LIMITS AND INEQUALITIES ASSOCIATED WITH THE EULER-MASCHERONI CONSTANT

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Abstract. (i) We present several limits associated with the Euler-Mascheroni constant. (ii) Let \( \gamma = 0.577215 \ldots \) be the Euler-Mascheroni constant, and let \( T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln (n + \frac{1}{2}) \) and \( P_n = \sum_{k=1}^{n} \frac{2}{2k-1} - \ln(4n) \). We determine the best possible constants \( \alpha, \beta, a \) and \( b \) such that the inequalities

\[
\frac{1}{48(n + \alpha)} \leq \gamma - T_n < \frac{1}{48(n + \beta)}
\]

and

\[
\frac{1}{24(n + a)} \leq P_n - \gamma < \frac{1}{24(n + b)}
\]

are valid for all integers \( n \geq 1 \).

1. Introduction and preliminaries

The Euler-Mascheroni constant \( \gamma = 0.577215664 \ldots \) is defined as the limit of the sequence

\[
D_n = H_n - \ln n,
\]

where \( H_n \) denotes the \( n \)th harmonic number, defined for \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \) by

\[
H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]

The first aim of this paper is to present several limits associated with the Euler-Mascheroni constant.

Theorem 1. Let

\[
e_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.
\]
Then

\[
\lim_{n \to \infty} \left( \frac{e_{H_n}}{n} \right) = e^\gamma, \quad (2)
\]

\[
\lim_{n \to \infty} \left( e_{H_{n+1}} - e_{H_n} \right) = e^\gamma \quad (3)
\]

and

\[
\lim_{n \to \infty} \left( 2n \left( e_{H_n} - e_{H_{n+1}} + e^\gamma \right) \right) = e^\gamma. \quad (4)
\]

Several bounds for \(D_n - \gamma\) have been given in the literature [3, 5, 14, 15, 16, 17, 20]. For example, the following bounds for \(D_n - \gamma\) was established in [14, 20]:

\[
\frac{1}{2(n + 1)} < D_n - \gamma < \frac{1}{2n} \quad (n \in \mathbb{N}).
\]

The convergence of the sequence \(D_n\) to \(\gamma\) is very slow. Some quicker approximations to the Euler-Mascheroni constant were established in [6, 7, 8, 9, 10, 11, 12, 13, 18, 19]. For example, DeTemple [10] studied in 1993 the sequence

\[
R_n = \sum_{k=1}^{n} \frac{1}{k} \ln \left( n + \frac{1}{2} \right),
\]

and proved

\[
\frac{1}{24(n + 1)^2} < R_n - \gamma < \frac{1}{24n^2}. \quad (5)
\]

Recently, Chen [7] obtained the following sharp form of the inequality (5): For all integers \(n \geq 1\), then

\[
\frac{1}{24(n + a)^2} \leq R_n - \gamma < \frac{1}{24(n + b)^2} \quad (6)
\]

with the best possible constants

\[
a = \frac{1}{\sqrt{24[-\gamma + 1 - \ln(3/2)]}} - 1 = 0.55106 \ldots \quad \text{and} \quad b = \frac{1}{2}.
\]

In 1997, Negoi [13] proved that the sequence

\[
T_n = \sum_{k=1}^{n} \frac{1}{k} \ln \left( n + \frac{1}{2} + \frac{1}{24n} \right) \quad (7)
\]

is strictly increasing and convergent to \(\gamma\). Moreover, the author proved that

\[
\frac{1}{48(n + 1)^3} < \gamma - T_n < \frac{1}{48n^3}. \quad (8)
\]

In view of the inequality (8) it is natural to ask: What is the smallest number \(\alpha\) and what is the largest number \(\beta\) such that the inequality

\[
\frac{1}{48(n + \alpha)^3} \leq \gamma - T_n \leq \frac{1}{48(n + \beta)^3}
\]
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holds for all integers \( n \geq 1 \). It is the second aim of this paper to answer this question.

