Fuzzy topological properties and hereditariness

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Abstract

Some known compactness notions and separation axioms already given in Chang-fuzzy topological spaces are extended to a more general context of categories where \( I \)-topological spaces on arbitrary \( I \)-sets are defined. The invariance under morphisms in these categories and the usual hereditary conditions of the considered topological properties in such a context are investigated.

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1. Introduction

Fuzzy compactness axioms have been defined in many different ways, since Chang [1] first defined fuzzy topological spaces on a set \( X \), now called \( I \)-topological spaces, and then introduced the notion of fuzzy compactness. Also a lot of fuzzy separation axioms have been defined that generalize the classical ones. These definitions have been changed many times accordingly to the properties one would expect or to the use one would make of them.

The invariance under fuzzy homeomorphisms, now called \( I \)-homeomorphisms, and the hereditary conditions have been successfully tested within the context of the category \( I-\text{TOP} \) of Chang-fuzzy topological spaces: not satisfactory results have been obtained in extending these requirements to more general categories including \( I \)-topological spaces on non-crisp fuzzy set. As an example Sarkar [20] showed that Chang-fuzzy compactness is not hereditary with respect to closed fuzzy, e.g. non-necessarily crisp, sets. Also Chaudhury and Das verified that the hereditariness of some fuzzy separation axioms fails to be true in a category of \( I \)-topological spaces on fuzzy sets.

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Nevertheless, questions concerning invariance and hereditariness have usually not been asked in a general context standing the difficulties arising in defining a topological structure on an arbitrary fuzzy set.

In many cases topological conditions, such as compactness, have been referred to fuzzy subsets in a fuzzy topological space rather than to fuzzy topological spaces in order to meet some advantages when dealing with hereditary conditions and something else. Such an approach is discussed and formalized in [3].

Our approach to the above problems in the present paper is based on a presentation of some categories of $I$-topological spaces on a fuzzy set given in [10] where also the possibility of considering induced $I$-topologies on any fuzzy subset is provided. We will need to use the properties of powerset operators given in [5].

The results in Sections 4 and 5 show that the most unsuitable category with respect to the investigation of hereditary conditions is the one used till now to this purpose (with minor exceptions: see e.g. [14,17]), where the induced $I$-topology on a fuzzy subset $M$ of a fuzzy set $Y$ by an $I$-topology $\delta$ on $Y$ is obtained by intersecting $M$ with all the open fuzzy sets in $\delta$, i.e. $\{A \land M \mid A \in \delta\}$.

More promising results are fulfilled in other categories.

2. Preliminaries

In the present paper, we shall only consider fuzzy sets as functions from a set $X$ to the lattice $I = [0, 1]$, namely $I$-sets, according to the definition of Zadeh [21], and we shall usually denote them by capital letters.

If $M$ is a subset of $X$ and $0 \leq \lambda \leq 1$, we shall denote by $\lambda_M$, or sometimes by $\hat{\lambda}_M$, the fuzzy set on $X$ that maps each element of $M$ to $\lambda$ and the remaining elements to 0.

The characteristic function of $M$, $1_M$, will be usually denoted by $M$ and called crisp fuzzy subset of $X$; in fact we identify the function $1_M$ with the subset $M$ of $X$, so we can write $M \in I^X$.

We shall consider the usual lattice structure on $I^X$, which is a completely distributive complete lattice with an order reversing involution $c: I^X \rightarrow I^X$, shortly o.r.i., defined by $c(a) = X - a$, $a \in I^X$; such a structure is also called fuzz in [6] and Hutton algebra in [18,19].

For any given $Y \in I^X$, we shall consider the powerset of $Y$ to be either one of the fuzzes, intervals in $I^X$, $\mathcal{S}_Y = [0, Y]$ and $\mathcal{E}_Y = [Y \land c(Y), Y]$, depending on the categorical context which will be clear later, with o.r.i. defined, respectively, by $c_Y(A) = Y - A$, $A \in \mathcal{S}_Y$, and $e_Y(A) = c(A) \land Y$, $A \in \mathcal{E}_Y$, (see [10]). More generally, we shall often denote by $\mathcal{F}_Y$ each one of the fuzzes $\mathcal{S}_Y$ and $\mathcal{E}_Y$; the minimum in $\mathcal{F}_Y$ will be usually denoted by $O$ or by $O_{\mathcal{F}_Y}$ when necessary.

The support in $\mathcal{F}_Y$ of an element $A \in \mathcal{F}_Y$ is the subset of $X$, $A_* = \{x \in X \mid A(x) > O_{\mathcal{F}_Y}(x)\}$.

Given any $A \in I^X$ we shall denote $A_x = \{x \in X \mid A(x) > \alpha\}$ the $\alpha$-level set of $A$, $0 \leq \alpha \leq 1$. $A_0$ is the support of $A$ and coincides with the support $A_*$ of $A$ in $\mathcal{S}_Y$, if $Y$ is any fuzzy set on $X$ including $A$.

The restriction of $A$ to $S$ is the fuzzy set $A|_S \in I^S$ such that $A|_S(s) = A(s)$ if $s \in X \cap S$, $A|_S(s) = 0$, otherwise.

We note that for any subset $Y$ of the set $X$, both the fuzzes $\mathcal{S}_Y$ and $\mathcal{E}_Y$ can be identified, via restriction, with the fuzz $I^Y$. 
If \( A, B, C \in I^X \), \( A \leq B \), we shall consider the lifting \( A \div B \in I^X \) of \( A \) from \( B \) and the compression \( A \cdot C \in I^X \) of \( A \) into \( C \) defined, respectively, for any \( x \in X \), by (see [4,10])

\[
(A \div B)(x) = \begin{cases} \frac{A(x)}{B(x)} & \text{if } B(x) \neq 0, \\ 0 & \text{if } B(x) = 0, \end{cases} \quad (A \cdot C)(x) = A(x) \cdot C(x).
\]

If \( Y \in I^X \), \( A, B \in \mathcal{F}_Y \), \( A \leq B \), we shall say that \( A \) is a bold fuzzy subset of \( B \) in \( \mathcal{F}_Y \) if \( A|_A = B|_A \) (see [10] where only the case \( Y = X \) is considered). Bold fuzzy subsets were called maximal in [2]. The bold extension of \( A \) in \( B \) is the fuzzy set \( B \land A \).

If \( Y \in I^X \), a fuzzy point in the fuzz \( \mathcal{F}_Y \) is a fuzzy set \( P \in \mathcal{F}_Y \) whose support in \( \mathcal{F}_Y \) is a singleton \( \{x\} \); \( x \) is the support and \( \lambda = P(x) \) is the value of the fuzzy point, which will be usually denoted by \( \lambda_x \) according to the above notation.

For the definition of \( I \)-topological spaces and subspaces and for the related categories and powerset operators we refer to [5,10]. We only summarize some fundamentals, after saying that the preliminary notions given above are recalled in the same way as in [5], with a slight modification which tends to meet, as all along the present paper, the terminology stated in [11]. Indeed, we shall need to merge notation of [5,10] and [11] in order to distinguish different categories related to the use of different powersets and powerset operators, some of which are not considered in any chapter of [11].

A fuzz function \( \phi^- : \mathcal{F} \to \mathcal{G} \) from a fuzz \( \mathcal{F} \) to a fuzz \( \mathcal{G} \) is a complete \( \lor \)-semilattice morphism such that its right adjoint \( \phi^- : \mathcal{G} \to \mathcal{F} \) defined by \( \phi^- (B) = \lor \{ A \in \mathcal{F} \mid \phi^- (A) \leq B \} \), \( B \in \mathcal{G} \), preserves the o.r.i.'s (see also [6]). \( \phi^- \) is a fuzz morphism if it is a complete-lattice morphism and preserves the o.r.i.'s.

Triples \((X,Y,\mathcal{F}_Y), Y \in I^X\), are objects of the so-called ground categories \( I \)-\( \text{SET}_d \), \( I \)-\( \text{SET}_\varepsilon \), \( I \)-\( \text{SET}_\gamma \) whose morphisms, from \((X,Y,\mathcal{F}_Y)\) to \((T,Z,\mathcal{F}_Z)\), are all the functions \( f \) from \( X \) to \( T \) such that \( Y \leq Z \circ f \) in \( I \)-\( \text{SET}_d \), all the fuzz functions from \( \varepsilon_Y \) to \( \varepsilon_Z \) in \( I \)-\( \text{SET}_\varepsilon \) and all the functions from \( Y_0 \) to \( Z_0 \) in \( I \)-\( \text{SET}_\gamma \).

Any morphism \( f : X \to T \) in \( I \)-\( \text{SET}_d \) determines the forward, \( f^-_d : \mathcal{F}_Y \to \mathcal{F}_Z \), and the backward, \( f^>_d : \mathcal{F}_Z \to \mathcal{F}_Y \), powerset operators defined by setting, for every \( A \in \mathcal{F}_Y \) and \( B \in \mathcal{F}_Z \), \( x \in X \) and \( t \in T \),

\[
f^>_d (A)(t) = \lor \{ A(x) \mid x \in X, f(x) = t \} \quad \text{and} \quad f^-_d (B)(x) = B(f(x)) \land Y(x),
\]

also the equality \( f^>_d (B) = \lor \{ A \in \mathcal{F}_Y \mid f^>_d (A) \leq B \} \) holds.

The forward powerset operator \( f^-_d \) defined by a morphism \( \phi^- \) in \( I \)-\( \text{SET}_\varepsilon \) is \( \phi^- \) itself and the corresponding backward powerset operator \( f^>_d \) is the function \( f^>_d \) defined above.

Every morphism \( f : Y_0 \to Z_0 \) in \( I \)-\( \text{SET}_\gamma \) determines the powerset operators \( f^-_\gamma : \mathcal{F}_Y \to \mathcal{F}_Z \) and \( f^>_\gamma : \mathcal{F}_Z \to \mathcal{F}_Y \) defined, \( \forall U \in \mathcal{F}_Y \), \( \forall V \in \mathcal{F}_Z \), by

\[
f^-_\gamma (U) = (f^-((U \div Y)|_{Y_0}))|_Z \land Z \quad \text{and} \quad f^>_\gamma (V) = (f^>((V \div Z)|_{Z_0}))|_X \land Y,
\]

where \( f^- : I^{Y_0} \to I^{Z_0} \) and \( f^- : I^{Z_0} \to I^{Y_0} \) are the usual fuzzy powerset operators defined, \( \forall A \in I^{Y_0}, \forall t \in Z_0 \), \( \forall B \in I^{Z_0}, \forall \lambda \in Y_0 \) by

\[
f^- (A)(t) = \lor \{ A(x) \mid x \in Y_0, f(x) = t \} \quad \text{and} \quad f^- (B)(x) = B(f(x)).
\]

The equality \( f^-_\gamma (V) = \lor \{ U \in \mathcal{F}_Y \mid f^-_\gamma (U) \leq V \} \) holds in this case too.
If $\mathcal{F} \in \{\mathcal{A}, \mathcal{C}, \mathcal{E}, \mathcal{G}\}$ we say that an $\mathcal{F}$-topological space on a carrier $Y \subseteq X$ is a quadruple $(X,Y,\mathcal{F}_Y,\delta)$ where $\mathcal{F}_Y$ is either $\mathcal{F}_Y$ (in case $\mathcal{F} \in \{\mathcal{A}, \mathcal{C}, \mathcal{G}\}$) or $\mathcal{E}_Y$ (in case $\mathcal{F} = \mathcal{E}$) and the $\mathcal{F}$-topology $\delta$ is a subset of $\mathcal{F}_Y$ closed under finite meets and arbitrary joins in $\mathcal{F}_Y$. $\mathcal{F}$-topological space are nothing else than $I$-topological spaces in case of a crisp carrier $Y \subseteq X$.

$\mathcal{F}$-topological spaces are the objects of the category $I$-$\text{TOP}_\mathcal{F}$; then they are called $\mathcal{A}$-topological spaces ($\mathcal{C}$-topological spaces, $\mathcal{E}$-topological spaces, $\mathcal{G}$-topological spaces, respectively) when they are considered as objects of the category $I$-$\text{TOP}_\mathcal{A}$ ($I$-$\text{TOP}_\mathcal{C}$, $I$-$\text{TOP}_\mathcal{E}$, $I$-$\text{TOP}_\mathcal{G}$, respectively).

