Optimal Broadcasting in Manhattan Street Networks

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Abstract

The Manhattan Street Network (MSN) is a mesh-structured, toroidal, two-connected, directed, regular network which resembles the geographical topology of the avenues and streets of Manhattan. Previous work on these networks, in most cases through simulations, has been mainly devoted to the study of the average distance and point-to-point routing schemes. In this paper we provide an analytical determination of the diameter and an algorithm which can broadcast optimally in a Manhattan Street Network.

1 Introduction

In these last years the study of a class of directed torus networks known as Manhattan Street Networks has received significant attention. These networks were introduced by Maxemchuk in 1985 [6] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan. Previous work has been devoted to the computation of the average distance and the generation of routing schemes. However the results are often given as conjectures supported by computer simulations [7, 4]. The

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study of spanning trees [1] in a MSN has allowed the computation of the diameter and the design of a broadcasting algorithm. More recently, Varvarigos in [8] evaluates the mean internodal distance, provides a shortest path routing algorithm and constructs edge-disjoint Hamiltonian cycles in the general case. In this paper we provide a direct analytical computation of the diameter, a scheme to broadcast in a Manhattan Street Network and a proof of the optimality of this broadcasting algorithm.

2 Notation and preliminary results

We consider bidimensional $N \times N$ Manhattan Street Networks with $N^2$ vertices. The vertices are labeled with pairs $(u_1, u_2)$ where $0 \leq u_1 \leq N - 1$ and $0 \leq u_2 \leq N - 1$. The edges of this graph are $\{(u_1, u_2) \rightarrow ((u_1 \pm 1) \mod N, u_2)\}$ and $\{((u_1, u_2) \rightarrow (u_1, (u_2 \pm 1) \mod N)\}$ where the sign is plus or minus depending on whether $u_1$ and $u_2$ are even or odd, see Figure 1. Therefore the MSN considered in this study are modeled by directed regular graphs with indegree and outdegree two which are vertex symmetric.

![Figure 1: The 8 x 8 Manhattan Street Network.](image)

Broadcasting in a graph is the process of spreading a message known initially by one vertex, subject to the following rules. The transfer of the message from one vertex to another (termed a call) takes one unit of time. A vertex can only call an adjacent vertex. A vertex can participate in at most
one call per unit of time. A broadcast scheme is a formal description of this process.

Given a connected digraph $G$ and a vertex $u$, the broadcast time of $u$, denoted $b(u)$, is the minimum number of time units required to broadcast a message originating at $u$. The broadcast time of the graph $G$ is defined $b(G) = \max\{b(u) | u \in G\}$. For any vertex $u$ in a connected graph with $|V|$ vertices, $b(u) \geq \lceil \log_2 |V| \rceil$, since during each time unit the number of vertices informed can at most double. For a vertex symmetric graph, the broadcast time is equal to the broadcast time of any of its vertices.

The following result will we used in Section 4 to prove the optimality of the broadcast scheme introduced in this paper.

**Theorem 1** For a graph $G$ with diameter $D$, $b(G) \geq D$. Moreover, if there exist three vertices $u$, $v_1$, and $v_2$ such that $v_1$ and $v_2$ are at distance $D$ of $u$, then $b(G) \geq D + 1$.

**Proof.** Given a graph $G$ with diameter $D$ let us assume that there exists a broadcasting protocol for it. By recurrence on $i$, we see that at step $i$ of this protocol, at most one vertex at distance $i$ from the originator could be informed. Hence the first assertion. Moreover, if there exist two vertices at distance $D$ from the originator, only one could be informed at time $D$ and we would need at least one more step to finish the broadcasting process. Then, $b(G) \geq D + 1$. □

# 3 Diameter of a MSN

The diameter of a MSN has been computed in [1] by considering spanning trees. We obtain the same result from the study of the distribution of vertices at each distance which is as follows ($n_k$ denotes the number of vertices at distance $k$ of a given vertex):

$N = 0 \mod 4, N > 4$: $n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11, n_i = 2N - |N - 2i| - |N - 2i + 4|$ (for $5 \leq i \leq N$), and $n_{N+1} = 2$.

$N = 2 \mod 4, N > 4$: $n_1 = 2, n_2 = 4, n_3 = 8, n_4 = 11, n_i = 2N - 2|N - 2i + 2|$ (for $5 \leq i \leq N - 1$), and $n_N = 2$.

Adding all the values up we obtain $N^2$ and therefore we can write the following result:
Theorem 2  The diameter of a $N \times N$ MSN is $N + 1$ for $N > 4, N = 0 \pmod{4}$ and $N$ for $N > 4, N = 2 \pmod{4}$.

From the distribution of vertices we can also compute the average distance which results in $\frac{N^3 + 2N^2 - 8}{2(N^2 - 1)}$, $N = 0 \pmod{4}$, $N > 4$ and $\frac{N^3 + 2N^2 - 4N - 4}{2(N^2 - 1)}$, $N = 2 \pmod{4}$, $N > 4$ which corresponds to the results given in [8].

4 Optimal broadcasting in a MSN

We give in this section a scheme to broadcast in any MSN. We prove that the scheme is optimal for a MSN with dimensions $N \times N$, $N > 4, N = 0 \pmod{4}$.

Algorithm Broadcast-MSN

1. The originator sends the message vertically

2. If a vertex receives the message from a vertical neighbor it first sends it vertically and then, in the following round, horizontally.

3. If a node receives the message from a horizontal neighbor it first sends it horizontally and then, in the following round, vertically.

This algorithm differs noticeably from the standard broadcasting algorithm for the undirected case in which, after a first change of direction there can be new changes of direction.

On the other hand, extensive tests using genetic programming to generate broadcasting schemes for a MSN have always produced an equivalent algorithm (in some cases with the horizontal and vertical calls reversed or swapped), see [3, 5]. Our algorithm informs all nodes in time $D + 1$ or $D + 2$ as it is stated in the next theorem.

Theorem 3  The broadcast time for a MSN, $N \geq 4$, $N = 0 \pmod{2}$, is $N + 2$. This broadcast time is optimal if $N = 0 \pmod{4}$.

Proof. We recall that the graph is vertex symmetric, and the broadcast can be initiated from any vertex. Using the broadcast algorithm in a MSN with dimensions $N \times N$, $N \geq 4, N = 0 \pmod{4}$, and starting from vertex $(0, 0)$, at step $N + 2(= D + 1)$ the last vertices to be informed are $(\frac{N}{2}, \frac{N}{2} + 1)$,
As the graph can not be informed in time $D$ because there are two vertices at this distance from the originator, the broadcast is optimal (see Theorem 1).

When $N = 2 \mod 4$, at step $N + 2 (= D + 2)$, the last vertices to be informed are $(\frac{N}{2} + 1, \frac{N}{2} + 1)$, $(\frac{N}{2} + 2, \frac{N}{2} + 1)$ and $(\frac{N}{2} + 2, \frac{N}{2} - 1)$. □

In Figure 2 we show the execution of the broadcast algorithm for a $8 \times 8$ MSN (diameter 9). The last four vertices reached (at time 10) are shown in different size.

![Figure 2: Optimal broadcasting in a 8 x 8 Manhattan Street Network.](image)

**References**