**Theorem 2.** For all integers \( n \geq 1 \), let the sequence \( T_n \) be defined by (7). Then

\[
\frac{1}{48(n + \alpha)^3} \leq \gamma - T_n < \frac{1}{48(n + \beta)^3}
\]

with the best possible constants

\[
\alpha = \frac{1}{\sqrt[3]{48[1 - \gamma + \ln(\frac{3\pi}{2})]}} - 1 = 0.27380525 \ldots \text{ and } \beta = \frac{83}{360} = 0.230555555 \ldots
\]

It is well-known \([1, \text{p.258}]\) that

\[
\psi \left( n + \frac{1}{2} \right) = -\gamma - 2 \ln 2 + \sum_{k=1}^{n} \frac{2}{2k - 1}.
\]

The third aim of this paper is to present the bounds for \( \sum_{k=1}^{n} \frac{2}{2k - 1} - \ln(4n) - \gamma \).

**Theorem 3.** Let \( n \in \mathbb{N} \). Then

\[
\frac{1}{24(n + a)^2} \leq \sum_{k=1}^{n} \frac{2}{2k - 1} - \ln(4n) - \gamma < \frac{1}{24(n + b)^2}
\]

with the best possible constants

\[
a = \frac{1}{\sqrt[2]{24(2 - 2\ln 2 - \gamma)}} - 1 = 0.06858 \ldots \text{ and } b = 0.
\]

Before we prove the main theorems, let us give some preliminary results.

The Euler-Mascheroni constant \( \gamma \) is deeply related to the gamma function \( \Gamma(z) \) thanks to the Weierstrass formula:

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^{-1} e^{z/k} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{-1, -2, -3, \ldots\}).
\]

The logarithmic derivative of the gamma function:

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln \Gamma(z) = \int_{1}^{z} \psi(t) \, dt
\]

is known as the psi (or digamma) function. The successive derivatives of the psi function \( \psi(z) \):

\[
\psi^{(n)}(z) := \frac{d^n}{dz^n} \{\psi(z)\} \quad (n \in \mathbb{N})
\]

are called the polygamma functions.

The following recurrence and asymptotic formulas are well known \([1, \text{pp.258-261}]\):

\[
\psi(z + 1) = \psi(z) + \frac{1}{z},
\]
\[ \psi(z) \sim \ln z - \frac{1}{2z} + \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^5} + \ldots \quad (z \to \infty \text{ in } |\arg z| < \pi) \]

and

\[ \psi'(z) \sim \frac{1}{z} + \frac{1}{6z^3} + \frac{1}{30z^5} + \frac{1}{42x^7} + \ldots \quad (z \to \infty \text{ in } |\arg z| < \pi), \]

from which we get

\[ \psi(x + 1) \sim \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \ldots \quad (x \to \infty) \quad (12) \]

and

\[ \psi'(x + 1) \sim \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{30x^5} + \frac{1}{42x^7} + \ldots \quad (x \to \infty). \quad (13) \]

It is also known [1, p.258] that

\[ \psi(n + 1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}. \quad (14) \]

It is easy to see that

\[ \ln \left( x + \frac{1}{2} + \frac{1}{24x} \right) = \ln x + \ln \left( 1 + \frac{1}{2x} + \frac{1}{24x^2} \right) \]

\[ = \ln x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \frac{1}{2x} + \frac{1}{24x^2} \right)^k \]

\[ = \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{48x^3} - \frac{7}{1152x^4} + \frac{11}{5760x^5} \]

\[ - \frac{13}{20730x^6} + O \left( \frac{1}{x^7} \right). \quad (15) \]

The following lemmas are needed in our present investigation.