The morphisms of such categories are those morphisms of the underlying ground categories $I$-$\text{SET}_\mathcal{A}$ (which corresponds to $I$-$\text{TOP}_\mathcal{A}$ and $I$-$\text{TOP}_\mathcal{C}$ as well), $I$-$\text{SET}_\mathcal{E}$ (which corresponds to $I$-$\text{TOP}_\mathcal{E}$) and $I$-$\text{SET}_\mathcal{G}$ (which corresponds to $I$-$\text{TOP}_\mathcal{G}$) that satisfy a suitable $\mathcal{F}$-continuity (namely $\mathcal{A}$-continuity, $\mathcal{C}$-continuity, $\mathcal{E}$-continuity and $\mathcal{G}$-continuity, respectively) condition: in fact the backward powerset operator related to a morphism in $I$-$\text{TOP}_\mathcal{A}$, $I$-$\text{TOP}_\mathcal{E}$, $I$-$\text{TOP}_\mathcal{G}$ must preserve open fuzzy sets and the backward powerset operator of a morphism in $I$-$\text{TOP}_\mathcal{C}$ must preserve closed fuzzy sets.

If $(X,Y,\mathcal{F}_Y,\delta)$ is any $\mathcal{A}$-topological (respectively, $\mathcal{C}$-topological, $\mathcal{E}$-topological, $\mathcal{G}$-topological) space and $M$ is a fuzzy set such that $M \in \mathcal{F}_Y$, then the $\mathcal{A}$-induced (respectively, $\mathcal{C}$-induced, $\mathcal{E}$-induced, $\mathcal{G}$-induced) topology on the carrier $M$, $\delta^\mathcal{A}_M$ (respectively $\delta^\mathcal{C}_M$, $\delta^\mathcal{E}_M$, $\delta^\mathcal{G}_M$), is defined by

\[
\begin{align*}
\delta^\mathcal{C}_M &= \{A \cap M \mid A \in \delta\}, \\
\delta^\mathcal{E}_M &= \{M \setminus (M \setminus (Y - A)) \mid A \in \delta\}, \\
\delta^\mathcal{G}_M &= \{A \setminus M \cup (X \setminus M) \setminus M \mid A \in \delta\}, \\
\delta^\mathcal{G}_M &= \{(A \setminus Y) \cdot M \mid A \in \delta\}.
\end{align*}
\]

We remark that only the condition $M_0 \leq Y_0$, weaker than $M \in \mathcal{F}_Y$, is needed in order to define $\delta^\mathcal{G}_M$.

It is proved in [10,5] that isomorphisms in the ground categories $I$-$\text{SET}_\mathcal{A}$, $I$-$\text{SET}_\mathcal{C}$, $I$-$\text{SET}_\mathcal{E}$ are morphisms whose forward powerset operator is a fuzz isomorphism (and the backward powerset operator is its inverse fuzz isomorphism).

It was also proved in [5] that the forward powerset operator associated to every morphism in each one of the considered ground categories maps fuzzy points into fuzzy points and that a similar statement holds for the backward powerset operator provided that the corresponding forward powerset operator is injective (see [5, Proposition 5.6]).

We recall that a family of fuzzy sets $\{A_j \mid j \in J\}$ in $\mathcal{F}$ is a cover of $A \in \mathcal{F}$, namely an $A$-cover, if $\bigvee \{A_j \mid j \in J\} \geq A$ and it is a shading of $A$, namely an $A$-shading, if $\forall x \in X \ j \in J$ exists such that $A_j(x) \geq A(x)$.

Clearly every $A$-shading is an $A$-cover and the converse is true if $A_j$ is crisp for all but finitely many $j \in J$.

It is proved in [5] that the backward powerset operator $x^\frown$ related to every morphism $x : (X,Y,\mathcal{F}_Y) \to (T,Z,\mathcal{F}_Z)$ in any one of the given ground categories maps $B$-covers and $B$-shadings, $B \in \mathcal{F}_Z$, into $x^\frown(B)$-covers and $x^\frown(B)$-shadings respectively; conversely, $x^\frown$ maps an $A$-cover, $A \in \mathcal{F}_Y$, into an $x^\frown(A)$-cover and it maps an $A$-shading into an $x^\frown(A)$-shading if $x^\frown$ is injective.

Also, with the above notation, the complemented backward powerset operator $x^\frown : \mathcal{F}_Z \to \mathcal{F}_Y$ defined by $x^\frown = c_Y \circ x^\frown \circ c_Z$ maps $B$-covers and $B$-shadings into $x^\frown(B)$-covers and $x^\frown(B)$-shadings, respectively (see [5]).
3. Some compactness conditions

In this section we restate, in the context of the categories $I$-$TOP_A$, $I$-$TOP_C$, $I$-$TOP_G$ and $I$-$TOP_E$ described in Section 2, some well-known definitions of compactness axioms already considered for $I$-topological spaces.

Then we review the behaviour of such axioms toward morphisms of the considered categories.

Each definition is referred to previous papers where the corresponding definitions can be found in the context of the category $I$-$TOP$ of Chang-fuzzy topological spaces.

So let $(X, Y, \mathcal{F}_Y, \delta)$ be an $\mathcal{F}$-topological space.

**Definition 3.1** (see Chang [1]). $(X, Y, \mathcal{F}_Y, \delta)$ is *cover-compact* if every open cover of $Y$ has a finite subcover.

**Definition 3.2** (see Gantner et al. [8]). $(X, Y, \mathcal{F}_Y, \delta)$ is *shading-compact* if every open shading of $Y$ has a finite subshading.

**Definition 3.3** (see Lowen [16]). $(X, Y, \mathcal{F}_Y, \delta)$ is *weak-compact* if for every open cover of $Y$ and for every $\varepsilon > 0$ a finite subfamily exists which is a cover of $Y - (\varepsilon_X \wedge (Y - O))$.

**Definition 3.4** (see Lowen [16]). $(X, Y, \mathcal{F}_Y, \delta)$ is *sub-compact* if for every open cover of $Y$ and for every $P \in \mathcal{F}_Y$ such that $P(x) < Y(x)$, $\forall x \in Y_s$, a finite subfamily exists which is a cover of $P$.

**Definition 3.5** (see Hutton [13]). $(X, Y, \mathcal{F}_Y, \delta)$ is *strong-compact* if for every closed fuzzy subset $P$ and for every open $P$-cover a finite subcover of $P$ exists.

Of course, the above definitions can be referred to every fuzzy subset $M \in \mathcal{F}_Y$, by requiring that the related subspace on $M$, in the considered category (see Section 2), satisfies the given conditions.

The following relationships between the above compactness axioms can be easily verified:

- strong-comp. $\Rightarrow$ cover-comp. $\Rightarrow$ shading-comp.,
- cover-comp. $\Rightarrow$ sub-comp. $\Rightarrow$ weak-comp.

**Lemma 3.6.** Every morphism $\pi : (X, Y, \mathcal{F}_Y, \delta) \rightarrow (T, Z, \mathcal{F}_Z, \tau)$ in any one of the categories $I$-$TOP_A$, $I$-$TOP_C$, $I$-$TOP_G$, $I$-$TOP_E$ induces a morphism $\pi' : (X, Y, \mathcal{F}_Y, \delta) \rightarrow (T, \mathcal{F}_Z, \mathcal{F}_Z^{-}(Y), \tau_{Z^{-}(Y)})$ such that $\pi'^{-}(Y) = \pi^{-}(Y)$, where $\tau_{Z^{-}(Y)}$, $\pi \in \{a, c, e, g\}$, is the induced topology in the considered category.

**Proof.** If $f : X \rightarrow T$, with $Y \subseteq Z \circ f$, is the function that determines the morphism $\pi$ either in $I$-$TOP_A$ or in $I$-$TOP_C$, then clearly $Y \subseteq \pi^{-}(Y) \circ f$, hence the same function determines a morphism $\pi' : (X, Y, \mathcal{F}_Y) \rightarrow (T, \mathcal{F}_Z^{-}(Y), \mathcal{F}_Z^{-}(Y))$ in $I$-$SET_A$ whose backward powerset operator is the restriction of $\mathcal{F}_Z^{-}$ to $\mathcal{F}_Z^{-}(Y)$; then the required $\mathcal{F}$-continuity or $G$-continuity condition is easily checked.

If $f : Y_0 \rightarrow Z_0$ determines the given morphism $\pi$ in the category $I$-$TOP_G$, then clearly $f(Y_0) = (\pi^{-}(Y))_0$ and the range reduction $f_0 = Y_0 \rightarrow (\pi^{-}(Y))_0$ determines the required morphism in $I$-$TOP_G$. 
In fact, it follows from $C \in t_\tau^\delta(Y)$, $B \in \tau$ and $C = (B \div Z) \cdot x^-(Y)$ that $(C \div x^-(Y))|_{f(y_0)} = (B \div Z)|_{f(y_0)}$ therence $(f_0)_\tau^\delta(C) = f_\tau^\delta(B) \in \delta$.

Eventually, let $\phi^- : \delta_Y \to \delta_Z$ be the fuzz function that determines $x$ in $I\text{-TOP}_\delta$. Then $\phi^- : \delta_Z \to \delta_Y$ is a fuzz morphism (see [5, Proposition 3.1]).

It is easily seen that $\phi^- \cdot (\phi^-(Y)) = Y$ and $\phi^-((\phi^-(Y)) \land c(\phi^-(Y))) = Y \land c(Y)$. Hence $\phi^-(Y) \leq B_1 \leq Z \Rightarrow \phi^-(B_1) = Y, Z \land c(Z) \leq B_2 \leq \phi^-(Y) \land c(\phi^-(Y)) \Rightarrow \phi^-(B_2) \leq Y \land c(Y)$, and consequently $\phi^-$ induces by restriction a fuzz morphism $\phi^- : \delta_Y \to \delta_Z$. Note, in particular, that $ec_{\phi^-(Y)}(B) = ec_Z(B) \land \phi^-(Y)$, if $B \in \delta_{\phi^-(Y)}$ (see [10, Remark 2.3]), hence $\forall B \in \delta_{\phi^-(Y)} \phi^-((ec_{\phi^-(Y)}(B)) = \phi^- (ec_Z(B) \land \phi^-(Y)) = \phi^- (ec_Z(B)) \land \phi^- (c(\phi^-(Y))) = ec_Y (\phi^-(B)) \land Y = \phi^- (B) \land Y = ec_Y (\phi^-(B)) = ec_Y (\phi^- (B))$.

Let $\phi^- : \delta_Y \to \delta_{\phi^-(Y)}$ be the left adjoint of $\phi^-$. Clearly $\phi^- (Y) \leq \phi^- (Y)$. Moreover, $\phi^- (Y) = \land \{B \in \delta_{\phi^-(Y)} \mid \phi^- (B) \geq Y\} = \land \{B \in \delta_{\phi^-(Y)} \mid \phi^- (B) \geq Y\} = \{B \in \delta_Z \mid \phi^- (B) \geq Y\} = \phi^- (Y)$ hence $\phi^- (Y) = \phi^- (Y)$.

It is clear that $\phi^- : (X, Y, \delta_Y) \to (T, \phi^-(Y), \delta_{\phi^-(Y)})$ is a morphism in $I\text{-SET}_\delta$ and we can see that $\phi^-$ is $\delta$-continuous with respect to $\delta$ and $t^\delta_{\phi^-(Y)}$, In fact, let $(B \land \phi^- (Y)) \lor (\phi^- (Y) \land c(\phi^-(Y))) \in t^\delta_{\phi^-(Y)}$, with $B \in \tau$. Since $\phi^- \cdot (B \land \phi^- (Y)) \lor (\phi^- (Y) \land c(\phi^-(Y))) = \phi^- (B \land \phi^- (Y)) \lor \phi^- (\phi^- (Y) \land c(\phi^-(Y))) = \phi^- (B \land \phi^- (Y)) \lor (Y \land c(Y)) = (\phi^- (B) \land Y) \lor (Y \land c(Y)) = \phi^- (B) \lor (Y \land c(Y)) = \phi^- (B) \in \delta$. □

Proposition 3.7. Cover-compactness and shading-compactness are preserved by any morphism in $I\text{-TOP}_\delta$, $I\text{-TOP}_\delta$, $I\text{-TOP}_\delta$, $I\text{-TOP}_\delta$.

