal{H}_i$ is obtained.
2. If the deletion of $x_{i+1}$ from an UC-order of $\mathcal{H}_{i+1}$ results in the vertex-order $x_1, \ldots, x_i$ of $\mathcal{H}_i$, then $x_{i+1}$ must be in the last position in the UC-order of $\mathcal{H}_{i+1}$.

The combination of these two assertions completes the proof of the theorem by induction.

**Remark.** A vertex-order $y_1, y_2, \ldots, y_n$ is non-UC if and only if there exist two vertices $y_k, y_j$ ($k < j$), that the subhypergraph induced by $\{y_1, y_2, \ldots, y_j\}$ has two proper colorings $c', c''$ such that $c'(y_j) = c''(y_k)$ and $c''(y_j) \neq c''(y_k)$.

**Proof of 1.** Assume that the original vertex-order $x_1, x_2, \ldots, x_i$ of $\mathcal{H}_i$ is mixed and we have a different sequence, say $y_1, y_2, \ldots, y_i$. It is not a UC-order, so there is a smallest $j$ ($1 < j \leq i$), for which the subhypergraph $\mathcal{H}_j'$ induced by $Y_j = \{y_1, y_2, \ldots, y_j\}$ is not uniquely colorable. Thus, there exists $y_k$ ($k < j$), a new color. Therefore, there exist two different proper colorings $c^*$ of $\mathcal{H}_j'$ beside the one determined by $c$, such that $y_k$ and $y_j$ have the same color in one of $c$ and $c^*$, and different colors in the other.

Suppose for a contradiction that the deletion of $x_{i+1}$ from a UC-order of $\mathcal{H}_{i+1}$ yields the above sequence $y_1, y_2, \ldots, y_i$. Then $\mathcal{H}_j'$ with vertices $y_1, y_2, \ldots, y_j$ is not UC, so it cannot be the starting sequence of the UC-order. Therefore, $x_{i+1}$ would have to appear earlier than $y_j$. We investigate $\mathcal{H}_j' \cup \{x_{i+1}\}$ and prove that it cannot be UC, either.

**Case 1:** The color of $x_{i+1}$ is a continuing one in the original sequence. If $x_{i+1} \in Y_j$, then let $c(x_{i+1}) = c(x_i), c^*(x_{i+1}) = c^*(x_i)$. These are suitable colorings, and the coloring of $x_{i+1}$ has no influence on the colors of $y_1, y_2, \ldots, y_j$. Thus, we have two different colorings, so $\mathcal{H}_j' \cup \{x_{i+1}\}$ is not UC. On the other hand, if $x_{i+1} \notin Y_j$, there is an edge containing $x_{i+1}$ in the subhypergraph so we can choose $c(x_{i+1})$ and $c^*(x_{i+1})$ as a totally new color. Therefore the extended $c$ and $c^*$ remain different colorings and $\mathcal{H}_j' \cup \{x_{i+1}\}$ is not UC.

**Case 2:** The color of $x_{i+1}$ is originally new. We can assign a totally new color to $x_{i+1}$ in $c^*$ too. It makes no influence on the coloring of $y_1, y_2, \ldots, y_j$, thus, we have got two different colorings for $\mathcal{H}_j' \cup \{x_{i+1}\}$.

**Case 3:** $x_{i+1}$ has a returning color in the original ordering. There are at most two influencing edges for $x_{i+1}$, namely $\{x_i, x_{i+1}\} \in \mathcal{C}$ and $\{x_i, x_{i+1}\} \in \mathcal{D}$. By the construction, the colors of $x_i$ and $x_{i+1}$ are different in c. If $x_{i+1} \notin Y_j$ or $x_i \notin Y_j$, then the $\mathcal{C}$-edge is not effective, so we let $x_{i+1}$ have a totally new color, and then $y_j$ still can have two different colors. Consequently, $\mathcal{H}_j' \cup \{x_{i+1}\}$ is not UC.

If $x_i$ and $x_{i+1}$ both are in $Y_j$ and their colors are different not only by $c$ but also by $c^*$, then $x_{i+1}$ can get the same color as $x_i$, and the subhypergraph does not become UC.

The only non-trivial case is when $x_i$ and $x_{i+1}$ have different colors in $c$ and the same color in $c^*$. In the former, $x_{i+1}$ still can get $c(x_i)$, while in the latter it can get a totally new color. In this way we have two different proper colorings, therefore $\mathcal{H}_j' \cup \{x_{i+1}\}$ cannot be UC.

