Stationary and nonstationary affine combination of subdivision masks

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Abstract

One of the difficult task in subdivision is to create new effective subdivision schemes. Therefore, aim of this paper is a systematic analysis of affine combination of known subdivision masks to generate new subdivision schemes with enhanced properties. This will be done in the stationary and the non stationary case for the univariate and bivariate settings.

Keywords: Stationary subdivision schemes; Non stationary subdivision schemes; Symbols; Affine combination

1. Introduction

The power of subdivision schemes has been extensively established in several contexts, like, just to mention two well known examples, the design of smooth curves and surfaces and the generation of refinable functions and wavelets. Subdivision schemes basically are iterative schemes based on simple refinement rules generating increasingly dense sequences of points convergent to a continuous curve or surface. In the linear case, the refinement rules are simply average rules based on what is called the mask of the subdivision scheme. This is a finitely supported sequence of coefficients. In the univariate case, a well-known example of subdivision scheme is given by B-spline subdivision schemes that can be used to generate spline curves.

Subdivision schemes range from stationary (i.e. the refinement rules do not depend on the recursion level) to nonstationary; from uniform (i.e. the refinement rules do not vary from point to point) to nonuniform; from binary (i.e. the number of points is ‘doubled’ at each iteration) to any a-ritity; from scalar (i.e. the mask is made of real numbers) to vector (i.e. the mask is a sequence of matrices). For an exhaustive review of subdivision schemes we refer the reader to the survey paper [6]. Even though in recent years subdivision schemes have gained popularity and important steps have been made in their analysis, more non-trivial examples are still needed. This is particularly true in the multivariate setting. Therefore, the aim of this paper is to move a first step in the systematic analysis of affine combination of known subdivision masks to generate new subdivision schemes with enhanced properties. This will be done in the stationary and the non stationary case for the univariate and the bivariate settings.

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In the simplest situation, starting from two symbols of convergent subdivision schemes, say \(a(z)\) and \(b(z)\), a new nonstationary subdivision scheme is defined by using the \(k\)-level symbols,

\[
c^k(z) = (1 - \lambda_k) a(z) + \lambda_k b(z), \quad 0 \leq \lambda_k \leq 1, \quad k \geq 0.
\]

If the parameter \(\lambda_k\) is constant with respect to \(k\), the linear combination is stationary as well as the subdivision scheme associated to \(\{c^k(z) = c(z)\}_{k \geq 0}\). This is actually the case of B-splines where \(a(z) = b(z), \ \lambda = 1/2\) and the smoothing process is nothing but an average process. Also, this is the case of GP functions where \(a(z)\) is a spline of degree \(n\) and \(b(z)\) a spline of degree \(n - 2\) [10].

The simple but effective idea behind affine combination is the use of the parameters \(\lambda_k\) to “improve” the combined subdivision schemes. The improvement can be with regard to smoothness of the limit function, to shortness of their support or to linear independence of their integer translates, for example.

Since convex combination of subdivision schemes has already been used by the author and co-authors in different contexts, this work also aims at providing a unified approach to the strategies used so far. In particular, we refer to the strategy for generating a new class of positive symmetric functions with support of length \(n\) and smoothness \(C^{n-1}\) (defined in [2]), to smooth the hat function in the most general way (used in [3]), to get Box-spline like linear independent functions on criss-cross triangulations (as done in [1]).

2. Subdivision schemes: some background

Any nonstationary subdivision scheme is defined by an infinite sequence of refinement masks \(\{a^k\}_{k \geq 0}\). We assume that any sequence \(a^k := \{a^k_\alpha\}_{\alpha \in \Z^s}\) is of real numbers and have finite support for all \(k \geq 0\). The \(k\)-level subdivision operator associated with the \(k\)-level mask \(a^k\) is

\[
S_a^k : \ell(\Z^s) \to \ell(\Z^s),
\]

\[
(S_a^k \lambda)_\alpha := \sum_{\beta \in \Z^s} a^k_{\alpha - 2\beta} \lambda_\beta, \ \alpha \in \Z^s,
\]

where \(\ell(\Z^s)\) denotes the linear space of real sequences indexed by \(\Z^s, s = 1\) and \(s = 2\) in the univariate and bivariate cases, respectively. The nonstationary subdivision scheme consists of the subsequent application of \(S_a^0, \ldots, S_a^k\) generating the scalar sequences

\[
\lambda^0 := \lambda, \ \lambda^{k+1} := S_a^k \lambda^k \text{ for } k \geq 0
\]

Obviously, the process could also be started from any fixed level \(m > 0\), that is by subsequent application of \(S_a^m, \ldots, S_a^{m+\ell}\) for \(\ell \geq 0\).

A subdivision scheme is \(L_\infty\)-convergent if, for any \(\lambda \in \ell^{\infty}(\Z^s)\), the linear space of bounded scalar sequences indexed by \(\Z^s\), there exists a continuous function \(f_\lambda\) (depending on the starting sequence \(\lambda\)) satisfying

\[
\lim_{k \to \infty} \|f_\lambda \left(\frac{x}{2^k}\right) - \lambda^k\|_\infty = 0
\]

and \(f_\lambda \neq 0\) for at least some initial data \(\lambda\). Here, the symbol \(f_\lambda(\cdot/2^k)\) abbreviates the scalar sequence \(\{f_\lambda(\alpha/2^k)\}_{\alpha \in \Z^s}\) and \(\|\lambda\|_\infty := \sup_{\alpha \in \Z^s} |\lambda_\alpha|\).

