MIMO Zero-Forcing Detection Performance for Correlated and Estimated Rician Fading with Lognormal Azimuth Spread and $K$-Factor

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Abstract—For multiple-input multiple-output (MIMO) wireless communications systems, we propose a new zero-forcing (ZF) detection approach that explicitly accounts for instantaneous channel state information (CSI) estimation error and spatial correlation. For this ZF approach we derive an average error probability (AEP) expression for transmit-correlated Rician fading. This AEP derivation exploits the effective signal-to-noise ratio that results by compounding ICSI estimation error and receiver noise and the approximation of the ensuing noncentral-Wishart distribution with a simpler central Wishart distribution. The derived AEP expression is then applied to evaluate MIMO ZF performance in Rayleigh and Rician fading for samples from recently-measured lognormal azimuth spread (AS) and Rician $K$-factor distributions, for pilot-based ICSI estimation. Numerical results depict the dependence of the AEP averaged over the AS and $K$ distributions on fading type, rank of the deterministic component of the channel matrix, and AS-$K$ correlation, for realistic scenarios.

Index Terms—Azimuth spread, correlation, $K$-factor, MIMO, Rician fading, zero-forcing detection.

I. INTRODUCTION

A. Background

Multiple-input multiple output (MIMO) concepts are promoted for next-generation wireless communications systems due to the capacity and symbol-detection performance gains promised by multi-antenna theory [1] [2]. However, these gains may not be always fully achievable in practice due to channel condition [3] (geometry; fading type, e.g., Rayleigh or Rician; fading parameters, e.g., azimuth spread (AS) and Rician $K$-factor) and transceiver impairments (instantaneous channel state information — ICSI — estimation error), as we attempt to show in this paper, for the low-complexity linear MIMO detection method known as zero-forcing (ZF).

B. Previous Work

ZF average bit-error rate expressions have been derived for $M$-QAM and $M$-PSK, transmit-correlated Rician fading, and perfectly-known ICSI in [4] [5] [6], by approximating the noncentral Wishart distribution of the matrix that enters the signal-to-noise ratio (SNR) expression of the ZF detector with a central Wishart distribution.

For estimated ICSI, ZF closed-form average bit-error rate expressions for $M$-QAM and $M$-PSK and uncorrelated Rayleigh fading appear in [7, Eqns. (20), (21)]. The disadvantages of the approach in [7] are that it employs two approximations and it requires knowledge of the mapping between the ICSI estimation method and its error. The advantage is that the ICSI estimation error is compounded with the receiver noise into the effective (i.e., actual) symbol-detection noise, and the analysis exploits the resulting effective SNR.

For estimated ICSI, an exact MIMO ZF detection performance analysis appears, also for uncorrelated Rayleigh fading, in [8]. As in [7], the ZF detection matrix is constructed from the ICSI estimate as if it were the true ICSI [8, Eqn. (11)]. However, only the ICSI estimation error due to nonintended streams is compounded with the receiver noise. Then, the effective signal-to-interference-plus-noise ratio is employed to derive closed-form average symbol-error rate expressions for $M$-PAM and QPSK.

To the best of our knowledge, MIMO ZF has yet to be evaluated comprehensively based on average error probability (AEP) expression for practical correlated spatial channel models. Such models have recently become available from the European WINNER II project, which measured actual channels for a wide range of environments [9]. WINNER II has revealed preponderantly-Rician fading with scenario-dependent AS and $K$ lognormal distributions and correlations. We recently employed these models to evaluate SIMO performance based on AEP expression [10] and (generic and IEEE 802.11n) MIMO performance based on simulations [11] [12].

C. Contributions and Approach

For transmit-correlated Rician fading and SNR-dependent ICSI estimation error model, a new ZF detection approach is proposed and systematically analyzed (for $M$-PSK), based on the effective SNR obtained by compounding the ICSI estimation error on all streams with the receiver noise. This analysis yields a simpler and more widely-applicable AEP expression than those from [4] [7] [8].