**Proof of 2.** We assume, from now on, that $\mathcal{H}_i$ is in its original order, $x_1, \ldots, x_i$, and $x_{i+1}$ is inserted in a way that the sequence remains a UC-order. We need to prove that $x_{i+1}$ is the last one.

**Case 1:** $x_{i+1}$ has a continuing color. The only influencing edge for $x_{i+1}$ is $\{x_i, x_{i+1}\} \in \mathcal{C}$. If $x_{i+1}$ occurs in the sequence earlier than $x_i$, then in $\mathcal{H}_{i+1} - x_i$ nothing prevents $x_{i+1}$ from getting the color $c(x_i)$, or a color different from $c(x_i)$. Thus, the vertex-order is not UC unless $x_{i+1}$ is set at the end. As it follows, $\mathcal{H}_{i+1}$ has only one UC-ordering, that means it is a UUC.

**Case 2:** $x_{i+1}$ has a returning color. Each edge containing $x_{i+1}$ also contains $x_i$. Similarly to the previous case, it will be a UC-order only if $x_{i+1}$ is set at the end of the sequence.

**Case 3:** $x_{i+1}$ has a new color. There is an edge $\{x_i; x_{i+1}\} \in \mathcal{D}$, but we have no other $\mathcal{D}$-edge containing $x_{i+1}$ and a vertex colored with $c(x_i)$. Let $X_j$ denote the set of vertices $x_j$ with color $c(x_i)$ but smaller subscript, $j < i$. According to condition $C_2$, $c(x_i)$ is not new, so that $X_i \neq \emptyset$. Since there is no edge containing both $x_{i+1}$ and any element of $X_i$,
it is our free choice to put \( x_{i+1} \) into the color class of \( X_i \) or assign it a distinct color, in the subhypergraph \( \mathcal{H}_{i+1} - x_i \). Thus, the vertex-order is UC only if \( x_{i+1} \) is set at the end, after \( x_i \). □

4. Extremal properties of the construction

In the proof of Theorem 2 we constructed only one UUC-graph for each feasible color-order. But in fact there are a lot of UUC-graphs with different structures. Here we describe two types of alternatives to the constructed M-graph.

Example 1. When the UUC-subgraph \( \mathcal{H}_i \) is supplemented with the vertex \( x_{i+1} \) having a returning color, the \( \mathcal{G} \)-edge \( \{x_j, x_i, x_{i+1}\} \) can be replaced with any set that contains \( x_j, x_{i+1} \) and some vertex \( x_k \) \((k < i + 1)\) having the same color as \( x_i \). The \( \mathcal{D} \)-edge \( \{x_i, x_{i+1}\} \) is unchanged. Obviously, these M-graphs are UUC, too. It may occur, that there exists an \( x_k \) for which \( c(x_k) = c(x_i) \) and \( j < k < i \) holds. For the next comparison of alternatives we assume this case and take \( \{x_j, x_k, x_{i+1}\} \) as the \( \mathcal{G} \)-edge. The obtained UUC will be referred as Example 1.

Example 2. If \( x_{i+1} \) has a new color, the \( \mathcal{D} \)-edge \( \{x_{d/i+1}, x_{i+1}\} \) can be replaced with any larger set that contains both of the original elements and some vertices (from \( \mathcal{H}_i \)) colored \( d \). Moreover, if \( x_{d/i+1} \) and \( x_{i+1} \) are not consecutive, then \( x_{d/i+1} \) can be replaced with any previous vertex having the same color, and \( \mathcal{H}_{i+1} \) still remains UUC.

Now, we list some types of measure for comparison, under which the construction of the previous section is minimal. They show that this construction results in the simplest structure for UUC in several aspects. Assuming that the M-graph \( \mathcal{H} \) has the unique UC-order \( x_1, x_2, \ldots, x_n \), we introduce the following concepts:

- The number of edges: \( N(\mathcal{H}) = |\mathcal{H}| = |\mathcal{G}| + |\mathcal{D}| = \sum_{H \in \mathcal{H}} 1 \).
- Edge-size sum: \( S(\mathcal{H}) = \sum_{H \in \mathcal{H}} |H| \).
- Edge-diameter sum: \( D(\mathcal{H}) = \sum_{H \in \mathcal{H}} \max\{j - k : x_j, x_k \in H\} \).
  That is, diameter of an edge \( H \) is the difference \( j - k \) where \( j \) is the largest and \( k \) is the smallest index occurring at the vertices of \( H \).
- Total edge-distance sum: \( Td(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{x_j \in H} \left(\max\{j : x_j \in H\} - 1\right) \).
  In each edge \( H \) the distances between the last vertex and the other ones are summed.
- Reverse-index sum: \( R(\mathcal{H}) = \sum_{H \in \mathcal{H}} \sum_{x_i \in H} (n + 1 - l) \).
  This means that in the unique UC-order \( x_1, x_2, \ldots, x_n \), new descending indices from \( n \) to \( 1 \) are introduced. So, each vertex \( x_i \) has reverse-index \( (n + 1 - l) \) and every edge is represented by the sum of reverse-indices assigned to its vertices.

Let us note that if the vertices of a mixed hypergraph \( \mathcal{H} \) are colored in the order \( x_1, x_2, \ldots, x_n \), then the reverse sequence \( (y_1, \ldots, y_n) = (x_n, \ldots, x_1) \) is termed the elimination order of \( \mathcal{H} \) (cf. [9]). That is, in \( R(\mathcal{H}) \) each edge is measured by the sum of the indices of its vertices in the elimination order of \( \mathcal{H} \).

We study all the UUC-graphs belonging to a given color-order. It will be proved, that the UUC-graph \( \mathcal{H}^* \) constructed in the previous section is minimal under each of the five measures.

**Proposition 5.** For any given color-order (according to conditions \( C_0, C_1, C_2 \)) the constructed \( \mathcal{H}^* \) is a minimal UUC-graph concerning the measures \( N \), \( S \) and \( D \). Moreover, \( \mathcal{H}^* \) is the only minimal UUC-graph under \( Td \) and \( R \).

Each of the measures is defined as the sum of the corresponding measures related to the edges of the UUC-graph. Now, we partition the set of edges according to the last vertices in them.

Let \( X_i(\mathcal{H}) \) be the subset of \( \mathcal{H} \) containing those edges, in which \( x_i \) is the vertex with the largest subscript. Obviously, each edge of \( \mathcal{H} \) belongs to exactly one set \( X_i(\mathcal{H}) \). Note, that two different UUC-graphs might have the same \( X_i \) for some \( i \) but not for every \( i \) \((1 \leq i \leq n)\).

Throughout this section, a given color-order \( c_1, c_2, \ldots, c_n \) will be considered, that satisfies the conditions \( C_0, C_1, C_2 \). We assume a UUC-graph \( \mathcal{H} \) and the constructed \( \mathcal{H}^* \), that belong to the above color-order. For short, we write \( X_i \) instead of \( X_i(\mathcal{H}) \); and \( X_i^* \) instead of \( X_i(\mathcal{H}^*) \). The measures, what we obtain by summing the values over the edges in \( X_i \), will be denoted by \( N_i, S_i, D_i, Td_i \) and \( R_i \).

The proof of Proposition 5 will be traced back to the following lemma.
Lemma 3. Let \(3 < i \leq n\). Considering all the UUC-graphs belonging to a given color-order \(c_1, \ldots, c_n\) (according to \(C_0, C_1, C_2\)) we have:

1. If \(c_i\) is a continuing color, \(X_i^*\) is the only minimal edge set under \(N_i, S_i, D_i, Td_i, R_i\).
2. If \(c_i\) is a returning color, \(X_i^*\) is minimal under \(N_i, S_i\) and \(D_i\), and this is the only minimal edge set concerning \(Td_i\) and \(R_i\).
3. If \(c_i\) is a new color, \(X_i^*\) is minimal under \(N_i\) and \(S_i\), and this is the only minimal edge set under \(D_i, Td_i\) and \(R_i\).

Proof of Lemma 3. If \(H\) is a UUC-graph, it has to satisfy the following requirement:

\((\ast)\) Because of the uniqueness of the UC-order, for each vertex \(x_i\) \((3 \leq i \leq n)\) there is an influencing edge containing both of \(x_i\) and \(x_{i-1}\). Otherwise the color of \(x_i\) could be determined before \(x_{i-1}\) and that yields another UC-order.