For any convergent nonstationary subdivision scheme with masks \(\{a^k\}_{k \geq 0}\), one gets a family of basic limit functions each defined by

\[
\phi_a^m := \lim_{\ell \to \infty} S_a^{m+\ell}, S_a^{m+\ell-1}, \ldots, S_a^m \delta_0 \quad m \geq 0 \quad \ell > 0
\]

where \(\delta_0\) is the delta-sequence, i.e. \(\delta_0 = \{\delta_\alpha, 0\}_{\alpha \in \Z^s}\). The functions \(\phi_a^m, m \geq 0\), are solutions to the functional equations

\[
\phi_a^m(x) = \sum_{\alpha \in \Z^s} a^m_\alpha \phi_a^{m+1}(2x - \alpha) \quad x \in \mathbb{R}, \quad m \geq 0
\]

which are also called nonstationary refinement equations (for example in [9]). If the mask is kept fixed over the iterations, i.e. \(\{a^k = a\}_{k \geq 0}\), the scheme is termed stationary. In that case, \(\phi_a^m = \phi_a\) for all \(m\) and the equations in (5)
reduce to the refinement equation for \( \phi_a := \lim_{k \to \infty} S_k \delta_0 \),
\[
\phi_a(x) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha \phi_a(2x - \alpha), \quad x \in \mathbb{R}^s.
\] (6)

Useful tools for the subdivision analysis are the symbols
\[
d^k(z) = \sum_{\alpha \in \mathbb{Z}^s} a^k_\alpha z^\alpha, \quad z \in \mathbb{C}^s \setminus \{0\},
\]
associated to the masks \( \{a^k\}_{k \geq 0} \). Since the masks are always supposed to be finitely supported all symbols are Laurent polynomials. Here, with an abuse of terminology, for convenience, we will refer to the subdivision scheme in (2) equivalently as the subdivision scheme \( \{a^k\}_{k \geq 0} \), the subdivision scheme \( \{a^k(z)\}_{k \geq 0} \) or the subdivision scheme \( \{S_k^a\}_{k \geq 0} \).

In the stationary situation we will simply use the words: the subdivision scheme \( a \), the subdivision scheme \( a(z) \) or the subdivision scheme \( S_a \).

We recall that, in the stationary case, necessary conditions for the subdivision convergence are given in terms of symbol properties as (see, for example [6]),
\[
a(1) = 2^s, \quad a((-1)^e) = 0, \quad e \in E^c := E \setminus \{0\}.
\] (7)

where \( E := \{0, 1\}^s \cap \mathbb{Z}^s \). Two nonstationary subdivision schemes with masks \( \{a^k\}_{k \geq 0} \) and \( \{b^k\}_{k \geq 0} \) are said to be asymptotically equivalent, in symbols \( \{a^k\}_{k \geq 0} \approx \{b^k\}_{k \geq 0} \), if for some fixed \( L \) it results
\[
\sum_{k=0}^{\infty} \max_{\alpha \in \mathbb{Z}^s} \sum_{\beta \in \mathbb{Z}^s} |a^k_{\alpha + L\beta} - b^k_{\alpha - 2\beta}| < \infty.
\] (8)

This concept allows one to derive convergence properties of a given nonstationary subdivision scheme from those of an asymptotic equivalent subdivision scheme known to be convergent. To this respect, we recall a useful result given in [6].

**Theorem A.** Let \( \{a^k\}_{k \geq 0} \approx \{b^k := b\}_{k \geq 0} \) be two asymptotically equivalent subdivisions schemes with the latter being a stationary subdivision. If the stationary subdivision \( b \) is convergent, then \( \{a^k\}_{k \geq 0} \) is convergent as well.

With the help of Laurent polynomial representation in [11] Levin gave sufficient conditions for convergence and smoothness investigation of nonstationary univariate subdivision schemes \( s = 1 \). These conditions are based on contractivity of the \( \ell \)-iterate of the subdivision operator. For completeness, we shortly recall its result.

**Theorem B.** Let \( \{a^k(z)\}_{k \geq 0} \) the Laurent polynomials associated with the masks \( \{a^k\}_{k \geq 0} \). Let \( \{b^k, r(z), 1 \leq r \leq N + 1\} \)
be the Laurent polynomials recursively defined as
\[
b^{k,0}(z) = a^k(z), \quad b^{k,r+1}(z) = \frac{2b^{k,r}(z)}{z+1}, \quad 0 \leq r < N.
\] (9)

If the scheme having Laurent polynomials \( \{b^{k,N}(z)\}_{k \geq 0} \) is convergent, the scheme with masks \( \{a^k\}_{k \geq 0} \) is \( C^N \).