**Lemma 1** ([4, Theorem 9]). Let \( k \geq 1 \) and \( n \geq 0 \) be integers. Then for all real numbers \( x > 0 \):

\[ S_k(2n; x) < (-1)^{k+1} \psi^{(k)}(x) < S_k(2n + 1; x), \quad (16) \]

where

\[ S_k(p; x) = \frac{(k - 1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{i=1}^{p} \left[ B_{2i} \prod_{j=1}^{k-1} (2i + j) \right] \frac{1}{x^{2i+k}}, \]

\( B_i \ (i = 0, 1, 2, \ldots) \) are Bernoulli numbers, defined by

\[ \frac{t}{e^t - 1} = \sum_{i=0}^{\infty} B_i \frac{t^i}{i!}. \]
It follows from (16) that
\[
\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \quad (x > 0),
\]
from which it follows that
\[
\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \psi'(x + 1) < \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} \quad (x > 0).
\]

Lemma 2. Let
\[
v(x) = 48 \left( \ln \left( x + \frac{1}{2} + \frac{1}{24x} \right) - \psi(x + 1) \right). \tag{18}
\]
Then, for \( x \geq 5, \)
\[
4 \left( -v'(x) \right)^2 > 3v(x)v''(x). \tag{19}
\]

Proof. We first show that for \( x > 0, \)
\[
v(x) < \frac{1}{x^3} - \frac{83}{120x^4} + \frac{11}{120x^5} + \frac{485}{3024x^6} + \frac{41}{4032x^7}, \tag{20}
v''(x) < \frac{12}{x^5} - \frac{83}{6x^6} + \frac{11}{4x^7} + \frac{485}{72x^8} + \frac{41}{72x^9}, \tag{21}
-v'(x) > \frac{3}{x^4} - \frac{83}{30x^5} + \frac{11}{24x^6} + \frac{485}{504x^7} + \frac{41}{576x^8} - \frac{28133}{17280x^9}. \tag{22}
\]

Define the function \( S \) by
\[
S(x) = v(x) - \left( \frac{1}{x^3} - \frac{83}{120x^4} + \frac{11}{120x^5} + \frac{485}{3024x^6} + \frac{41}{4032x^7} \right).
\]
From (12) and (15), we conclude that
\[
\lim_{x \to \infty} S(x) = \lim_{x \to \infty} \left( -\frac{28133}{138240x^8} + O(x^{-9}) \right) = 0.
\]
Differentiation and applying the right-hand inequality of (17) yields
\[
S'(x) = \frac{48(24x^2 - 1)}{x(24x^2 + 12x + 1)} + \frac{3}{x^4} - \frac{83}{30x^5} + \frac{11}{24x^6} + \frac{485}{504x^7} + \frac{41}{576x^8} - 48\psi'(x + 1) > \frac{48(24x^2 - 1)}{x(24x^2 + 12x + 1)} + \frac{3}{x^4} - \frac{83}{30x^5} + \frac{11}{24x^6} + \frac{485}{504x^7} + \frac{41}{576x^8} - 48\psi'(x + 1)
\]
\[
= \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}
\]
\[
= \frac{388x + 41}{576x^8(24x^2 + 12x + 1)} > 0 \quad (x > 0).
\]
This yields
\[ S(x) < \lim_{x \to \infty} S(x) = 0 \quad (x > 0). \]
This proves (20).

The proofs of (21) and (22) are similar, we leave it to readers. Consequently,
\[
4 \left( -v'(x) \right)^2 - 3v(x)v''(x) \\
> 4 \left( 3 \frac{x^4}{2} - \frac{83}{30x^3} + \frac{11}{24x^6} + \frac{485}{504x^7} + \frac{41}{576x^8} - \frac{28133}{17280x^9} \right)^2 \\
- 3 \left( \frac{1}{x^3} - \frac{83}{120x^4} + \frac{11}{120x^5} + \frac{485}{3024x^6} + \frac{41}{4032x^7} \right) \left( \frac{12}{x^5} - \frac{83}{6x^6} + \frac{11}{4x^7} + \frac{485}{72x^8} + \frac{41}{72x^9} \right)
\]
\[
= \frac{1}{3657830400x^{18}} (525430177982161 + 1266203791908180(x - 5) \\
+ 1147261865648100(x - 5)^2 + 549749077117560(x - 5)^3 \\
+ 157168724355792(x - 5)^4 + 27908026040160(x - 5)^5 \\
+ 303720340160(x - 5)^6 + 186635352960(x - 5)^7 \\
+ 4987858176(x - 5)^8) > 0 \quad (x \geq 5).
\]
Therefore, the inequality (19) holds for \( x \geq 5 \). \( \square \)

2. Proofs of Theorems 1–3

We are now in a position to prove our Theorems 1–3.