Proof. Let $\alpha : (X, Y, \delta_Y, \tau) \to (T, Z, \delta_Z, \tau)$ be any morphism in any one of the considered categories. By using Lemma 3.6 we may also assume $\alpha^- (Y) = Z$. Let $Y$ be cover-compact (shading-compact, respectively) and let $\{B_j \mid j \in J\}$ be an open cover (shading, respectively) of $Z$ in $\tau$.

Then by excluding the case of $I\text{-TOP}_\delta$, $\{x^- (B_j) \mid j \in J\}$ is an open cover (shading, respectively) of $x^- (Z) = Y$.

If $\{x^- (B_j) \mid j \in J_0\}$ is a finite subcover (shading, respectively) of $Y$ then $\{x^- (x^- (B_j)) \mid j \in J_0\}$ is a finite cover (shading, respectively) of $Z$ hence, a fortiori by the adjunction inequalities (see [5]), $\{B_j \mid j \in J_0\}$ is a finite subcover (shading, respectively) of $Z$.

If $x$ is a morphism in $I\text{-TOP}_\delta$, then $\{x^-_c (B_j) \mid j \in J\}$ is an open cover (shading, respectively) of $x^-_c (Z) = Y$.

If $\{x^-_c (B_j) \mid j \in J_0\}$ is a finite subcover (shading, respectively) of $Y$ then $\{x^- (x^-_c (B_j)) \mid j \in J\}$ is a finite cover (shading, respectively) of $x^- (Y) = Z$ hence, a fortiori by Corollary 6.4 of [5], $\{B_j \mid j \in J_0\}$ is a finite subcover (shading, respectively) of $Z$. □

The following is a trivial consequence.

Corollary 3.8. Cover-compactness and shading-compactness are invariant under isomorphisms in $I\text{-TOP}_\delta$, $I\text{-TOP}_\delta$, $I\text{-TOP}_\delta$, and $I\text{-TOP}_\delta$.

Proposition 3.9. Weak-compactness is preserved by morphisms in $I\text{-TOP}_\delta$ and $I\text{-TOP}_\delta$. 
Proof. Let \( \varphi : (X, Y, \mathcal{F}_Y, \delta) \to (T, Z, \mathcal{F}_Z, \tau) \) be a morphism in \( I\text{-TOP}_\delta \) and let \( Y \) be weak-compact. By using Lemma 3.6 we may also assume \( \varphi^{-1}(Y) = Z \).

If \( \{B_j \mid j \in J\} \) is an open \( Z \)-cover and \( \varepsilon > 0 \), then \( \{\varphi^{-1}(B_j) \mid j \in J\} \) is an open cover of \( Y \); consequently a finite subfamily \( \{\varphi^{-1}(B_j) \mid j \in J_0\} \) exists which is a cover of \( Y - (\varepsilon_X \land Y) \) hence the family \( \{\varphi^{-1}(\varphi^{-1}(B_j)) \mid j \in J_0\} \) is a cover of \( \varphi^{-1}(Y - (\varepsilon_X \land Y)) \).

Now \( \varphi^{-1}(Y - (\varepsilon_X \land Y)) = Z - (\varepsilon_T \land Z) \); in fact let \( f : X \to T \) be the function identified with the morphism \( \varphi \); then \( f(Y_0) = Z_0 \) (the proof is trivial) hence \( \varphi^{-1}(Y - (\varepsilon_X \land Y))(t) = 0 = (Z - (\varepsilon_T \land Z))(t) \) if \( t \in T - Z_0 \) and, if \( t \in Z_0, \varphi^{-1}(Y - (\varepsilon_X \land Y))(t) = \vee \{((Y(x) - \varepsilon) \lor 0 \mid x \in X, f(x) = t\} = \vee \{Y(x) \mid x \in X, f(x) = t\} - \varepsilon \lor 0 = Z(t) - (\varepsilon \land Z(t)) \).

It follows from \( \varphi^{-1}(\varphi^{-1}(B_j)) \subseteq B_j, \forall j \in J \), (see the adjunction inequalities in [5]) that \( \{B_j \mid j \in J \} \) is a cover of \( Z - (\varepsilon_T \land Z) \).

Now let us assume that \( \varphi \) is a morphism in \( I\text{-TOP}_\delta \) and let us assume the same notation and the further conditions as in the preceding case. Then we have that \( \{\varphi^{-1}(B_j) \mid j \in J\} \) is an open cover of \( Y \) and we can consider a finite subfamily \( \{\varphi^{-1}(B_j) \mid j \in J_1\} \) which is a cover of \( Y - (\varepsilon_X \land Y) \) and consequently \( \{\varphi^{-1}(\varphi^{-1}(B_j)) \mid j \in J_1\} \) is a cover of \( \varphi^{-1}(Y - (\varepsilon_X \land Y)) = Z - (\varepsilon_T \land Z) \). Once more it follows from \( \varphi^{-1}(\varphi^{-1}(B_j)) \subseteq B_j, \forall j \in J_1 \), that \( \{B_j \mid j \in J_1\} \) is a cover of \( Z - (\varepsilon_T \land Z) \). \( \square \)

As a consequence, we have the following:

Corollary 3.10. Weak-compactness is preserved by any isomorphism in \( I\text{-TOP}_\delta \) and \( I\text{-TOP}_\varepsilon \).

Example 3.1 (Weak-compactness is not preserved under isomorphisms in \( I\text{-TOP}_\delta \)). Let \( X = [0, 1] \), \( Y = 1_x \), and let \( \delta \) be the \( \delta \)-topology on the carrier \( Y \), in the fuzz \( \delta_Y = I^X \), whose non-trivial open sets are the fuzzy sets \( A' \), \( t \in (0, 1) \), defined by

\[
A'(x) = \begin{cases} 
1 & \text{if } x \in [0, t) \\
0 & \text{if } x \in [t, 1) 
\end{cases}
\]

Furthermore, let us consider the fuzzy set \( Z \) on \( X \) defined by \( Z(x) = 1 - x/2 \) and let \( \tau \) be the \( \delta \)-topology, in the fuzz \( \delta_Z = [c(Z), Z] \), whose non-trivial open sets \( B' \), \( t \in (0, 1) \), are defined by

\[
B'(x) = \begin{cases} 
1 - \frac{x}{2} & \text{if } x \in [0, t) \\
\frac{x}{2} & \text{if } x \in [t, 1) 
\end{cases}
\]

For every \( A \in \delta_X \) and for every \( x \in [0, 1) \) we put \( \varphi'^{-1}(A)(x) = (\frac{1}{2} - A(x))x + A(x) \) and so we obtain an isomorphism in the category \( I\text{-TOP}_\delta \)

\[
\phi'^{-1} : (X, Y, \delta_Y, \delta) \to (X, Z, \delta_Z, \tau),
\]

In fact, the backward powerset operator maps any fuzzy set \( B \in \delta_Z \) to the fuzzy set defined by \( \phi'^{-1}(B)(x) = (2B(x) - x)/(2(1 - x)) \).

Now it is easily seen that \((X, Y, \delta_Y, \delta)\) is not weak-compact although \((X, Z, \delta_Z, \tau)\) is.
Example 3.2 (Weak-compactness is not preserved under isomorphisms in $I$-$\text{TOP}_\mathcal{G}$). Let $X = \{1/n \mid n \in N\}$, $Y = 1_X$ and let $\delta$ be the $\mathcal{G}$-topology on the carrier $Y$ whose non-trivial open sets are the fuzzy sets $A^p$, $p \in N$, defined by

$$A^p\left(\frac{1}{n}\right) = \begin{cases} 0 & \text{if } n \geq p, \\ 1 & \text{if } n < p. \end{cases}$$

Moreover, let $Z$ be the fuzzy set on $X$ such that $Z(1/n) = 1/n$, $\forall n \in N$, and let $\tau$ be the $\mathcal{G}$-topology on the carrier $Z$ whose non-trivial open sets $B^p$, $p \in N$, are defined by

$$B^p\left(\frac{1}{n}\right) = \begin{cases} 0 & \text{if } n \geq p \\ \frac{1}{n} & \text{if } n < p. \end{cases}$$

In $I$-$\text{TOP}_\mathcal{G}$ $(X, Y, \mathcal{S}_Y, \delta)$ is isomorphic to (indeed it is the normalized $\mathcal{G}$-topological space of, see [3,10]) $(X, Z, \mathcal{S}_Z, \tau)$ by means of the identity function of $X$.

It is easy to see that $(X, Z, \mathcal{S}_Z, \tau)$ is weak-compact while $(X, Y, \mathcal{S}_Y, \delta)$ is not.

Remark. It can be proved that in $I$-$\text{TOP}_\mathcal{G}$ weak-compactness is preserved by morphisms whose domain space $(X, Y, \mathcal{S}_Y, \delta)$ has a carrier $Y$ such that $\wedge \{Y(x) \mid x \in Y_0\} > 0$. Actually all crisp fuzzy sets have this property.

We note that Definitions 3.3 and 3.4 constitute two different ways of extending to an $\mathcal{F}$-topological space, $\mathcal{F} \in \{\mathcal{A}, \mathcal{C}, \mathcal{E}, \mathcal{G}\}$, on a non-crisp carrier the notion of compactness given in [16]. Now the first way brings to bad results, since weak-compactness is a topological property neither in $I$-$\text{TOP}_\mathcal{G}$ nor in $I$-$\text{TOP}_\mathcal{G}$.

A more satisfactory behaviour in such a sense has the sub-compactness notion given by Definition 3.4 which is in any case a topological property and furthermore, it is preserved by morphisms in all the categories but $I$-$\text{TOP}_\mathcal{G}$, as the following results show.

Lemma 3.11. Let $\alpha: (X, Y, \mathcal{S}_Y) \to (T, Z, \mathcal{S}_Z)$ be a morphism in any one of the categories $I$-$\text{SET}_\mathcal{A}$, $I$-$\text{SET}_\mathcal{C}$, $I$-$\text{SET}_\mathcal{E}$, or $I$-$\text{SET}_\mathcal{G}$ such that $\alpha^{-1}(Y) = Z$ and let $Q \in \mathcal{S}_Z$ satisfy the condition $Q(t) < Z(t)$, $\forall t \in Z_0$.

Then a fuzzy set $P \in \mathcal{S}_Y$ exists such that $P(x) < Y(x)$, $\forall x \in Y_0$, and $\alpha^{-1}(P) = Q$.

Proof. Let $f : X \to T$ and $f : Y_0 \to Z_0$ be functions that determine the morphism $\alpha$ in $I$-$\text{SET}_\mathcal{A}$ and $I$-$\text{SET}_\mathcal{C}$, respectively. Of course $f(Y_0) = Z_0$ since $\alpha^{-1}(Y) = Z$.

If $\alpha$ is a morphism in $I$-$\text{SET}_\mathcal{C}$, we define $P \in \mathcal{S}_Y$ by setting $P(x) = 0$, if $x \in X - Y_0$, and $P(x) = \vee \{0, Y(x) - Z(f(x)) + Q(f(x))\}$, if $x \in Y_0$. Then clearly $P(x) < Y(x)$, $\forall x \in Y_0$.

Furthermore, $\forall t \in T - Z_0$, $\alpha^{-1}(P)(t) = 0 = Q(t)$ and $\forall t \in Z_0$ we have $\alpha^{-1}(P)(t) = \vee \{P(x) \mid x \in X, f(x) = t\} = \vee \{0, Y(x) - Z(f(x)) + Q(f(x))\} = \vee \{0, \vee \{Y(x) - Z(t) + Q(t) \mid x \in X, f(x) = t\}\} = \vee \{0, Q(t)\} = Q(t).$

If $\alpha$ is a morphism in $I$-$\text{SET}_\mathcal{C}$, we define $P \in \mathcal{S}_Y$ by setting once more $P(x) = 0$ if $x \in X - Y_0$ and assuming $\forall x \in Y_0$, $P(x) = Y(x)Q(f(x))/Z(f(x))$. Clearly $P(x) < Y(x)$, $\forall x \in Y_0$ and if $t \in T - Z_0$, $\alpha^{-1}(P)(t) = 0 = Q(t)$.