### 3. Univariate stationary affine combination

In the stationary case we consider two symbols \( a(z) \) and \( b(z) \), and define a new subdivision scheme having symbol
\[
c(z) = (1 - \lambda) a(z) + \lambda z b(z).
\] (10)

For the subdivision scheme based on \( c(z) \) we can prove a convergence result based on the following Lemma where a mask \( a \) is said to be contractive if it satisfies
\[
\sum_{\beta \in \mathbb{Z}} |a_{\alpha - 2\beta}| < 1, \quad \text{for all } \alpha \in \mathbb{Z}.
\]

**Lemma 1.** Let \( a \) and \( b \) be contractive masks and \( 0 \leq \lambda \leq 1 \). Then the mask \( c \) with symbol defined as in (10) is contractive as well.
Proof. It holds
\[
\sum_{\beta \in \mathbb{Z}} |c_{\alpha-2\beta}| \leq (1 - \lambda) \sum_{\beta \in \mathbb{Z}} |a_{\alpha-2\beta}| + \lambda \sum_{\beta \in \mathbb{Z}} |b_{\alpha+1-2\beta}| < 1 \text{ for all } \alpha \in \mathbb{Z}. \quad \square
\]

An important consequence of the previous Lemma is that any “smoothed” mask \( c(z) := ((1 + z)/2)a(z) \) inherits contractivity by the mask \( a \). The viceversa is not always true since \( c \) may be contractive even if \( a \) is not. This is the case of \( a(z) = -(1/2)z^{-1} + (1/4) - (1/2)z \).

**Proposition 2.** Let \( a(z) \) and \( b(z) \) be symbols of convergent subdivision schemes such that \( a(z) = (z + 1)\tilde{a}(z) \) and \( b(z) = (z + 1)\tilde{b}(z) \) with \( \tilde{a} \) and \( \tilde{b} \) contractive masks. Then the subdivision scheme associated with \( c(z) \) as in (10) where \( 0 \leq \lambda \leq 1 \) is convergent as well.

**Proof.** Since \( c(z) = (z + 1)((1 - \lambda)\tilde{a}(z) + \lambda z\tilde{b}(z)) = (z + 1)\bar{c}(z) \), with \( \bar{c} \) contractive (due to Lemma 1), the claim easily follows. \( \square \)

Concerning the use of the parameter \( \lambda \) to increase the regularity of the combined subdivision scheme we can prove the following

**Proposition 3.** Let \( a(z) = ((z + 1)^m/2^m)\tilde{a}(z) \) and \( b(z) = ((z + 1)^m/2^m)\tilde{b}(z) \) be convergent subdivision schemes with \( \tilde{a}(z) \) and \( \tilde{b}(z) \) Laurent polynomials such that \( \tilde{a}(-1) + \tilde{b}(-1) \neq 0 \). Then, the subdivision scheme
\[
c(z) = (1 - \lambda) a(z) + \lambda z b(z), \quad \text{with } \lambda := \frac{\tilde{a}(-1)}{\tilde{a}(-1) + \tilde{b}(-1)},
\]
(11)
satisfies the necessary conditions for convergence \( c(1) = 2, c(-1) = 0 \) and contains \( m + 1 \) smoothing factors being of the form
\[
c(z) = \frac{(z + 1)^{m+1}}{2^{m+1}} \bar{c}(z),
\]
where \( \bar{c}(z) \) is a Laurent polynomial.

**Proof.** Since \( c(\pm 1) = (1 - \lambda)a(\pm 1) + \lambda b(\pm 1) \) the first claim easily follows. Next, factorizing \( ((z + 1)^m/2^m) \) we have
\[
c(z) = \frac{(z + 1)^m}{2^m} \left( (1 - \lambda)\tilde{a}(z) + \lambda z\tilde{b}(z) \right)
\]
and the choice of \( \lambda \in (0, 1) \) in (11) gives \( (1 - \lambda)\tilde{a}(-1) - \lambda \tilde{b}(-1) = 0 \) which is the existence of an other factor \( (z + 1) \). \( \square \)

**Remark 4.** In case we look for a convex combination of subdivision masks and \( \text{sign} (\tilde{a}(-1) \cdot \tilde{b}(-1)) < 0 \) one can consider the linear combination \( c(z) = (1 - \lambda) a(z) + \lambda b(z) \) for which the previous theorem holds true with \( 0 < \lambda < 1 \).

### 3.1. Examples

There are several examples of subdivision schemes generated via convex or affine combination of masks though not always seen that way. For example in [8] a similar analysis is conducted for some special cases of perturbed masks.

A first example we consider here is given by the special situation where \( a(z) = b(z) \) and \( \lambda = 1/2 \) so that
\[
c(z) = \frac{z + 1}{2} a(z),
\]
and the linear combination is an average process resulting in a smoothing process. Here the smoothing factor \( ((z + 1)/2) \) increases by one the regularity of the limit function as proved in [11]. B-splines are in this situation since for \( a(z) = b(z) = ((z + 1)^{n+1}/2^n) \), the symbol of a \( C^{n-1} \) B-splines, it turns out that \( c(z) = ((z + 1)^{n+2}/2^{n+1}) \) is the symbol of a \( C^n \) B-spline.
A second example is given by the four point scheme (we refer to [5] for a first discussion of this scheme) where 
\( a(z) = ((z + 1)^2) \) is the symbol of a degree 1 \( C^0 \) spline, 
\( b(z) = -(z^3 + (3/2)z^2 + 1 + 3z^{-1} - z^{-3}) \) is the symbol of a nonconvergent scheme, \( \lambda = (1/16) \) and the combined scheme 
\[
c(z) = \frac{1}{16} \left(-z^{-2} + 9 + 16z + 9z^2 - z^4\right),
\]
is actually \( C^1 \).