This AEP expression is used to evaluate ZF symbol-detection performance for fading models and AS and $K$ distributions and correlation that characterize actual scenarios as described in [9]. The AEP expression is first computed for samples from the lognormal AS and $K$ distributions with scenario-dependent mean, variance, and correlation from [9, Table 4-5]. Then, the AEP is numerically averaged over these
AS and $K$ samples. The new AEP expression can reduce simulation effort substantially.

**D. Notation**

Scalars, vectors, and matrices are represented in lowercase italics, boldface lowercase, and boldface uppercase, respectively; e.g., $x$, $\mathbf{x}$, and $\mathbf{X}$; $\mathbf{x} \sim \mathcal{N}(\mathbf{x}, \mathbf{X})$ indicates that $\mathbf{x}$ is a complex-valued circularly-symmetric random vector [1, p. 39] of multivariate Gaussian distribution with mean $\mathbf{x}$ and covariance $\mathbf{X}$; $\psi \sim \mathcal{N}(0, 1)$ indicates that scalar $\psi$ is a real-valued random variable of Gaussian distribution with zero-mean and unit variance; subscripts $i$ and $j$, identify, respectively, the deterministic (mean) and random components of a scalar or vector; index $\cdot$ indicates a normalized variable; $i = 1 : N$ stands for the enumeration $i = 1, 2, \ldots N$; the superscripts $\cdot^T$ and $\cdot^H$ stand for transpose and Hermitian (complex-conjugate) transpose; $[\cdot]_i$ and $[\cdot]_{i,j}$ indicate the $i$th and $i,j$th element of a vector and a matrix, respectively; $E\{\cdot\}$ denotes statistical average.

**E. Paper Organization**

Section II introduces statistical models for transmitted signal, channel fading, AS, and $K$. Section III describes the new ZF detection approach for imperfect ICSI and shows the AEP derivation. Section IV presents relevant numerical results.

**II. SYSTEM MODEL**

**A. Signal and Channel Fading Models**

We consider a single-user MIMO wireless communication system over a frequency-flat Rician fading channel. Let us assume that there are $N_T$ and $N_R$ antenna elements at the transmitter and receiver, respectively, with $N_T \leq N_R$. Letting $\mathbf{x} = [x_1 \ x_2 \ \ldots \ x_{N_T}]^T$ denote the $N_T$-dimensional vector with the transmitted symbols ($M$-PSK modulation, $E\{|xx^H|\} = I_{N_R}$), the $N_R$-dimensional vector with the received signals can be represented as [1, p. 63]:

$$
\mathbf{r} = \sqrt{\frac{E_s}{N_T}} \mathbf{H} \mathbf{x} + \mathbf{n},
$$

where $E_s/N_T$ is the energy transmitted per symbol, $\mathbf{H}$ is the $N_R \times N_T$ complex-valued channel matrix whose elements are assumed to have unit variance, and $\mathbf{n}$ is temporally- and spatially-white, circularly-symmetric, zero-mean, complex Gaussian, i.e., $\mathbf{n} \sim \mathcal{N}(0, \mathbf{N}_0 \mathbf{I})$ [1].

Let us denote the deterministic (i.e., mean) and random components of $\mathbf{H}$ as $\mathbf{H}_d$ and $\mathbf{H}_r$, respectively, i.e., $\mathbf{H} = \mathbf{H}_d + \mathbf{H}_r$. When $\mathbf{H}_r$ is non-zero and the elements of $\mathbf{H}_r$ are complex-valued Gaussian random variables, $|([\mathbf{H}]_{i,j})|$ has a Rician distribution, whereas for zero channel mean, $|(|\mathbf{H}_d|_{i,j})|$ has a Rayleigh distribution [2].