Now let us consider $t \in Z_0$; then $\alpha^{-1}(P)(t) = \vee \{P(x)Y(x) \mid x \in Y_0, f(x) = t\} \cdot Z(t) = \vee \{(1/Y(x))Y(x)Q(f(x))/Z(f(x)) \mid x \in Y_0, f(x) = t\} \cdot Z(t) = (Q(t)/Z(t))Z(t) = Q(t).$ □
Proposition 3.12. The sub-compactness condition is preserved by morphisms in the categories \( I-\text{TOP}_&, I-\text{TOP}_¥, I-\text{TOP}_¥ \).

Proof. Let \( \alpha : (X, Y, \mathcal{F}_¥, \delta) \rightarrow (T, Z, \mathcal{G}_Z, \tau) \) be a morphism in one of the categories \( I-\text{TOP}_&, I-\text{TOP}_¥, I-\text{TOP}_¥ \) and let \( (X, Y, \mathcal{F}_¥, \delta) \) be sub-compact. By Lemma 3.6 we can reduce to the case when \( \alpha^{-}(Y) = Z \).

Let us consider any open cover \( \{B_j \mid j \in J\} \) of \( Z \) in \( \tau \) and let \( Q \in \mathcal{G}_Z \) be any fuzzy set such that \( Q(t) < Z(t), \forall t \in Z_0 \).

If \( \alpha \) is \( \mathcal{F}_¥ \)-continuous or \( \mathcal{G}_¥ \)-continuous, \( \{\alpha^{-}(B_j) \mid j \in J\} \) is an open cover of \( Y \); an open cover of \( Y \) is given by \( \{\alpha^{-}(B_j) \mid j \in J\} \) when \( \alpha \) is \( \mathcal{G}_¥ \)-continuous. Now let us consider a fuzzy set \( P \in \mathcal{F}_¥ \) satisfying the conditions \( \alpha^{-}(P) = Q \) and \( P(x) < Y(x), \forall x \in Y_0 \); some finite subfamily \( \{\alpha^{-}(B_j) \mid j \in J_0\} \) covers \( P \) and consequently the finite subfamily \( \{\alpha^{-}(\alpha^{-}(B_j)) \mid j \in J_0\} \) covers \( Q = \alpha^{-}(P) \).

By the adjunction inequalities (see [5]) we conclude that the finite family \( \{B_j \mid j \in J_0\} \) covers \( Q \).

Similarly, if the finite family \( \{\alpha^{-}(B_j) \mid j \in J_1\} \) covers \( P \), then it follows from \( \alpha^{-}(\alpha^{-}(B_j)) \leq B_j \) and \( \alpha^{-}(P) = Q \) that \( \{B_j \mid j \in J_1\} \) covers \( Q \). \( \Box \)

Lemma 3.13. Let \( \alpha^{-} : (X, Y, \mathcal{F}_¥) \rightarrow (T, Z, \mathcal{G}_Z) \) be an isomorphism in \( I-\text{SET}_¥ \) and let \( Q \in \mathcal{G}_Z \), \( Q(t) < Z(t), \forall t \in Z_+ \). If \( P = \alpha^{-}(Q) \), then \( P(x) < Y(x), \forall x \in Y_+ \).

Proof. If for some \( x \in Y_+ \) we had \( P(x) = Y(x) = \lambda \), then \( \lambda \) would be a bold fuzzy point in \( P \) and in \( Y \) as well. It would follow from the bijectivity of \( \alpha^{-} \) that \( \mu = \alpha^{-}(\lambda) \in \mathcal{G}_Z \) should be a bold fuzzy point in \( \alpha^{-}(Y) = Z \) (see [5]). Hence \( \mu = Q(t) = Z(t) \) and \( t \in Z_+ \), which is impossible. \( \Box \)

Proposition 3.14. Sub-compactness is preserved under isomorphisms in any one of the categories \( I-\text{TOP}_&, I-\text{TOP}_¥, I-\text{TOP}_¥, I-\text{TOP}_¥ \).

Proof. The statement follows from Proposition 3.12 for the categories \( I-\text{TOP}_&, I-\text{TOP}_¥, I-\text{TOP}_¥ \).

As for \( I-\text{TOP}_¥ \), let us consider an isomorphism \( \alpha : (X, Y, \mathcal{F}_¥, \delta) \rightarrow (T, Z, \mathcal{G}_Z, \tau) \) and assume that \( (X, Y, \mathcal{F}_¥, \delta) \) is sub-compact. Let \( \{B_j \mid j \in J\} \) be an open cover of \( Z \) and let \( Q \in \mathcal{G}_Z \) be any fuzzy set such that \( Q(t) < Z(t), \forall t \in Z_+ \). Then \( \{\alpha^{-}(B_j) \mid j \in J\} \) is an open cover of \( Y \) and, by Lemma 3.13, a finite \( J_0 \subseteq J \) exists such that \( \{\alpha^{-}(B_j) \mid j \in J_0\} \) covers \( \alpha^{-}(Q) \). Since \( \alpha^{-} \) is bijective, it is easy to verify that \( \{B_j \mid j \in J_0\} \) covers \( Q \). \( \Box \)

Proposition 3.15. Strong-compactness is preserved by morphisms in \( I-\text{TOP}_¥ \).

Proof. Let \( \alpha \) be a morphism from \( (X, Y, \mathcal{F}_¥, \delta) \rightarrow (T, Z, \mathcal{F}_Z, \tau) \) such that \( \alpha^{-}(Y) = Z \) (use, once more, Lemma 3.6 to get this condition) and let \( (X, Y, \mathcal{F}_¥, \delta) \) be strong-compact. Let us consider any closed fuzzy set \( Q \in \mathcal{F}_Z \) and any open \( Q \)-cover \( \{B_j \mid j \in J\} \).

Then \( \{\alpha^{-}(B_j) \mid j \in J\} \) is an open cover of \( \alpha^{-}(Q) \) hence it is an open cover of \( \alpha^{-}(Q) \leq \alpha^{-}(Q) \) (see [5]) which is a closed fuzzy set in \( \mathcal{F}_¥ \).

If \( \{\alpha^{-}(B_j) \mid j \in J_0\} \) is a finite cover of \( \alpha^{-}(Q) \), then it follows from \( \alpha^{-}(\alpha^{-}(B_j)) = B_j \) and \( \alpha^{-}(\alpha^{-}(Q)) = Q \) (see once more [5]) that \( \{B_j \mid j \in J_0\} \) is a \( Q \)-cover. \( \Box \)
Proposition 3.16. Let \( \alpha \) be any morphism from \((X, Y, \mathcal{F}_Y, \delta)\) to \((T, Z, \mathcal{F}_Z, \tau)\) either in \( I\text{-}\text{TOP}_\delta \) or in \( I\text{-}\text{TOP}_\delta \) and let \((X, Y, \mathcal{F}_Y, \delta)\) be strong-compact. Then \((T, Z, \mathcal{F}_Z, \tau)\) is strong-compact, provided that we assume, in case of the category \( I\text{-}\text{TOP}_\delta \), that \( \alpha^- \) is surjective.

Proof. Let \( Q \) be any closed fuzzy set in \( \mathcal{F}_Z \) and let \( \{B_j \mid j \in J\} \) be an open \( Q \)-cover. Then \( \{\alpha^-(B_j) \mid j \in J\} \) is an open cover of the closed fuzzy set \( \alpha^-(Q) \).

If \( \{\alpha^-(B_j) \mid j \in J_0\} \) is a finite cover of \( \alpha^-(Q) \), then clearly \( \{B_j \mid j \in J_0\} \) is a finite \( Q \)-cover, since \( \alpha^- \) is surjective (use Lemma 3.6 that allows this condition in case of \( I\text{-}\text{TOP}_\delta \)). \( \square \)

Note that in \( I\text{-}\text{TOP}_\delta \) the assumption that \( \alpha^- \) is surjective is stronger than the condition \( \alpha^-(Y) = Z \), while these conditions are equivalent in \( I\text{-}\text{TOP}_\delta \) (see [5]).

Proposition 3.17. Strong-compactness is invariant under isomorphisms in \( I\text{-}\text{TOP}_\delta \), \( I\text{-}\text{TOP}_\delta \), \( I\text{-}\text{TOP}_\delta \), \( I\text{-}\text{TOP}_\delta \).

Proof. The proof is straightforward in \( I\text{-}\text{TOP}_\delta \) since the backward powerset operator related to any isomorphism preserves closed fuzzy sets and open fuzzy sets as well: in fact it is a fuzz isomorphism (see [5]).

As for the remaining categories, the statement is a trivial consequence of Propositions 3.15 and 3.16. \( \square \)

Example 3.3 (Strong-compactness is not preserved by morphisms in \( I\text{-}\text{TOP}_\delta \)). Let \( X = [0, 1], Y(x) = x \forall x \in X, T = [\frac{1}{2}, 1], Z(t) = t \forall t \in T, \delta = \{A^\lambda \mid \lambda \in [\frac{1}{4}, \frac{1}{2}]\} \cup \{0, Y\}, \tau = \{0, Z\} \cup \{B^\mu \mid \mu \in [0, \frac{1}{4}]\}\) where, \( \forall x \in X, \forall t \in T \)

\[
A^\lambda(x) = \begin{cases}
0 & \text{if } 0 \leq x \leq \lambda, \\
x - \lambda & \text{if } \lambda < x \leq \frac{1}{2}, \\
1 & \text{if } \frac{1}{2} < x \leq 1,
\end{cases}
B^\mu(t) = \begin{cases}
\mu & \text{if } t = \frac{1}{2}, \\
t & \text{if } \frac{1}{2} < t \leq 1.
\end{cases}
\]

Let \( \alpha : (X, Y, \mathcal{F}_Y) \to (T, Z, \mathcal{F}_Z) \) be the morphism in \( I\text{-}\text{SET}_\delta \) defined by the function \( f : X \to T \) that maps \( x \in X \) into \( x \lor \frac{1}{2} \); then \( \alpha^-(Y) = Z \).

Now \( (X, Y, \mathcal{F}_Y, \delta) \) is a strong-compact \( \delta \)-topological space; in fact the non-trivial closed fuzzy sets in \( \mathcal{F}_Y \), \( P^k, \frac{1}{4} \leq k \leq \frac{1}{2} \), are defined by

\[
P^k(x) = \begin{cases}
x & \text{if } 0 \leq x < k, \\
k & \text{if } k \leq x \leq \frac{1}{2}, \\
0 & \text{otherwise},
\end{cases}
\]

hence every open cover of \( P^k \) has to contain \( Y \).

Nevertheless \( (T, Z, \mathcal{F}_Z, \tau) \) is not strong-compact; in fact the fuzzy point of support \( \frac{1}{2} \) and value \( \frac{1}{4} \), which is a closed fuzzy set in \( \mathcal{F}_Z \), has an open cover \( \{B^\mu \mid 0 \leq \mu < \frac{1}{4}\} \) that admits no finite subcover.

Eventually, it is easily seen that \( \alpha \) is a morphism between these spaces in \( I\text{-}\text{TOP}_\delta \).
4. Hereditariness of compactness notions

In this section, we test the hereditary condition of the compactness notions, introduced in Section 3, with respect to closed subspaces in the categories $\mathbf{I-TOP}_\tau$, $\mathbf{I-TOP}_c$, $\mathbf{I-TOP}_c$, $\mathbf{I-TOP}_c$.

Sarkar [20] has already shown that cover-compactness and shading-compactness are not hereditary with respect to closed non-crisp subspaces $Q$ with the induced topology we denoted by $\delta^Q_0$ in Section 2. The hereditary condition of compactness notions with respect to a closed non-crisp subspace $Q$ with the induced topology $\delta^Q_0$, $\delta^Q_0$ or $\delta^Q_0$ has not been considered by anyone, to our knowledge.

The results given below will show that $\mathbf{I-TOP}_\tau$ and $\mathbf{I-TOP}_c$ are the most suitable categories if the hereditariness of the considered compactness axioms is expected.

We will test the hereditariness also for those compactness conditions that fail to be preserved by isomorphisms in some one of the considered categories (see Section 3).

**Lemma 4.1.** Let $\alpha: (X, Y, \mathcal{S}_Y) \to (T, Z, \mathcal{S}_Z)$ be a morphism in $\mathbf{I-SET}_\tau$ or in $\mathbf{I-SET}_c$ and let $\{B_j \mid j \in J\}$ be any family of fuzzy sets in $\mathcal{S}_Y$. Then $\{x^c(B_j) \mid j \in J\}$ is a cover (shading) of $Y$ if and only if $\{B_j \mid j \in J\}$ is a cover (shading) of the bold extension of $x^c(Y)$ in $Z$, namely $Z \wedge (x^c(Y))_0$.