An other example is given by GP-functions where 
\( a(z) = ((z + 1)^{n+1}/2^n) \) is a spline of degree \( n \) while 
\( b(z) = ((z + 1)^n/2^{n-2}) \) is a spline of degree \( n - 2 \), \( \lambda = (1 - (1/2^\ell)) \), \( \ell \in \mathbb{N} \) and the combined symbol 
\[
c(z) = \frac{1}{2\ell} \frac{(z + 1)^{n-1}}{2^n} \left(z^2 + 2(2^{\ell+1} - 1)z + 1\right),
\]
is associated with a \( C^{n-2} \) totally positive function (see [10] for all details). Though there is no gain in the regularity of the limit function, the additional degree of freedom given by \( \ell \) allows one to use GP-functions more effectively in several applications (see, for instance [4]).

As an application of Proposition 3, we continue by considering the four point case. We have 
\( a(z) = ((z + 1)^2/4) \cdot 2 \) and 
\( b(z) = ((z + 1)^2/4) \cdot (-z + 2 - (3/2)z^{-1} + 2z^{-2} - z^{-3}) \) so that \( m = 2 \). Now, since \( \tilde{a}(-1) = 2 \) and \( \tilde{b}(-1) \) = 30 from (11) we get that the parameter \( \lambda = (1/16) \) produces an extra smoothing factor \( (z + 1) \) i.e. 
\( c(z) = ((z + 1)^3/8)(-1/2z + (3/2) + (3/2)z^{-1} - (1/2)z^{-2}) \) generating a \( C^1 \) limit function.

As a second application we consider a convex combination of two GP-functions with the same “degree” \( n \) that is 
\[
a(z) = \frac{(z + 1)^n}{2^n} \left(\frac{z^2 + (2^{\ell + 2} - 2)z + 1}{2^{\ell + 1}}\right),
\]
\[
b(z) = \frac{(z + 1)^n}{2^n} \left(\frac{z^2 + (2^{\ell + 2} - 2)z + 1}{2^{\ell + 1}}\right).
\]

According to Proposition 3, we take the combination coefficient 
\[
\lambda = \frac{(1 - 2^{\ell_1})2^{\ell_2}}{2^{\ell_1} + 2^{\ell_2} - 2^{\ell_1 + \ell_2 + 1}},
\]
and the combined scheme \( c(z) \), turns out to be 
\[
c(z) = \frac{(z + 1)^{n+1}}{2^{n+1}} \left(c_2z^2 + c_1z + c_0\right),
\]
with 
\[
c_2 = \lambda \quad c_1 = \frac{1 - \lambda}{2^{\ell_1}} + \frac{(2^{\ell_2 + 2} - 3)\lambda}{2^{\ell_2}} \quad c_0 = \frac{(2^{\ell_2 + 2} - 3)(1 - \lambda)}{2^{\ell_1}} + \frac{(4 - 2^{\ell_2 + 2})\lambda}{2^{\ell_2}}.
\]

In Figs. 1 and 2 the refinable functions associated with \( a \), \( b \), and \( c \) are given for \( n, \ell_i, \ i = 1, 2 \) and \( \lambda \) specified in the figure’s caption.

![Fig. 1. Refinable function for \( n = 2, \ell_1 = 1 \) (left), \( \ell_2 = 2 \) (right).]
3.2. Generalization

To generalize the just discussed strategy we now consider a convex combination based on three subdivision masks and two “directions” $z$ and $z^{-1}$,

$$d(z) = \lambda_1 a(z) + \lambda_2 z b(z) + \lambda_3 z^{-1} c(z), \quad \sum_{i=1}^{3} \lambda_i = 1.$$  

This type of convex combination is discussed in [3, Introduction] where a symbol $a(z)$ is constructed by multiplication of two symbols, that is $a(z) := h(z) \sigma(z)$, with $h(z) = (z^{-1}/2)(1 + z)^2$ – the symbol associated with the hat function – and $\sigma(z)$ the so called smoothing factor. In the symmetric case $\sigma(z) = 1 + (\lambda/2)(z^{-1} - 2 + z)$ so that

$$d(z) = (1 - \lambda) a(z) + \frac{\lambda}{2} z b(z) + \frac{\lambda}{2} z^{-1} c(z), \quad a(z) = b(z) = c(z).$$

while in the nonsymmetric one $\sigma(z) = 1 + (1/2)((1 - \lambda)z^{-1} - 1 + \lambda z)$ so that

$$d(z) = \frac{1}{2} a(z) + \frac{\lambda}{2} z b(z) + \frac{1 - \lambda}{2} z^{-1} c(z), \quad a(z) = b(z) = c(z).$$

As before, the combination parameters can be used to increase the regularity of the limit as discussed in the following Proposition.