Proof of Theorem 1. Define the sequence \( u_n \) by
\[ u_n = e^{H_n}. \]

Then
\[ \ln u_n = \psi(n + 1)n \ln \left( 1 + \frac{1}{n} \right) + \gamma n \ln \left( 1 + \frac{1}{n} \right). \quad (23) \]

It is known that
\[ \ln \left( 1 + \frac{1}{n} \right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + O \left( \frac{1}{n^5} \right). \quad (24) \]

Upon substituting from (12) and (24) into (23), we get
\[
\ln u_n = \ln(n) + \gamma + \frac{-\frac{1}{2} \ln(n) + \frac{1}{2} - \frac{1}{2} \gamma}{n} + \frac{\frac{1}{3} \gamma - \frac{1}{3} + \frac{1}{3} \ln n}{n^2} + O \left( \frac{1}{n^3} \right),
\]
which implies
\[
    u_n = e^n + e^n \left( \frac{1}{2} \ln(n) + \frac{1}{2} - \frac{1}{2} \gamma \right) + e^n \left( \frac{1}{3} \gamma - \frac{1}{3} + \frac{1}{3} \ln n + \frac{1}{2} \left( - \frac{1}{2} \ln(n) + \frac{1}{2} - \frac{1}{2} \gamma \right)^2 \right) + O \left( \frac{1}{n^2} \right). \tag{25}
\]
From (25), we imply (2).

We write \( u_{n+1} - u_n \) as asymptotic representation:
\[
    u_{n+1} - u_n = e^n - \frac{e^n}{2n} + O \left( \frac{1}{n^2} \right). \tag{26}
\]
From (26), we imply (3) and (4).

**Proof of Theorem 2.** The inequality (9) can be written as
\[
    \alpha \geq \frac{1}{\sqrt[3]{48 \left( \ln (n + \frac{1}{2} + \frac{1}{3} \pi n) - \psi(n + 1) \right)}} - n \beta.
\]
In order to prove (9) we define the function \( f \) by
\[
    f(x) = (v(x))^{-1/3} - x,
\]
where \( v(x) \) is as in Lemma 2.

Differentiation yields
\[
    f'(x) = - \frac{v'(x)}{3(v(x))^{4/3}} - 1 \quad \text{and} \quad f''(x) = \frac{4(-v'(x))^2 - 3v(x)v''(x)}{9(v(x))^{7/3}}.
\]
From (12), (13) and (15), we conclude that
\[
    \lim_{x \to \infty} f'(x) = \lim_{x \to \infty} \left( - \frac{4909}{64800x^2} + O(x^{-3}) \right) = 0.
\]
By (19), we obtain \( f''(x) > 0 \) for \( x \geq 5 \). This implies that
\[
    f'(x) < \lim_{x \to \infty} f'(x) = 0 \quad (x \geq 5). \tag{27}
\]
From (27) and \( f(1) = 0.27380525 \ldots, f(2) = 0.2598408 \ldots, f(3) = 0.25212076 \ldots, f(4) = 0.24749774 \ldots, f(5) = 0.24447118 \ldots, \) we conclude that the sequence
\[
    f(n) = \frac{1}{\sqrt[3]{48 \left( \ln (n + \frac{1}{2} + \frac{1}{3} \pi n) - \psi(n + 1) \right)}} - n \quad (n \in \mathbb{N})
\]
is strictly decreasing. This leads to

$$\lim_{n \to \infty} f(n) < f(1) = \frac{1}{\sqrt{48[1 - \gamma + \log(\frac{37}{32})]} - 1 = 0.27380525 \ldots}$$

It remains to prove that

$$\lim_{n \to \infty} f(n) = \frac{83}{360}.$$  \hspace{1cm} (28)