**Proof.** First we assume that $\alpha$ is a morphism in $\mathbf{I-SET}_\tau$.

If $\{B_j \mid j \in J\}$ is a cover (shading) of $Z \wedge (x^c(Y))_0$ then $\{x^c(B_j) \mid j \in J\}$ is a cover (shading) of $x^c(Z \wedge (x^c(Y))_0)$ (see [5]) as well as a cover (shading) of $Y$ since the following holds

\[(Z \wedge (x^c(Y))_0) \wedge c_Z(Z \wedge (x^c(Y))_0) = 0\]

\[\Rightarrow x^c(Z \wedge (x^c(Y))_0) \wedge x^c(c_Z(Z \wedge (x^c(Y))_0)) = x^c(0) = 0\]

\[\Rightarrow x^c(Z \wedge (x^c(Y))_0) = c_Y(x^c(c_Z(Z \wedge (x^c(Y))_0)))\]

\[\Rightarrow x^c(Z \wedge (x^c(Y))_0) \geq x^c(x^c(Y)) \geq Y.\]

Conversely, let $\{x^c(B_j) \mid j \in J\}$ be a cover (shading) of $Y$.

If $t \notin (x^c(Y))_0$ then $(Z \wedge (x^c(Y))_0)(t) = 0 \leq B_j(t) \forall j \in J$.

So let $t \in (x^c(Y))_0$ and let us consider any $x \in Y_0$ such that the point $x^c((Y(x))_0)$ has support $t$. Then in the “cover case” we have $Y(x) = \vee \{x^c(B_j)(x) \mid j \in J\} \Rightarrow Y(x) \Rightarrow (Y \wedge (Z(t) - \vee \{B_j(t) \mid j \in J\}))$ hence $\vee \{B_j(t) \mid j \in J\} = Z(t) = (Z \wedge (x^c(Y))_0)(t)$.

Similarly in the “shading case” an index $j_0 \in J$ exists such that $Y(x) = x^c(B_j)(x) = (Y(x) - Y(x) \wedge (Z(t) - B_{j_0}(t)))$ hence $B_{j_0}(t) = Z(t) = (Z \wedge (x^c(Y))_0)(t)$.

To conclude, let us assume that $\alpha$ is a morphism in $\mathbf{I-SET}_c$.

As for the converse, let us assume that $\{x^c(B_j) \mid j \in J\}$, $B_j \in \mathcal{S}_Z$, is a cover (shading) of $Y$.

If $t \notin (x^c(Y))_0$ then $(Z \wedge (x^c(Y))_0)(t) = 0 \leq B_j(t) \forall j \in J$.

If $t \in (x^c(Y))_0$ and if we consider any $x \in Y_0$ such that the point $x^c((Y(x))_0)$ has support $t$, then in the “cover case” we obtain $Y(x) = \vee \{x^c(B_j)(x) \mid j \in J\} = (Y(x) \wedge (Z(t) - \vee \{B_j(t) \mid j \in J\}))$, hence $\vee \{B_j(t) \mid j \in J\} = Z(t) = (Z \wedge (x^c(Y))_0)(t)$.

In the “shading case” an index $j_0 \in J$ exists such that $Y(x) = x^c(B_{j_0}(t) = (B_{j_0}(t)/Z(t))Y(x)$, hence $B_{j_0}(t) = Z(t) = (Z \wedge (x^c(Y))_0)(t)$. \(\Box\)
Proposition 4.2. Cover-compactness and shading-compactness are inherited by closed subspaces in $I\text{-}\text{TOP}_\mathcal{F}$ and in $I\text{-}\text{TOP}_\mathcal{G}$.

Proof. Let $(X, Y, \mathcal{F}_Y, \delta)$ be a cover-compact (shading-compact) $\mathcal{F}$-topological space and let $Q \in \mathcal{F}_Y$ be closed, i.e. $c_Y(Q) \subseteq \delta$. Let us denote by $J$ the inclusion monomorphism in $I\text{-}\text{TOP}_\mathcal{F}$ related to the subspace $(X, Q, \mathcal{F}_Q, \delta_Q)$.

If $\{J^{-}_c(A_k) \mid k \in K\}$, $A_k \in \delta \forall k \in K$, is any open cover (shading) of $Q$ in $(X, Q, \mathcal{F}_Q, \delta_Q)$, then by Lemma 4.1 $\{A_k | k \in K\}$ is an open cover (shading) of $Y \land Q_0$, hence $\{A_k | k \in K \} \cup \{c_Y(Q)\}$ is an open cover (shading) of $Y$.

By the assumption, a finite subset $K_0 \subseteq K$ exists such that $\{A_k | k \in K_0 \} \cup \{c_Y(Q)\}$ is a cover (shading) of $Y$ and consequently $\{A_k | k \in K_0 \}$ is a cover (shading) of $Y \land Q_0$ and, by Lemma 4.1, $\{J^{-}_c(A_k) \mid k \in K_0\}$ is a finite cover (shading) of $Q$.

The proof runs in a similar way within the category $I\text{-}\text{TOP}_\mathcal{G}$. □

Proposition 4.3. Weak-compactness is closed hereditary in $I\text{-}\text{TOP}_\mathcal{F}$ and in $I\text{-}\text{TOP}_\mathcal{G}$.

Proof. Let $(X, Y, \mathcal{F}_Y, \delta)$ be weak-compact and let $Q \in \mathcal{F}_Y$ be a closed fuzzy set. Let us denote by $J$ the inclusion monomorphism of the subspace with carrier $Q$ and let $\{J^{-}_c(A_k) \mid k \in K\}$, $A_k \in \delta \forall k \in K$, be any open cover of $Q$ in such a subspace.

Then $\{A_k \mid k \in K\}$ is a cover of $Y \land Q_0$ by Lemma 4.1 hence $\{A_k \mid k \in K \} \cup \{c_Y(Q)\}$ is a cover of $Y$.

If we consider any real number $0 < \varepsilon < 1$, then we can see that $\varepsilon < Q(x)$, $\forall x \in (Q - (\varepsilon_X \land Q))_0$.

Let $\{A_k \mid k \in K_0 \} \cup \{c_Y(Q)\}$ be a finite cover of $Y - (\varepsilon_X \land Y)$.

Since $(Y - (\varepsilon_X \land Y))(x) > c_Y(Q)(x) \forall x \in (Q - (\varepsilon_X \land Q))_0$, $\{A_k \mid k \in K_0\}$ is a cover of $(Y - (\varepsilon_X \land Y)) \land (Q - (\varepsilon_X \land Q))_0$ and consequently $\{J^{-}_c(A_k) \mid k \in K_0\}$ is a cover of $J^{-}_c((Y - (\varepsilon_X \land Y)) \land (Q - (\varepsilon_X \land Q))_0)$.

Now $J^{-}_c((Y - (\varepsilon_X \land Y)) \land (Q - (\varepsilon_X \land Q))_0) \supseteq Q - (\varepsilon_X \land Q)$; in fact for any $x \in (Q - (\varepsilon_X \land Q))_0$ we have in the category $I\text{-}\text{TOP}_\mathcal{F}$ $J^{-}_c((Y - (\varepsilon_X \land Y)) \land (Q - (\varepsilon_X \land Q))_0)(x) = Q(x) - \varepsilon = (Q - (\varepsilon_X \land Q))(x)$ and in the category $I\text{-}\text{TOP}_\mathcal{G}$ (we recall that $J^{-}_c = J^{-}_c$) $J^{-}_c((Y - (\varepsilon_X \land Y)) \land (Q - (\varepsilon_X \land Q))_0)(x) = \lor \{Q(x) - \varepsilon = (Q - (\varepsilon_X \land Q))(x) \}$.

We conclude that, in each one of the considered categories, $\{J^{-}_c(A_k) \mid k \in K_0\}$ is a finite cover of $Q - (\varepsilon_X \land Q)$. □

Proposition 4.4. Sub-compactness is closed hereditary in $I\text{-}\text{TOP}_\mathcal{F}$ and in $I\text{-}\text{TOP}_\mathcal{G}$.

Proof. Let us use the same notation as in the proof of Proposition 4.3 and assume that $(X, Y, \mathcal{F}_Y, \delta)$ is sub-compact. Let $\{J^{-}_c(A_k) \mid k \in K\}$, $A_k \in \delta \forall k \in K$, be any open cover of the closed set $Q$ with the induced $\mathcal{F}$-topology, $\mathcal{F} \in (\mathcal{G}, \mathcal{C})$.

Then, by Lemma 4.1, $\{A_k | k \in K\}$ is a cover of $Y \land Q_0$ hence $\{A_k | k \in K \} \cup \{c_Y(Q)\}$ is a cover of $Y$.

Let $P \in \mathcal{F}_Q$ be any fuzzy set such as $P(x) < Q(x), \forall x \in Q_0$, and let us define a fuzzy set $P' \in \mathcal{F}_Y$ by setting $P'(x) = 0$ if $x \notin Q_0$ and, if $x \in Q_0$, $P'(x) = \lor \{P(x) + Q(x)/2, Y(x) - Q(x)/P(x)\}$. Since $P'(x) < Y(x)$, $\forall x \in Q_0$, a finite subcover $\{A_k \mid k \in K_0\} \cup \{c_Y(Q)\}$ of $P'$ exists. Moreover $\{A_k \mid k \in K_0\}$ is a cover of $P'$; in fact $P'(x) > c_Y(Q)(x) \forall x \in P'$. □
Thence $\{J^-(A_k)\}_{k \in K_0}$ is a cover of $J^-(P')$.

Now in the category $\mathcal{I}$-$\textsc{Top}$ we have $J^-(P')(x) = c_{\mathcal{I}}(J^-(c_{\mathcal{I}}(P')))(x) = Q(x) - Y(x) + P'(x) \geq P(x), \quad \forall x \in Q_0$ and in the category $\mathcal{I}$-$\textsc{Top}_g$ we have $J^-(P')(x) = J^-(P')(x) = (P'(x)/Y(x))Q(x) \geq P(x), \quad \forall x \in Q_0$.

Of course $J^-(P')(x) \geq 0 = P(x)$ if $x \notin Q_0$, in any case. Hence $J^-(P') \geq P$ and consequently $\{J^-(A_k)\}_{k \in K_0}$ is a $P$-cover. □

**Proposition 4.5.** Let $(X, Y, \mathcal{S}_Y, \delta)$ be any strong-compact $\mathcal{G}$-topological space, $M \in \mathcal{S}_X$, $M_0 \leq Y_0$ and let the bold extension of $M$ in $Y$, $M_0 \wedge Y$, be closed.

Then $(X, M, \mathcal{S}_M, \delta_M')$ is strong-compact.

**Proof.** Let $J$ be the inclusion monomorphism of the given subspace and let $J^-$ be the related backward powerset operator in $\mathcal{I}$-$\textsc{Top}_g$. Furthermore, let $\{A_j \mid j \in J\}$ be any open cover of any closed fuzzy subset $P \in \mathcal{S}_M$ in $\delta_M'$.

Then for every $j \in J$ an open fuzzy set $B_j \in \delta$ exists such that $A_j = J^-(B_j)$ and a closed fuzzy set $Q \in \mathcal{S}_Y$ exists such that $P = J^-(Q)$.

Clearly $\{B_j \mid j \in J\} \cup \{c_{\mathcal{I}}(M_0 \wedge Y)\}$ is an open cover of $Q$ in $\delta$, hence a finite subcover $\{B_j \mid j \in J_0\} \cup \{c_{\mathcal{I}}(M_0 \wedge Y)\}$ exists and $\{A_j \mid j \in J_0\}$ is a finite cover of $P$ since $J^-(c_{\mathcal{I}}(M_0 \wedge Y)) = 0$. □

**Example 4.1** (Weak-compactness and sub-compactness are not closed hereditary in $\mathcal{I}$-$\textsc{Top}_g$). Let $X = [0, 1]$ and let $\delta$ be the $\mathcal{G}$-topology on $Y = 1_X$ whose non-trivial open fuzzy sets $A \in \mathcal{S}_Y$ are determined by the condition $A(x) \leq \frac{1}{2}, \quad \forall x \in X$.