**Proposition 5.** Let $a_i(z) = (z + 1)^m/2^m \tilde{a}_i(z), \; i = 1, 2, 3$ be convergent subdivision schemes with $\tilde{a}_i(z), \; i = 1, 2, 3$ Laurent polynomials such that $D \neq 0$, with

$$D := \sum_{i=1}^{2} \sum_{j=i+1}^{3} i \tilde{a}_i(-1) \tilde{a}_j(-1) + (-1)^j \left( \tilde{a}_i(-1) \tilde{a}_j(-1) - \tilde{a}_i(-1) \tilde{a}_j(-1) \right), \quad (14)$$

where $a^\sim(z) := \sum_{\alpha \in \mathbb{Z}} a_\alpha z^\alpha$. Then, the combined subdivision scheme with symbol

$$d(z) := \lambda_1 a_1(z) + \lambda_2 z a_2(z) + \lambda_3 z^{-1} a_3(z), \quad \sum_{i=1}^{3} \lambda_i = 1, \quad (15)$$

does not the necessary conditions for convergence and contains $m + 2$ smoothing factors being of the form

$$d(z) = \frac{(z + 1)^{m+2}}{2^{m+2}} \tilde{d}(z)$$

with $\tilde{d}(z)$ a Laurent polynomial.
Proof. As in the proof of Proposition 3, the first claim easily follows while for the second one, we write \( d(z) \) as

\[
\hat{d}(z) := \lambda_1 \hat{a}_1(z) + \lambda_2 z \hat{a}_2(z) + \lambda_3 z^{-1} \hat{a}_3(z).
\]

To guarantee the existence of two extra factors \((z + 1/2)\) for \( \hat{d}(z) \) we require that \( \hat{d}(z) \) and its first \( z \)-derivative are zero at \(-1\). For convenience, for any mask \( a \) we introduce the Laurent polynomial \( a(z) := \sum_{a \in \mathbb{Z}} a \alpha z^\alpha \) and express the derivative of the symbol \( \hat{d}(z) \) in terms of it as \((\hat{d}(z))' = (\hat{d}(z))' / z\). Requiring \( \hat{d}(-1) = 0 \) and \((\hat{d}(-1))' = 0\) leads to the linear system \( Mx = T \) with

\[
M := \begin{pmatrix}
\hat{a}_1(-1) + \hat{a}_3(-1) & \hat{a}_3(-1) - \hat{a}_2(-1) \\
\hat{a}_3(-1) - \hat{a}_3^2(-1) - \hat{a}_1^2(-1) & \hat{a}_2(-1) + \hat{a}_2^2(-1) + \hat{a}_3(-1) - \hat{a}_3^2(-1)
\end{pmatrix}
\]

\[
x := \begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix}, \quad T := \begin{pmatrix}
\hat{a}_3(-1) \\
\hat{a}_3(-1) - \hat{a}_3^2(-1)
\end{pmatrix}
\]

where \( \det M = D \neq 0 \) due to the assumptions. This concludes the proof. □

The assumption of convergence is not always necessary as the following example shows: Let us consider the subdivision masks

\[
a_1(z) := \frac{1}{2} \left( z^{-2} + 2 + z^2 \right), \quad a_2(z) := -z^{-1} + 4 - z, \quad a_3(z) := \frac{1}{8} \left( z^{-1} + 8 + 7z \right),
\]

and the combined scheme

\[
d(z) = \frac{(z + 1)^2}{24} \left( 5z^{-2} + 6z^{-1} + 1 \right),
\]

constructed as in (15) with, following Proposition 5, \( \lambda_1 = (3/12), \lambda_2 = (1/2), \lambda_3 = (8/12) \). Though the subdivision schemes associated with \( a_1(z) \) and \( a_2(z) \) are not convergent while the one associated with \( a_3(z) \) is only \( C^0 \), the combined scheme is indeed \( C^1 \) since \((5z^{-2} + 6z^{-1} + 1/12)\) is a contractive mask. The graphs of the results obtained running 5 steps of the corresponding subdivision schemes are shown in Figs. 3 and 4, respectively.

4. Univariate nonstationary convex combination

A nonstationary subdivision scheme is defined via a nonstationary convex combination of two symbols, say \( a(z) \) and \( b(z) \). The \( k \)-level symbol is then

\[
e^k(z) = (1 - \lambda_k) a(z) + \lambda_k z b(z), \quad 0 \leq \lambda_k \leq 1.
\]

We also assume that

\[
\lambda_0 := 0, \quad \lim_{k \to \infty} \lambda_k = 1.
\]
Under this assumption, it is convenient to consider \( a(z) \) and \( b(z) \) with different smoothness and different support: in fact, (17) mean that we start the nonstationary iterative process with the symbol \( a(z) \) and “end” it with the symbol \( b(z) \). Thus, we expect to generate functions having shape given by the starting mask and smoothness by the limit one. This idea is better formulated in the following Theorem.