Proof of 1. The unique coloring of a continuing \(x_i\) requires the existence of at least one influencing \(\mathcal{H}\)-edge for \(x_i\). This edge certainly contains \(x_i\) and a previous vertex with the same color \(c_j\). There might be other vertices in it, these have to be colored differently from \(x_i\) and from each other. But in the latter case the unique colorability of \(x_i\) demands further edges influencing for \(x_i\). As a consequence of these facts and \((\ast)\), if \(X_i\) has only one edge, it has to be the 2-element \(\mathcal{H}\)-edge \(\{x_{i-1}, x_i\}\). Therefore, \(N_i \geq 1, S_i \geq 2, D_i \geq 1, Td_i \geq 1\); and \(R_i \geq (n + 1 - i) + (n + 1 - i + 1) = 2n - 2i + 3\). Thus, \(X_i^*\) is the only minimal edge-set under each measure.

Proof of 2. Assume, that \(x_i\) is returning. It must have an influencing \(\mathcal{H}\)-edge forcing the color of \(x_i\) to be the same as the color of a previous \(x_j\) \((j < i - 1)\). If this edge consists only of \(x_i\) and \(x_j\) then the actual vertex could be colored right after \(x_j\), what contradicts the uniqueness of the UC-order. Thus, the influencing \(\mathcal{H}\)-edge surely has a third vertex with different color, and there exists a \(\mathcal{D}\)-edge distinguishing the color of \(x_i\) and of the third vertex. Hence, in every UUC-graph: \(N_i \geq 2, S_i \geq 3 + 2\).

We get the smallest diameters if the \(\mathcal{H}\)-edge contains the last vertex with color \(c_i\) before \(x_i\) (its index is denoted by \(c_i / i\)) and a third vertex—between the previous two ones—having the same color as \(x_{i-1}\). So the diameter of the \(\mathcal{H}\)-edge is \((i - c_i / i)\) and the \(\mathcal{D}\)-edge can have the smallest diameter 1. One can see, that the constructed \(H^*\) has these minimum values regarding \(N_i, S_i\) and \(D_i\). Taking Example 1 into consideration, obviously, \(X_i^*\) is not the unique structure that is minimal under these measures.

To obtain the smallest value of \(Td_i\) and \(R_i\), we have to choose \(x_{i-1}\) as the third vertex of the \(\mathcal{H}\)-edge. In this way, the edges in the minimal \(X_i\) are entirely determined, therefore \(X_i^*\) is the only minimal edge-set under \(Td_i\) and \(R_i\).

Proof of 3. Forcing \(x_i\) to have a new color, it needs \(c_i - 1\) influencing \(\mathcal{D}\)-edges, one for each preceding color. For every color \(d < c_i\) there must be a \(\mathcal{D}\)-edge containing \(x_i\) and at least one vertex colored with \(d\). Such an edge has minimum two elements; the minimum diameter and total distance is \(i - (d/i)\) (where \(x_{d/i}\) is the last vertex with color \(d\) before \(x_i\)); and its reverse-index sum is at least \((n + 1 - i) + (n + 1 - (d/i))\). The edge set \(X_i^*\) has these minimum values. There exist other structures \(X_i\) being minimal under \(N_i\) and \(S_i\) (see Example 2) but there is no other minimal set concerning \(D_i, Td_i\) or \(R_i\). □
Finally, we study the connection between the last two measures, under which $\mathcal{H}^*$ has been proved to be the unique minimal UUC-graph. Consider an edge $H$ in a UUC-graph $\mathcal{H}$ and compare its total distance and reverse-index sum. The largest subscript occurring in $H$ is denoted by $j$.

$$R(H) = \sum_{x_l \in H} (n + 1 - l) = \sum_{x_l \in H} ((n + 1 - j) + (j - l))$$

$$= |H|(n + 1 - j) + \sum_{x_l \in H} (j - l) = |H|(n + 1 - j) + Td(H).$$

Collecting all the edges ending at $x_j$ (these are the elements of $X_j$) and summing their sizes we get $S_j$. With this notation, first we obtain a formula for $R_j$, and then summing them we get the value of $R$ concerning the entire UUC:

$$R_j(\mathcal{H}) = S_j(\mathcal{H}) \cdot (n + 1 - j) + Td_j(\mathcal{H}),$$

$$R(\mathcal{H}) = \sum_{j=1}^{n} S_j(\mathcal{H}) \cdot (n + 1 - j) + Td(\mathcal{H}).$$

Alternatively, using the elimination order $(y_1, \ldots, y_n) = (x_n, \ldots, x_1)$, we have

$$R(\mathcal{H}) = \sum_{i=1}^{n} \ i \cdot d(y_i),$$

where $d(y_i)$ denotes the degree of vertex $y_i$ (that is, the number of edges incident with it).

References