By using the asymptotic formulas (12) and (15), we conclude that

$$\frac{1}{\sqrt{48 \left( \ln (n + \frac{1}{2} + \frac{1}{24n}) - \psi(n + 1) \right)} - n = \frac{83}{360} + O(n^{-1})} = 1 + O(n^{-1}),$$

which implies (28). The proof of Theorem 2 is complete. \hspace{1cm} \(\square\)

Proof of Theorem 3. By (10), the inequality (11) can be written as

$$a \geq \frac{1}{\sqrt{24\left[\psi(n + \frac{1}{2}) - \ln n\right]} - n > b.} \hspace{1cm} (29)$$

In order to prove (29) we define the function \(h\) by

$$h(x) = \frac{1}{\sqrt{24\left[\psi(x + \frac{1}{2}) - \ln x\right]} - x} \quad (x > 0).$$

It is known (see [8, p.86] and [7, p.163]) that for \(x > \frac{1}{2}\),

$$\frac{1}{24(x - \frac{1}{2})^2} - \frac{7}{960(x - \frac{1}{2})^4} < \psi(x) - \ln \left( x - \frac{1}{2} \right) \hspace{1cm} (30)$$

and

$$\frac{1}{x - \frac{1}{2}} - \psi'(x) < \frac{1}{12(x - \frac{1}{2})^3} - \frac{7}{240(x - \frac{1}{2})^5} + \frac{31}{1344(x - \frac{1}{2})^7}. \hspace{1cm} (31)$$

(We remark that the inequalities (30) and (31) were derived from [2].) It is well-known that let \(x \geq -1\), then for \(\alpha < 0\) or \(\alpha > 1\),

$$(1 + x)^\alpha \geq 1 + \alpha x, \hspace{1cm} (32)$$

the equal sign holds if and only if \(x = 0\).
Differentiation and applying the inequalities (30), (31) and (32), we obtain for $x \geq 2$:

$$- \left( 24 \left( \psi \left( x + \frac{1}{2} \right) - \ln x \right) \right)^{3/2} \cdot h'(x)$$

$$= 12 \left( \psi' \left( x + \frac{1}{2} \right) - \frac{1}{x} \right) + \left( 24 \left( \psi \left( x + \frac{1}{2} \right) - \ln x \right) \right)^{3/2}$$

$$> - \frac{1}{x^3} + \frac{7}{20x^5} - \frac{31}{112x^7} + \left( \frac{1}{x^3} - \frac{7}{40x^5} \right)^{3/2}$$

$$= - \frac{1}{x^3} + \frac{7}{20x^5} - \frac{31}{112x^7} + \frac{1}{x^3} \left( 1 - \frac{7}{40x^2} \right)^{3/2}$$

$$> - \frac{1}{x^3} + \frac{7}{20x^5} - \frac{31}{112x^7} + \frac{1}{x^3} \left( 1 - \frac{3}{2} \frac{7}{40x^2} \right)$$

$$= 41 + 196(x-2) + 49(x-2)^2 \quad \text{as} \quad x \to \infty.$$
we obtain

\[ \frac{1}{\sqrt{24\left[\psi(x + \frac{1}{2}) - \ln x\right]}} - x = \frac{1 - x\sqrt{24\left[\psi(x + \frac{1}{2}) - \ln x\right]}}{\sqrt{24\left[\psi(x + \frac{1}{2}) - \ln x\right]}} = \frac{1 - x\sqrt{\frac{1}{x^5} - \frac{7}{40x^7} + O(x^{-6})}}{\sqrt{\frac{1}{x^5} - \frac{7}{40x^7} + O(x^{-6})}} = \frac{x - x\sqrt{1 - \frac{7}{40x^2} + O(x^{-4})}}{\sqrt{1 - \frac{7}{40x^2} + O(x^{-4})}} = \frac{x^2 + O(x^{-2})}{x + O(x^{-1})} \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty, \]

which implies (33). This completes the proof of Theorem 3.

Remark 1. Some calculations in this work were performed by using the Maple software for symbolic calculations.

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References


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