**Theorem 6.** Let \( a(z) = ((z + 1)^m/2^m)\hat{a}(z) \) generate a \( C^{m-1} \) function with support \([l_a, r_a]\). Let \( zb(z) = ((z + 1)^{m+r}/2^{m+r})\hat{b}(z) \) generate a \( C^{m+r-1} \) function with support \([l_b, r_b]\). Moreover, let \([l_a, r_a] \subset [l_b, r_b]\). Consider the nonstationary subdivision scheme \( \{c^k(z)\}_{k \geq 0} \) where

\[
c^k(z) = (1 - \lambda_k) a(z) + \lambda_k z b(z), \quad 0 \leq \lambda_k \leq 1, \quad \lambda_0 := 0, \quad \lim_{k \to \infty} \lambda_k = 1. \tag{18}
\]

If \( \sum_{k=1}^{\infty} 2^k (1 - \lambda_k) < \infty \), the nonstationary subdivision scheme with symbols given in (16) generates a \( C^{m+r-1} \) basic limit function \( \phi^0_k \) with support \([l_a + l_b)/2, (r_a + r_b)/2]\).

**Proof.** Consider the nonstationary subdivision scheme having symbols

\[
\hat{c}^k(z) = (1 - \lambda_k) \hat{a}(z) + \lambda_k \left( \frac{z + 1}{2} \hat{b}(z) \right), \quad 0 \leq \lambda_k \leq 1, \quad \lambda_0 := 0, \quad \lim_{k \to \infty} \lambda_k = 1, \tag{19}
\]

where \( \hat{b}(z) \) is a \( C^{r-1} \) scheme. It follows that

\[
\hat{b}(z) - \hat{c}^k(z) = (\hat{b}(z) - \hat{a}(z)) (1 - \lambda_k),
\]

and since \( \sum_{k=1}^{\infty} 2^k (1 - \lambda_k) < \infty \), by assumption, \( \hat{c}^k(z) \) and \( \hat{b}(z) \) are asymptotically equivalent. This implies, by [7, Theorem 8], that the nonstationary scheme \( \hat{c}^k(z) \) is convergent and the limit function is at least \( C^{r-1} \). Next, since \( \hat{c}^k(z) = \left( \frac{z+1}{2} \right)^m \hat{c}^k(z), k \geq 0 \), Theorem B allows us to conclude the proof.

As regards to the support, we can use the results given in [6, p. 78]: assuming \([l(k), r(k)], k \geq 0\), are the supports of the \( k \)-level masks associated with a nonstationary subdivision scheme, the support of the basic limit function \( \phi^0_k \) is proved to be included in

\[
[L, R] := \left[ \sum_{k=0}^{\infty} 2^{-k-1} l(k), \sum_{k=0}^{\infty} 2^{-k-1} r(k) \right].
\]

Since \( c^0(z) = a(z) \), hence \( l(0) = l_a \) and \( r(0) = r_a \). For \( k > 0 \) we have \( l(k) = l_b \) and \( r(k) = r_b \). Thus, the left endpoint is

\[
L = \sum_{k=0}^{\infty} 2^{-k-1} l(k) = \frac{l_a}{2} + \frac{l_b}{2} \sum_{k=1}^{\infty} 2^{-k} = \frac{l_a + l_b}{2};
\]

\[
\text{Fig. 4. Result after 5 steps of the subdivisions } a_3(z) \text{ (left) and } d(z) \text{ (right).}
\]
Fig. 5. Linear $C^0$-splines on $[-1, 1]$ (left), quadratic $C^1$-spline on $[-1, 2]$ (right).

Fig. 6. $C^1$ limit function with support $[-1, 3/2]$ with $\lambda_0 = 0$, $\lambda_k = 2^{-1/k^2}$, $c^k(z)$ as in (16).

while the right endpoint is

$$R = \sum_{k=0}^{\infty} 2^{-k-1} r(k) = \frac{r_a}{2} + \frac{r_b}{2} \sum_{k=1}^{\infty} 2^{-k} = \frac{r_a + r_b}{2}. \quad \Box$$

**Remark 7.** The previous Theorem states that, indeed, the limit function has the same smoothness of $\phi_b$ but with a shorted support. This is an improvement with respect to the ratio smoothness/support size no longer true if we start the process with a mask different from $a$ that is if we consider the nonstationary subdivision scheme $\{c^k(z)\}_{k \geq m}$ and $m > 0$.

**Fig. 6** shows the result obtained when using a non stationary convex combination of a linear and a quadratic B-splines shown in **Fig. 5**. The used convex combination parameters are listed in the caption.

### 5. Bivariate affine combination

This section is a first step in the analysis of multivariate affine combination: a bivariate stationary affine combination of masks is considered together with a proper ‘factorization’ of the combined symbol.

For $z = (z_1, z_2) \in \mathbb{C} \setminus \{(0, 0)\}$ and $F$ a (finite) set of bivariate directions containing $e_0 = (0, 0)$, let $a_e(z)$, $e \in F$, symbols of bivariate subdivision schemes. We consider the combined scheme

$$c(z) := \sum_{e \in F} \lambda_e z^e a_e(z), \quad \text{where} \quad \sum_{e \in F} \lambda_e = 1. \quad (20)$$
Similarly to the univariate case, for $F = \{e_0 := (0, 0), e_1 := (1, 0), e_2 := (0, 1), e_3 := (1, 1)\}$ (that is in case $F \equiv E$), whenever $a_e(z) = a(z)$ and $\lambda_e = (1/4)$, for all $e \in F$, the convex combination reduces to a bivariate average process since (20) becomes

$$c(z) := \frac{(1+z_1)(1+z_2)}{4} a(z).$$

(21)

The above average turns out to be a bivariate smoothing process as stated in the following Lemma where the norm of a subdivision operator based on a finitely supported mask $a$ is $\|S_a\| = \sup_{a \in \mathbb{Z}^2} \sum_{\beta \in \mathbb{Z}^2} |a_{a-2\beta}|$.