Evidently $(X, Y, \mathcal{S}_Y, \delta)$ is weak-compact and sub-compact: indeed it is cover-compact.

If $Q = (\frac{1}{2})_X$ then $(X, Q, \mathcal{S}_Q, \delta_Q)$ is neither sub-compact nor weak-compact; in fact, the open $Q$-cover $\{(\frac{1}{2})_X \mid x \in X\}$ has no finite subcover of $P = (\frac{1}{2})_X = Q - ((\frac{1}{2}_X) \wedge Y)$.

We note that $(X, Q, \mathcal{S}_Q, \delta_Q)$ also provides an example of a closed subspace of a cover-compact, hence shading-compact, $\mathcal{G}$-topological space, namely $(X, Y, \mathcal{S}_Y, \delta)$, which is not shading-compact, hence it is not cover-compact, either.

**Example 4.2** (Strong-compactness is not closed hereditary in $\mathcal{I}$-$\textsc{Top}_g$). Let us consider the $\mathcal{G}$-topology $\delta$ on $Y = 1_X$, $X = [0, 1]$, whose non-trivial open fuzzy sets $A$ are defined by the condition $A(x) \leq \frac{1}{2}, \quad \forall x \in X$, and let $Q = (\frac{1}{2})_X$. Then $(X, Y, \mathcal{S}_Y, \delta)$ is strong-compact since every open cover of any closed fuzzy set in $\delta$ must contain $Y$. However $(X, Q, \mathcal{S}_Q, \delta_Q)$ is not strong compact, in fact the open cover $\{k_1 \mid 0 < k_\frac{1}{2}, k \leq \frac{1}{2}\}$ of the closed fuzzy set $(\frac{1}{2})_X$ in $\delta_Q$ has no finite subcover. We note that $(X, Q, \mathcal{S}_Q, \delta_Q)$ is strong-compact, in fact $\delta_Q$ has no non-trivial open fuzzy sets.

**Example 4.3** (Strong-compactness is not closed hereditary in $\mathcal{I}$-$\textsc{Top}_g$). We consider once more on the carrier $Y = 1_X$, $X = [0, 1]$, the $\mathcal{G}$-topology $\delta$ all of whose non-trivial fuzzy open sets $A$ satisfy the condition $\frac{1}{2} \leq A(x) \leq \frac{1}{2}, \quad \forall x \in X$. Since every open cover of $Y$ must contain $Y$ and every open cover of each other closed fuzzy set $P$ in $\delta$ contains an element greater than $P$, we can see that $(X, Y, \mathcal{S}_Y, \delta)$ is strong-compact.
Nevertheless, if \( Q = (\frac{1}{2})_X \), then \((X, Q, \mathcal{F}_Q, \delta_Q)\) is not strong-compact since the open cover \( \{ k_X | 0 < k < \frac{1}{2} \} \) in \( \delta_Q \) of the closed fuzzy set \( (\frac{1}{4})_X \) has no finite subcover.

Also we note that in this case \((X, Q, \mathcal{F}_Q, \delta_Q)\) is trivially strong-compact.

Example 4.4 (Cover-compactness, shading-compactness, weak-compactness, sub-compactness are not closed hereditary in \( I\text{-TOP}_\sigma \)). Let \( X = I \), \( Y = 1_X \) and, \( \forall A \in \mathcal{E}_X \), \( A \in \delta \) iff \( A = 1_X \) or \( A(x) \leq \frac{2}{5} \ \forall x \in X \).

Then \((X, Y, \mathcal{E}_Y, \delta)\) is cover-compact hence it is shading-compact, weak-compact and sub-compact.

Now let us consider the closed fuzzy set \( Q = (\frac{3}{7})_X \) in \( \delta \); then \( \mathcal{E}_Q = [(\frac{1}{7})_X, (\frac{3}{7})_X] \) and \( \delta_Q = \mathcal{E}_Q \).

Clearly \((X, Q, \mathcal{E}_Q, \delta_Q)\) is not weak-compact: in fact, the open cover of \( Q \) whose elements are all bold fuzzy points in \( \mathcal{E}_Q \) has no finite subcover of \((\frac{1}{2})_X = Q - ((\frac{1}{6})_X \land (Q \land c(Q))))\). Thence such a closed subspace is neither sub-compact nor shading-compact nor cover-compact.

5. Some separation axioms

In this section, we consider a few separation axioms already introduced and studied in \([7,9,15]\).

Indeed too many different definitions have been given by several authors in the traditional context of fuzzy topology: at least one new separation axiom is considered in most of the papers on fuzzy topology.

Sometimes new definitions are careless introduced and a typical example can be found in \([7, \text{Section 2}]\): in fact the author of that paper noted at the beginning of Section 2 that one of the Hausdorff axioms already given in the literature implies that every fuzzy set is open. Nevertheless, Definitions 2.5 and 2.10 of \([7]\) contain some separation axioms that never occur for a Chang-fuzzy topological space which has non-crisp open fuzzy subsets. In particular, we mean the following.

Definition 5.1 (see Fora \([7]\)). A Chang-fuzzy topological space \((X, \delta)\) is said to be

1. \( T_c \) iff every bold fuzzy point in \( X \) is closed;
2. regular iff for every fuzzy point \( \lambda \) in \( X \) and for every closed fuzzy set \( P \) such that \( \lambda < c(P)(x) \), open fuzzy sets \( A, B \) exist such that \( \lambda < A(x) \), \( P \leq B \) and \( A \land B = O \);
3. \( T_3 \) iff it is regular and \( T_c \).

Definition 5.2 (see Fora \([7]\)). \((X, \delta)\) is said to be

1. \( T_s \) iff all fuzzy points in \( X \) are closed;
2. normal iff for every pair of closed fuzzy sets \( P, Q \) such that \( P \leq c(Q) \) there exist open fuzzy sets \( A, B \) such that \( P \leq A \), \( Q \leq B \) and \( A \land B = O \);
3. \( T_4 \) iff it is normal and \( T_s \).

Let us assume that a closed non-crisp fuzzy set \( P \) exists in \((X, \delta)\); then \( 0 < P(x_0) < 1 \) for some \( x_0 \in X \).

If \( 0 < \lambda < 1 - P(x_0) \) and if \( A, B \) are open fuzzy sets such that \( \lambda < A(x_0) \) and \( P \leq B \) then \( A(x_0) \land B(x_0) > 0 \).
Hence \((X, \delta)\) cannot be either regular or \(T_3\).

Now let us assume that \((X, \delta)\) is \(T_5\) and that a closed non-crisp fuzzy set \(P\) exists in \((X, \delta)\). Then, if \(0 < P(x_0) < 1\) and \(0 < \lambda \leq 1 - P(x_0)\), \(\lambda_x\) is closed and the relation \(P \leq c(\lambda_x)\) holds. So if we consider arbitrary open fuzzy sets \(A, B\) such that \(\lambda_x \leq A\), \(P \leq B\) then \(A(x_0) \wedge B(x_0) > 0\) and consequently \((X, \delta)\) cannot be \(T_4\).

In order to illustrate the different behaviour of separation axioms toward hereditary conditions in the categories of \(\mathcal{F}\)-topological spaces, we described in Section 2, we need to introduce no new definition: more simply, we shall extend some known axioms given in the context of \(I\)-\(\text{TOP} [7,9,15]\) to the more general context of the categories \(I\)-\(\text{TOP}_\mathcal{F}\), \(\mathcal{F} \in \{\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{G}\}\).

We shall see that the most suitable categories seem to be \(I\)-\(\text{TOP}_{\mathcal{F}}\) and \(I\)-\(\text{TOP}_{\mathcal{G}}\) if the hereditary conditions of separation axioms are expected to hold. However, subspaces in the categories \(I\)-\(\text{TOP}_{\mathcal{F}}\) and \(I\)-\(\text{TOP}_{\mathcal{G}}\) do not frequently inherit separation axioms: an example that illustrates this assertion in \(I\)-\(\text{TOP}_{\mathcal{G}}\) has already been exhibited in [2].

Separation axioms in subspaces of the fuzzy real line and fuzzy intervals with the same induced topology as we consider in the category \(I\)-\(\text{TOP}_{\mathcal{G}}\) have been studied in [17]; nevertheless, the hereditary conditions have not been investigated in [17].

Let us consider any \(\mathcal{F}\)-topological space \((X, Y, \mathcal{F}_Y, \delta)\) in any one of the categories \(I\)-\(\text{TOP}_{\mathcal{A}}\), \(I\)-\(\text{TOP}_{\mathcal{C}}\), \(I\)-\(\text{TOP}_{\mathcal{D}}\), \(I\)-\(\text{TOP}_{\mathcal{G}}\) and let, as usual, \(\kappa_Y\) be the o.r.i. in the fuzz \(\mathcal{F}_Y\), \(cl_Y\) the closure operator in \(\mathcal{F}_Y\) with respect to \(\delta\). Then we restate some definitions of [7,9].

**Definition 5.3.** \((X, Y, \mathcal{F}_Y, \delta)\) is said to be

1. \(T_2\) iff for any two non-bold fuzzy points \(\lambda_x, \mu_y, x \neq y, A, B \in \delta\) exist such that \(\lambda < A(x), \mu < B(y), A \wedge B = O\);
2. \(FT_2\) iff for any two fuzzy points \(\lambda_x, \mu_y, x \neq y, A, B \in \delta\) exist such that \(\lambda \leq A(x), \mu \leq B(y), A \leq \kappa_Y(B)\);
3. \(T_{2_\omega}\) iff for any two non-bold fuzzy points \(\lambda_x, \mu_y, x \neq y, A, B \in \delta\) exist such that \(\lambda < A(x), \mu < B(y), A \leq \kappa_Y(B)\);
4. \(FT_{2_\omega}\) iff for any two fuzzy points \(\lambda_x, \mu_y, x \neq y, A, B \in \delta\) exist such that \(\lambda \leq A(x), \mu \leq B(y), cl_Y(A) \leq \kappa_Y(cl_Y(B))\);
5. \(T_{1_\omega}\) iff for any two non-bold fuzzy points \(\lambda_x, \mu_y, x \neq y, A, B \in \delta\) exist such that \(\lambda < A(x), \mu < B(y), cl_Y(A) \leq \kappa_Y(cl_Y(B))\);
6. \(FR\) iff for every fuzzy point \(\lambda_x\) and every closed fuzzy set \(P\) such that \(\lambda \leq \kappa_Y(P)(x), A, B \in \delta\) exist such that \(\lambda \leq A(x), P \leq B, A \leq \kappa_Y(B)\);
7. \(FT_3\) iff it is \(T_3\) and \(FR\);
8. \(\omega-R\) iff for every fuzzy point \(\lambda_x\) and every closed set \(P\) such that \(\lambda < \kappa_Y(P)(x), A, B \in \delta\) exist such that \(\lambda \leq A(x), P \leq B, A \leq \kappa_Y(B)\);
9. \(T_{3_\omega}\) iff it is \(T_c\) and \(\omega-R\);
10. \(FN\) iff for any two closed fuzzy sets \(P, Q\) such that \(P \leq \kappa_Y(Q), A, B \in \delta\) exist such that \(P \leq A, Q \leq B, A \leq \kappa_Y(B)\);
11. \(FT_4\) iff it is \(T_s\) and \(FN\).

We recall that the \(FN\) axiom introduced in [9] in the category \(I\)-\(\text{TOP}\) and also called \(\omega-N\) in [7] has an analogous formulation in the point-free context [12].
It is easy to prove that in every one of the categories $I$-$\text{TOP}_\mathscr{A}$, $I$-$\text{TOP}_\mathcal{F}$, $I$-$\text{TOP}_\mathcal{G}$, $I$-$\text{TOP}_\mathcal{H}$ the following implications hold most of which have already been verified (see [7,9]) in the category $I$-$\text{TOP}$:

\[ FT_4 \Rightarrow FT_3 \Rightarrow FT_{\frac{3}{2}} \Rightarrow FT_2 \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ T_{3\alpha} \Rightarrow T_{\frac{3}{2}\alpha} \Rightarrow T_{2\alpha} \subseteq T_2. \]

Now we recall (see [5]) that if $\alpha$ is any isomorphism from $(X, Y, \mathcal{F}_Y, \delta)$ to $(T, Z, \mathcal{F}_Z, \tau)$ in any one of the categories $I$-$\text{TOP}_\mathscr{A}$, $I$-$\text{TOP}_\mathcal{F}$, $I$-$\text{TOP}_\mathcal{G}$, $I$-$\text{TOP}_\mathcal{H}$ then $\alpha^{-1}$ and $\alpha^{-1} = \alpha_c^{-1}$ are fuzz isomorphisms and they are inverse of each other. Consequently $\alpha^{-1}$ and $\alpha^{-1}$ map open and closed fuzzy subsets into open and closed fuzzy subsets respectively, hence they commute with the closure operators in $\mathcal{F}_Y$ and $\mathcal{F}_Z$. Furthermore, $\alpha^{-1}$ and $\alpha^{-1}$ map (bold) fuzzy points into (bold) fuzzy points.