**Lemma 8.** Let the subdivision scheme $S_a$ with symbol $a(z)$ be $C^\ell$. Then, the subdivision scheme with symbol $c(z) := ((1+z_1)/2)((1+z_2)/2)a(z)$ is $C^{\ell+1}$. Furthermore, assuming the norm of the subdivision operator $S_a$ satisfies $\|S_a\| \leq K$, it results $\|S_c\| \leq K$ as well.

**Proof.** Consider the divided difference schemes in $z_1$ and $z_2$ directions having symbols $((1+z_i)/2)c(z)$, $i = 2, 1$, respectively. Since the subdivision scheme associated with $c(z)$ is $C^\ell$ and multiplication by $((1+z_i)/2)$ correspond to a directional convolution with $\chi_{[0,1]}$ the first part of the claim easily follows. Concerning the norm of subdivision operator, the claim follows by writing it in term of the mask elements as $\|S_a\| = \sup_{a \in \mathbb{Z}^2} \sum_{\beta \in \mathbb{Z}^2} |a_{a-2\beta}|$. Now, since with $e_1 = (1, 0)$, $e_2 = (0, 1)$, $e_3 = (1, 1)$ we have

$$\sup_{a \in \mathbb{Z}^2} \sum_{\beta \in \mathbb{Z}^2} |c_{a-2\beta}| = \sup_{a \in \mathbb{Z}^2} \sum_{\beta \in \mathbb{Z}^2} \left|a_{a-2\beta} + a_{a-e_1-2\beta} + a_{a-e_2-2\beta} + a_{a-e_3-2\beta}\right| 4 < 1$$

so that the last part of the claim follows too. \qed

A convex combination of Box-splines subdivision masks is used in [1] to get Box-spline like linear independent functions on criss-cross triangulations. There, the authors investigated a combined scheme of type

$$(1 - \gamma)a(z) + \gamma b(z)$$

with $a(z)$ and $b(z)$ symbols of Box-splines on criss-cross triangulations and $0 < \gamma < 1$ a real parameter. This convex combination turns out to generate linear independent functions even though $b(z)$ generates linear dependent ones. In terms of (20) it corresponds to the choices

$$F = \{e_0, e_3\}, \ a_{e_0}(z) = B_{1111}(z) \quad a_{e_3}(z) = \chi_{[0,1]^2}(z), \ \lambda_{e_0} = \gamma, \ \lambda_{e_3} = (1 - \gamma),$$

$$F = \{e_3, 2e_3\}, \ a_{e_3}(z) = B_{2211}(z) \quad a_{2e_3}(z) = B_{22}(z), \ \lambda_{e_3} = \gamma, \ \lambda_{2e_3} = (1 - \gamma).$$

Other examples of bivariate convex combination of subdivision masks can be taken from [3]. For example the smoothed Courant elements defined in [3, Section 1] uses the set $F := E \cup \{(-1,0), (0,-1), (-1,-1)\}$, the coefficients

$$\lambda_{(0,0)} = 1 - \mu, \ \lambda_e = \frac{\mu}{6}, \ e \in F \setminus (0,0)$$

and $a_e(z) = (1/2)z_1^{-1}z_2^{-1}(1+z_1)(1+z_2)(1+z_1z_2)$, for all $e \in F$.

In the general situation, the idea is again to use the combination parameters to get a combined subdivision scheme with enhanced properties. One possibility is to set the combination parameters for improving the smoothness of the limit function or, in the simplest situation, to get a convergent scheme from non-convergent ones. This means to extract an extra smoothing factor from $c(z)$. Note that in the bivariate case the smoothing factor may be a matrix factor as stated in the following Theorem.
Theorem 9. Let $a_e(z)$ such that $a_e(1, 1) = 4$ for all $e \in F$. Furthermore, let $a_e((-1, -1)^f) \neq 0$ for some $e \in F$ and $f \in E$ where $E := \{e_1, e_2, e_3\}$. Assuming the linear system

$$
\begin{align*}
\sum_{e \in F'} \lambda_e \left( (-1, 1)^e a_e(-1, 1) - a_{e_0}(-1, 1) \right) &= -a_{e_0}(-1, 1) \\
\sum_{e \in F'} \lambda_e \left( (1, -1)^e a_e(1, -1) - a_{e_0}(1, -1) \right) &= -a_{e_0}(1, -1) \\
\sum_{e \in F'} \lambda_e \left( (-1, -1)^e a_e(-1, -1) - a_{e_0}(-1, -1) \right) &= -a_{e_0}(-1, -1)
\end{align*}
$$

(22)

is solvable (with respect to the coefficients $\lambda_e, e \in F$), there exists $c(z)$ as in (20) and a difference matrix symbol $B(z) := \begin{pmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{pmatrix}$ satisfying

$$
D(z - 1)c(z) = B(z)D(z^2 - 1), \quad D(z - 1) := \begin{pmatrix} z_1 - 1 & 0 \\ 0 & z_2 - 1 \end{pmatrix}.
$$

Proof. Let us start by writing $c(z)$ as depending only on coefficients $\lambda_e, e \in F'$ where $F' := F \setminus \{(0, 0)\}$

$$
c(z) = a_{e_0}(z) + \sum_{e \in F'} \lambda_e \left( z^e a_e(z) - a_{e_0}(z) \right), \quad e_0 := (0, 0).
$$

(23)

Since $c(1, 1) = 4$, we set $\lambda_e, e \in F'$ in order $c(z)$ also satisfies $c((-1, -1)^f) = 0$ for all $f \in E'$. This is possible by solving the linear system (22). Doing so, $(z_i - 1)c(z), i = 1, 2$ can be decomposed (see [6]) into

$$(z_i - 1)c(z_1, z_2) = b_{11}(z_1, z_2)(z_1^2 - 1) + b_{12}(z_1, z_2)(z_2^2 - 1), \quad i = 1, 2,
$$

and the claim follows. □

As an example, let us consider 5 symbols associated to convergent and non-convergent subdivision schemes (for 3 of them the associated refinable functions are discontinuous). These are

$$
\begin{align*}
a_{e_0}(z) &= \frac{(1 + z_1)(1 + z_2)(z_1^{-1} + 2 + z_1)}{4}, \\
a_{e_1}(z) &= \frac{(1 + z_1)(1 + z_2)z_1^{-1}(z_2^{-1} + 2 + z_2)}{4}, \\
a_{e_2}(z) &= z_2^{-1}(1 + z_1)(1 + z_2), \\
a_{e_3}(z) &= \frac{z_1^{-1}z_2^{-1}(1 + z_1)(1 + z_2) \left( z_1(1 + z_2^{-1}) + z_2^2(1 + z_2) \right)}{4}, \\
a_{e_4}(z) &= \frac{z_1^{-1}z_2^{-2} \left( z_1^{-1}z_2^{-1} + 2 + z_1z_2 \right) (1 + z_2)(1 + z_2)}{4}.
\end{align*}
$$

For $F := \{e_0, e_1, e_2, e_3, e_4 := (1, 2)\}$ we consider the convex combination

$$
c(z) = \sum_{e \in F} \lambda_e z^e a_e(z)
$$

that is $c(z) = ((1 + z_1)(1 + z_2)/4)c(z)$ where

$$
c(z) = \lambda_{e_0}(z_1^{-1} + 2 + z_1) + \lambda_{e_1}(z_2^{-1} + 2 + z_2) + \lambda_{e_2}4 + \lambda_{e_3} \left( z_1(1 + z_2^{-1}) + z_2^2(1 + z_2) \right) + \lambda_{e_4} \left( z_1^{-1}z_2^{-1} + 2 + z_1z_2 \right).
$$
Next we use Theorem 9 and obtain that the choice

$$\lambda_{e_0} = \frac{3}{10}, \lambda_{e_1} = \frac{3}{10}, \lambda_{e_2} = \frac{-3}{10}, \lambda_{e_3} = \frac{2}{5}, \lambda_{e_4} = \frac{3}{10},$$

generates the symbol

$$\tilde{c}(z_1, z_2) = \frac{1}{80} \left( z_2^{-1} (31z_1^{-1} + 31 + 9z_1) + (31z_1^{-1} + 31 + 49z_1 + 9z_1^2) \right) +$$

$$\frac{1}{80} z_2 (31 + 31z_1 + 9z_1^2),$$

such that

$$\begin{pmatrix} 1 - z_1 & 0 \\ 0 & 1 - z_2 \end{pmatrix} \tilde{c}(z_1, z_2) = \frac{1}{80} \begin{pmatrix} b_{11}(z_1, z_2) & b_{12}(z_1, z_2) \\ b_{21}(z_1, z_2) & b_{22}(z_1, z_2) \end{pmatrix} \begin{pmatrix} 1 - z_1^2 & 0 \\ 0 & 1 - z_2^2 \end{pmatrix}$$

where

$$b_{11}(z_1, z_2) = (z_2^{-1} + 1) \left( 18z_1 + 31 + 31z_1^{-1} \right),$$

$$b_{12}(z_1, z_2) = -31 + 13z_1^2 + 18z_1^3,$$

$$b_{21}(z_1, z_2) = 0,$$

$$b_{22}(z_1, z_2) = 31 + 31z_1 + 18z_1 - 1^2 + 18z_1z_2^{-1} + 31z_2^{-1} + 31z_1^{-1}z_2^{-1}.$$
Proof. The scheme $c(z)$ is $C^1$ if the two divided difference schemes $(2c(z)/(1 + z_i)) = ((1 + z_j)/2)c(z)$, $i, j = 1, 2, i \neq j$ are convergent. This is equivalent to the fact that the difference of divided difference schemes $((1 + z_i)/2)B(z)$, $i = 1, 2$ are contractive a fact that we easily get from the contractivity of the mask $B$ and by the use of Lemma 8. □

References