Thence the following result can be easily checked.

**Proposition 5.4.** All the separation axioms defined in 5.3 and the $T_s$ and $T_c$ axioms are preserved by isomorphisms in each one of the categories $I$-$\text{TOP}_\mathscr{A}$, $I$-$\text{TOP}_\mathcal{F}$, $I$-$\text{TOP}_\mathcal{G}$, $I$-$\text{TOP}_\mathcal{H}$.

As for the hereditary conditions, we shall obtain different results in different categories depending on the properties of the trace operator (i.e. the backward powerset operator related to the inclusion monomorphism, see [10, Definition 7.1]). In fact such operator is a fuzz morphism in $I$-$\text{TOP}_\mathcal{G}$ and $I$-$\text{TOP}_\mathcal{H}$ and no longer in $I$-$\text{TOP}_\mathscr{A}$ and $I$-$\text{TOP}_\mathcal{F}$.

**Proposition 5.5.** The axioms $T_c, T_s, T_2, FT_2, T_{2\alpha}, FT_3, T_{3\alpha}$ are hereditary in $I$-$\text{TOP}_\mathcal{G}$ and in $I$-$\text{TOP}_\mathcal{H}$.

**Proof.** Let $(X, Y, \mathcal{F}_Y, \delta)$ be any $\mathcal{F}$-topological space and $(X, M, \mathcal{F}_M, \delta_M)$ any one of its subspaces either in $I$-$\text{TOP}_\mathcal{G}$ or in $I$-$\text{TOP}_\mathcal{H}$. If $J$ is the inclusion morphism, then $J^{-1}$ is injective and $t = J^{-1}$ is a fuzz function; furthermore, it is an easy consequence of results in [10, 5] that every (bold) fuzzy point in $\mathcal{F}_M$ is the trace of a (bold) fuzzy point in $\mathcal{F}_Y$.

Now it is straightforward to prove the assertion. □

**Lemma 5.6.** Let $(X, M, \mathcal{F}_M, \delta_M)$ be a subspace of $(X, Y, \mathcal{F}_Y, \delta)$ in $I$-$\text{TOP}_\mathcal{G}$ or in $I$-$\text{TOP}_\mathcal{H}$ (hence $\delta_M$ is either $\delta_M^\mathcal{A}$ or $\delta_M^\mathcal{B}$), let $t : \mathcal{F}_Y \rightarrow \mathcal{F}_M$ be the trace operator and let $cl_M$ and $cl_Y$ be the closure operator in the subspace and in the whole space, respectively.

Then we have

(i) $cl_M(t(B)) \leq t(cl_Y(B)), \quad \forall B \in \mathcal{F}_Y$;
(ii) $cl_M(t(A)) = t(cl_Y(A)), \quad \forall A \in \mathcal{F}_M$.

**Proof.** (i) If $B \in \mathcal{F}_Y$ then $t(cl_Y(B))$ is a closed fuzzy set in $\mathcal{F}_M$ that clearly contains $t(B)$.

(ii) If $A \in \mathcal{F}_M$ then the condition of (i) holds; moreover, if $P$ is any closed fuzzy set in $\mathcal{F}_M$ such that $P \supseteq t(A)$ and $Q$ is a closed fuzzy set in $\mathcal{F}_Y$ such that $P = t(Q)$, then it follows from $t(Q) \supseteq t(A)$ and $A \in \mathcal{F}_M$ that $Q \supseteq A$, hence $Q \supseteq cl_Y(A)$.

Consequently, $P = t(Q) \supseteq t(cl_Y(A))$. □
Proposition 5.7. The axioms \( FT_{2(1/2)} \) and \( T_{2(1/2)_0} \) are hereditary in \( I\text{-TOP}_\delta \) and \( I\text{-TOP}_\delta'. \)

Proof. Let \( \lambda_x, \mu_y \) be two fuzzy points in the subspace \((X, M, \mathcal{F}_M, \delta_M)\) as required by the axioms we need to test and let \( \lambda'_x, \mu'_y \) be the fuzzy points in the whole space \((X, Y, \mathcal{F}_Y, \delta)\) such that \( \lambda_x = t(\lambda'_x), \mu_y = t(\mu'_y) \).

If \( A', B' \in \delta \) satisfy the required conditions in the axioms \( FT_{2(1/2)} \) and \( T_{2(1/2)_0} \), respectively, in the whole space with respect to the fuzzy points \( \lambda'_x \) and \( \mu'_y \), then \( A = t(A') \) and \( B = t(B') \) satisfy the corresponding conditions in the subspace; in fact the trace operator is monotone and, by Lemma 5.6, \( cl_M(A) = cl_M(t(A')) \leq t(cl_Y(A')) \leq \kappa_Y(\kappa_Y(t(\mathcal{L}_Y(B')))) = \kappa_M(t(cl_Y(B'))) \leq \kappa_M(cl_M(t(B'))) = \kappa_M(cl_M(B)). \)

\( \square \)

Proposition 5.8. The axiom \( FT_4 \) is closed hereditary in \( I\text{-TOP}_\delta \).

Proof. Let \((X, Y, \mathcal{F}_Y, \delta)\) be \( FT_4 \) and let \( M \in \mathcal{F}_M \) be closed.

Let \( P, Q \in \mathcal{F}_M \) be closed in \((X, M, \mathcal{F}_M, \delta_M), P \leq ec_M(Q), \) and \( P = t(P'), Q = t(Q'), P', Q' \) closed in \( \delta \). Then \( P'' = P' \land M, Q'' = Q' \land M \) are closed fuzzy sets in \( \delta_M \) and \( t(P'') = P, t(Q'') = Q, P'' \leq ec_Y(Q'') \). If \( A'', B'' \in \delta \), \( P'' \leq A'', Q'' \leq B'', A'' \leq ec_Y(B'') \) then \( A = t(A'') \) and \( B = t(B'') \) are open fuzzy sets in \( \delta_M \) that clearly contain \( P \) and \( Q \), respectively, and \( A \leq ec_M(B). \)

\( \square \)

Proposition 5.9. Let \((X, Y, \mathcal{F}_Y, \delta)\) be any \( FT_4 \) topological space, \( M \in \mathcal{F}_X, M_0 \leq Y_0 \) and let the bold extension of \( M \) in \( Y, Y \land M_0, \) be closed. Then \((X, M, \mathcal{F}_M, \delta_M^b)\) is \( FT_4 \).

Proof. Let \( P, Q \in \mathcal{F}_M \) be closed in \( \delta_M^b, P \leq ec_M(Q) \) and let \( P = t(P'), Q = t(Q'), P', Q' \) closed in \( \delta \).

If \( P'' = P' \land M_0, Q'' = Q' \land M_0 \) then \( t(P'') = P, t(Q'') = Q \).

Moreover \( P \leq ec_Y(Q) \Rightarrow t(P'') \leq t(ec_Y(Q'')) \Rightarrow P'' \leq ec_Y(Q''). \)

Let \( A'', B'' \in \delta \), \( P'' \leq A'', Q'' \leq B'', A'' \leq ec_Y(B'') \) and let us consider \( A = t(A''), B = t(B'') \). Then \( P \leq A, Q \leq B \) and \( A \leq ec_M(B). \)

\( \square \)

Proposition 5.10. The \( T_2 \) axiom is hereditary in \( I\text{-TOP}_{\alpha} \) and \( I\text{-TOP}_{\delta}. \)

Proof. Let \((X, Y, \mathcal{F}_Y, \delta)\) satisfy the \( T_2 \) axiom and \( M \in \mathcal{F}_M \). If \( \lambda_x, \mu_y, x \neq y \), are non-bold fuzzy points in \( \mathcal{F}_M \) then we consider the non-bold fuzzy points in \( \mathcal{F}_Y, \lambda'_x, \mu'_y, \lambda''_x, \mu''_y \) such that \( t(\lambda'_x) = \lambda_x, t(\mu'_y) = \mu_y \) in \( I\text{-TOP}_{\alpha} \) and \( J^-\alpha(\lambda''_x) = \lambda_x, J^-\alpha(\mu''_y) = \mu_y \) in \( I\text{-TOP}_{\delta} \) (we note that such points are uniquely determined and their values are \( \lambda' = \lambda, \mu' = \mu \) in \( I\text{-TOP}_{\alpha} \) while \( \lambda'' = Y(x) - M(x) + \lambda, \mu'' = Y(y) - M(y) + \mu \) in \( I\text{-TOP}_{\delta} \)).

In the category \( I\text{-TOP}_{\alpha} \), if \( A', B' \in \delta \) satisfy the conditions \( \lambda' < A'(x), \mu' < B'(y), A' \land B' = 0 \), then \( A = t(A'), B = t(B') \) satisfy the conditions \( \lambda = t(\lambda'_x)(x) < t(A'(x)) = A(x), \mu = t(\mu'_y)(y) < t(B'(y)) = B(y) \) and \( A \land B = t(A' \land B') = 0 \). Hence \((X, M, \mathcal{F}_M, \delta_M^b)\) satisfies \( T_2 \).

Similarly the \( T_2 \) axiom can be verified for \((X, M, \mathcal{F}_M, \delta_M^b)\).

\( \square \)

Example 5.1 (No one of the axioms \( T_{2_0}, FT_2, T_{2(1/2)_0}, FT_{2(1/2)}, T_{3_0}, FT_3 \) is hereditary in \( I\text{-TOP}_{\alpha} \); the \( FT_4 \) axiom is not closed hereditary in \( I\text{-TOP}_{\alpha} \)). Let \( X = [0, 1], Y = X \) and let us consider the topology \( \delta \) in \( \mathcal{F}_Y \) whose open fuzzy sets \( A \) are characterized by the following properties:
\( A(0) > \frac{1}{2} \Rightarrow A(x) \geq (1-x)/2 \) for all but a finite number of \( x \) in \([0, 1]\) and \( A(1) > \frac{1}{2} \Rightarrow A(x) \geq x/2 \) for all but a finite number of \( x \) in \([0, 1]\). It is useful to note that closed fuzzy sets \( P \) are characterized by \( P(0) < \frac{1}{2} \Rightarrow P(x) \leq (x+1)/2 \) for all but a finite number of \( x \) in \([0, 1]\) and \( P(1) < \frac{1}{2} \Rightarrow P(x) \leq (2-x)/2 \) for all but a finite number of \( x \) in \([0, 1]\).

Hence every fuzzy point is closed.

We can prove that \((X, \mathcal{S}_Y, \delta)\) is \( F T_4 \) (hence it is \( F T_3, T_{3\alpha}, FT_{2(1/2)}, T_{2(1/2)\alpha}, FT_2, T_{2\alpha} \)).

In fact let us consider any two closed fuzzy sets \( P, Q \) such that \( P \leq c_Y(Q) \). If \( Q(0) \geq \frac{1}{2}, \ Q(1) \geq \frac{1}{2} \) then \( P(0) \leq \frac{1}{2} \) and \( P(1) \leq \frac{1}{2} \) and the open fuzzy sets \( A = P \) and \( B = c_Y(P) \) satisfy the required conditions.

If \( Q(0) \leq \frac{1}{2}, \ Q(1) \leq \frac{1}{2} \) then \( Q \in \delta \) and the open fuzzy sets \( A = c_Y(Q) \) and \( B = Q \) satisfy once more the required conditions.

If \( Q(0) > \frac{1}{2} \) and \( Q(1) < \frac{1}{2} \), then \( P(0) < \frac{1}{2} \). However, we may have \( P(1) \leq \frac{1}{2} \) hence \( Q \in \delta \) and the assertion follows in this case too. Also we may have \( P(1) > \frac{1}{2} \); then we consider the open fuzzy sets \( F, G \in \delta \) defined by \( F(x) = x/2 \) and \( G(x) = (1-x)/2 \) and the open fuzzy sets \( A = (F \lor P) \land c_Y(Q), B = (G \lor P) \land c_Y(Q) \).

Clearly \( P \leq A, Q \leq B \) and since \( A \leq c_Y(Q) \) we have moreover \( c_Y(B) = (c_Y(G) \lor P) \land c_Y(Q) \leq (F \lor P) \land c_Y(Q) = A \).

If \( Q(0) < \frac{1}{2} \) and \( Q(1) > \frac{1}{2} \) we can produce a similar proof. Consequently \((X, Y, \mathcal{S}_Y, \delta)\) is \( FN \).

Since \((X, Y, \mathcal{S}_Y, \delta)\) is clearly \( T_4 \) we conclude that it is \( F T_4 \).

Now we consider the fuzzy subset \( M \in \mathcal{S}_Y \) defined, \( \forall x \in X \), by

\[
M(x) = \begin{cases} 
1 - 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\
2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1. 
\end{cases}
\]

We can prove that \((X, M, \mathcal{S}_M, \delta^M)\) is not \( T_{2\alpha} \) (hence it is not \( F T_2, T_{2(1/2)\alpha}, FT_{2(1/2)}, T_{3\alpha}, FT_3, FT_4 \)).

In fact let us consider the non-bold fuzzy points \((\frac{3}{4})_0\) and \((\frac{3}{4})_1\) and let \( A, B \in \delta^M, \frac{3}{4} < A(0), \frac{3}{4} < B(1) \). Then \( A', B' \in \delta \) exist such that \( A = A' \land M, B = B' \land M \).

It follows from \( A'(0) > \frac{1}{2} \) and \( B'(1) > \frac{1}{2} \) that \( A'(x) \geq (1-x)/2 \) and \( B'(x) \geq x/2 \) for all but a finite number of \( x \) in \([0, 1]\).

Consequently, for all but a finite number of \( x \) in \((\frac{3}{4}, \frac{1}{2})\) we have \( A(x) \geq (1-x)/2 \land M(x) = M(x) \) and \( B(x) \geq x/2 \land M(x) = M(x) \), thus some elements \( x \in (\frac{3}{4}, \frac{1}{2}) \) exist such that \( A(x) > 0 = M(x) - B(x) = c_M(B)(x) \).

We conclude that the axioms \( T_{2\alpha}, FT_2, T_{2(1/2)\alpha}, FT_{2(1/2)}, T_{3\alpha}, FT_3 \) are not hereditary in \( I\text{-TOP}_{\delta} \) and, since clearly \( M \) is closed in \( \delta \), we also can say that the axiom \( FT_4 \) is not closed hereditary in \( I\text{-TOP}_{\delta} \).

**Proposition 5.11.** The \( T_{2\alpha} \) and \( FT_2 \) axioms are hereditary in \( I\text{-TOP}_{\delta} \).

**Proof.** Let \((X, Y, \mathcal{S}_Y, \delta)\) be a \( T_{2\alpha} \) \( \mathcal{E} \)-topological space and \( M \in \mathcal{S}_Y \).

If we consider any two non-bold fuzzy points \( \lambda_x, \mu_y \) \( x \neq y \), in \( \mathcal{S}_M \) then \( \lambda' = Y(x) - M(x) + \lambda < Y(x) \) and \( \mu' = Y(y) - M(y) + \mu < Y(y) \); so \( A', B' \in \delta \) exist such that \( \lambda' < A'(x), \mu' < B'(y) \), \( A' \leq c_Y(B') \).

Let \( A = J^-(A'), B = J^-(B') \), where \( J \) is the inclusion monomorphism of the subspace \((X, M, \mathcal{S}_M, \delta^M)\) and \( J^- \) is the related complemented backward powerset operator (see [5]). Then \( A, B \in \delta^M \) and it follows from \( J^- \leq J^- \) (see [5]) that \( A \leq A', B \leq B' \).
Now we have \( A(x) = M(x) - ((Y(x) - A'(x)) \land M(x)) = (M(x) - Y(x) + A'(x)) \land 0 > (M(x) - Y(x) + A') = \lambda \) and similarly \( B(y) > \mu \).

Moreover, \( \forall t \in X \), we have \( A(t) = M(t) - ((Y(t) - A'(t)) \land M(t)) = (M(t) - Y(t) + A'(t)) \land 0 \leq (M(t) - B'(t)) \lor 0 \leq M(t) - B(t) = c_M(B)(t) \).

The proof of the hereditariness of \( FT_2 \) is similar. \( \square \)

**Example 5.2** (\( FT_{2(1/2)} \) and \( T_{2(1/2)0} \) are not hereditary in \( I\text{-TOP}_\emptyset \)). Let \( X \) be a set and \( \tau \) be an ordinary topology on \( X \) such that \( X \) is \( T_2 \) and it is not \( T_{2(1/2)} \). Moreover, let us consider \( Y = 1_X \) and let \( \delta \) be the \( \mathcal{C} \)-topology on \( Y \) whose open fuzzy sets, \( A \in \delta \), are determined by the conditions
\[ \{A(x) \mid x \in X\} \subseteq [0, \frac{1}{2}, 1] \land \{x \in X \mid A(x) = 1\} \text{ is open in } \tau. \]

Then \( (X, Y, \mathcal{F}_Y, \delta) \) is an \( FT_{2(1/2)} \) (hence a \( T_{2(1/2)0} \)) space.

Now let us consider \( M = (\frac{1}{2})_X \). The induced topology \( \delta^*_M \) has open fuzzy sets \( A \in \mathcal{F}_M \) and closed fuzzy sets \( P \in \mathcal{S}_M \) characterized, respectively, by
\[ \{A(x) \mid x \in X\} \subseteq [0, \frac{1}{2}, 1] \land \{x \in X \mid A(x) = \frac{1}{2}\} \in \tau, \]
\[ \{P(x) \mid x \in X\} \subseteq [0, \frac{1}{2}, 1] \land \{x \in X \mid P(x) = 0\} \in \tau. \]

If \( z \neq t \) are elements of \( X \) such that any closed neighbourhoods of \( z \) and \( t \) in \( \tau \) have a non-empty intersection, then we cannot find closed neighbourhoods of \( \lambda^*_z, \mu^*_t \) in \( \delta^*_M \) with the conditions required by the \( T_{2(1/2)0} \) axiom, whenever \( 0 < \lambda, \mu < \frac{1}{2} \). In fact any two \( A, B \in \delta^*_M \) such that \( \lambda < A(z), \mu < B(t) \) take value \( \frac{1}{2} \) on two open sets \( U, V \in \tau \) such that \( z \in U, t \in V \), respectively. Then every closed fuzzy set in \( \delta^*_M \) containing \( A \) takes value \( \frac{1}{2} \) on \( cl_\tau(U) \) and every closed fuzzy set in \( \delta^*_M \) containing \( B \) takes value \( \frac{1}{2} \) on \( cl_\tau(V) \). It follows from \( cl_\tau(U) \land cl_\tau(V) \neq \emptyset \) that \( cl_M(A) \neq cl_M(B) \).

Consequently \( (X, M, \mathcal{F}_M, \delta^*_M) \) is not \( T_{2(1/2)0} \) and it is not \( FT_{2(1/2)} \), either.

The lower separation axioms defined in [7,9] for Chang-fuzzy topological spaces can be easily extended to \( \mathcal{F} \)-topological spaces \( (X, Y, \mathcal{F}_Y, \delta) \) in the categories \( I\text{-TOP}_s, I\text{-TOP}_\emptyset, I\text{-TOP}_s, I\text{-TOP}_\emptyset \) by using fuzzy points, hence by using elements of the support \( Y_s \), and they turn out to be hereditary properties. Also \( T_s \) and \( T_e \) are hereditary. Nevertheless, we shall conclude this section by showing that the extension to the context of this paper of a weaker version of the \( T_2 \) axiom, formulated in [15], which is satisfied by the fuzzy real line and the fuzzy intervals, is not hereditary in most of the considered categories.

**Definition 5.12** (see Kubiak [15]). \( (X, Y, \mathcal{F}_Y, \delta) \) is \( L-T_2 \) if for any \( x, y \in Y_s, x \neq y \), there exist \( A, B \in \delta \) such that \( A(x) < A(y), B(y) < B(x), A \leq \kappa_Y(B) \).

**Proposition 5.13.** The \( L-T_2 \) axiom is hereditary in \( I\text{-TOP}_\emptyset \).

**Proof.** Let \( (X, Y, \mathcal{F}_Y, \delta) \) be \( L-T_2 \), \( M \in \mathcal{F}_X, M_0 \leq Y_0, x, y \in M_0, x \neq y \). Let \( A, B \in \delta \) satisfy the conditions of Definition 5.12.

Then the traces \( A' = t(A) = (A \div Y) \cdot M, B' = t(B) = (B \div Y) \cdot M \) are open fuzzy sets in \( \delta^*_M \) and clearly they satisfy the required conditions. \( \square \)
Example 5.3 (The $L$-$T_2$ axiom is not hereditary in $I$-$\mathbf{TOP}_\delta$, $I$-$\mathbf{TOP}_\gamma$, $I$-$\mathbf{TOP}_\epsilon$). Let $X = Y = [0, 1]$ and let $\delta, \gamma, \epsilon$ be the $\mathcal{F}$-topologies in $\mathcal{F}_Y$ whose non-trivial open fuzzy sets are defined by

- $A \in \delta \iff A(x) \geq \frac{1}{3}, \forall x \in X$,
- $A \in \gamma \iff A(x) \leq \frac{1}{3}, \forall x \in X$,
- $A \in \epsilon \iff A(x) \leq \frac{1}{3}, \forall x \in X$.

Of course $(X, Y, \mathcal{F}_Y, \delta) \in I$-$\mathbf{TOP}_\delta$, $(X, Y, \mathcal{F}_Y, \gamma) \in I$-$\mathbf{TOP}_\gamma$, $(X, Y, \mathcal{F}_Y, \epsilon) \in I$-$\mathbf{TOP}_\epsilon$ are $L$-$T_2$ spaces.

Nevertheless, if $M = (\frac{1}{3})_X$, $N = (\frac{1}{2})_X$, $P = (\frac{3}{4})_X$, every one of the subspaces $(X, M, \mathcal{S}_M, \delta_M)$, $(X, N, \mathcal{S}_N, \gamma_N)$, $(X, P, \mathcal{S}_P, \epsilon_P)$ has no non-trivial open fuzzy set; hence none of them satisfies the $L$-$T_2$ axiom.

6. Concluding remarks

To our knowledge the present paper is the first attempt to investigate extensively the problem of hereditariness in the traditional context of fuzzy topology taking into account the possibility of considering spaces whose carrier is a non-necessarily crisp fuzzy subset of a set $X$; also the invariance under isomorphisms of topological conditions in the same context has been usually ignored, and some results of Section 3 show that this is not a trivial question too.

We needed to test not so many, already considered, fuzzy topological conditions in order to discover many different situations and in this sense, we obtained results that illustrate all the possible circumstances, although we did not answer all the possible questions concerning invariance and hereditariness of the considered axioms: the purpose of this paper does not justify big efforts for solving problems whose solution nothing would add to the general economy of our present investigation.

In fact, if someone assumes that the axioms considered in Sections 4 and 5 are of some interest in their own right, then we remark that some problems remain unsolved that are as follows:

1. Are sub-compactness and strong-compactness preserved by morphisms in $I$-$\mathbf{TOP}_\delta$?
2. Is strong-compactness closed hereditary in $I$-$\mathbf{TOP}_\delta$?
3. Are the $T_{30}$ and $FT_3$ axioms hereditary in $I$-$\mathbf{TOP}_\delta$?
4. Is the $FT_4$ axiom closed hereditary in $I$-$\mathbf{TOP}_\delta$?

We also remark that hereditariness of strong-compactness as well as of the $FT_4$ axiom holds in the category $I$-$\mathbf{TOP}_\delta$ with respect to fuzzy subsets whose bold extension is closed (see Propositions 4.5 and 5.9).

Nevertheless we ask the following.

5. Are strong-compactness and the $FT_4$ axiom closed hereditary in the traditional sense in $I$-$\mathbf{TOP}_\delta$?

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References