In general, the channel matrix for Rician fading can be written as [1]:

$$
\mathbf{H} = \mathbf{H}_d + \mathbf{H}_r = \sqrt{\frac{K}{K+1}} \mathbf{H}_{d,n} + \sqrt{\frac{1}{K+1}} \mathbf{H}_r,\quad (2)
$$

where $\mathbf{H}_{d,n}$ is the normalized deterministic component (mean) of $\mathbf{H}$, i.e., with $||\mathbf{H}_{d,n}||^2 = N_T N_R$, and $\mathbf{H}_r$ is the normalized random component of $\mathbf{H}$, i.e., with $E\{|[\mathbf{H}_{r,n}]_{i,j}^2\} = 1$, $\forall i = 1 : N_R, j = 1 : N_T$. Since $E\{|[\mathbf{H}_{r,n}]^2\} = N_T N_R$, the channel matrix it is properly normalized, i.e., $E\{||\mathbf{H}||^2\} = N_T N_R$ [13]. Thus, the power ratio of the deterministic and the random components of $\mathbf{H}$ is

$$
\frac{E\{||\mathbf{H}_d||^2\}}{E\{||\mathbf{H}_r||^2\}} = \frac{K}{K+1},
$$

which is known as the Rician $K$-factor [1].

Channel and antenna geometry determine the rank of $\mathbf{H}_{d,n}$ and, thus, MIMO system performance. For transmitter–receiver distance much larger than antenna interelement distance (such as in typical sub/urban and rural scenarios) this rank is likely low or unitary [1, p. 41]. Then, performance has been found to degrade with increasing $K$ [1, p. 79] [4, Fig. 4]. For transmitter–receiver distance similar to antenna interelement distance (such as in indoor scenarios) this rank is likely high or even $N_T$. Then, performance has been found to improve with increasing $K$ [1, p. 79] [4, Fig. 4].

Let us assume that there is transmit correlation but no receive correlation, as in the downlink from a high base-station antenna to a mobile station immersed in a rich-scattering environment. Then, we can use the MIMO correlation model $\mathbf{H}_{d,n} = \mathbf{H}_w \mathbf{R}_T^{1/2}$, where $\mathbf{R}_T$ is the $N_T \times N_T$ transmit correlation matrix, and $\mathbf{H}_w$ has independent and identically distributed circularly-symmetric, zero-mean, complex Gaussian, unit-variance elements [1]. The rows of $\mathbf{H}_d$ are mutually independent whereas each of them has correlation matrix $\mathbf{R}_T$.

**B. Statistical Models for AS and $K$**

The AS (which approximates the root mean square of the power azimuth spectrum — PAS) and the $K$-factor determine the mean and variance of the channel fading gains (i.e., the channel matrix elements) and, thus, performance. Multi-antenna performance assessments have typically assigned arbitrary or extreme values to $K$ and spatial correlation [1] [2].

However, comprehensive measurements undertaken by the European WINNER II project [9] indicate that AS and $K$ can both be modeled as lognormal random variables with scenario-dependent means and variances. For example, for the indoor office/residential scenario denoted as A1 in [9], the AS (in degrees) and $K$ have been modeled as:

$$
\begin{align*}
\text{AS} &= 10^{0.31 \chi + 1.64}, \quad \chi \sim \mathcal{N}(0, 1), \\
K &= 10^{0.1(6 \psi + 7)}, \quad \psi \sim \mathcal{N}(0, 1).
\end{align*}
$$

The AS and $K$ distributions for the other line-of-sight scenarios measured in [9] are reproduced in Table I. This table also shows the AS–$K$ correlation, i.e., the correlation coefficient of $\chi$ from (4) and $\psi$ from (5). Scenario A1 experiences the strongest (negative) such correlation.

The Laplacian PAS, which has been found to model accurately received power in actual channels [9], is expressed in terms of the AS in [14, Eqns. (4.2), p. 136]. Then, AS affects the elements of $\mathbf{R}_T$ as shown in [14, Eqns. (4.3)-(4.4), pp. 136-137].
TABLE I
BASE-STATION AS AND K STATISTICS [9, TABLE 4-5]

<table>
<thead>
<tr>
<th>Scenario</th>
<th>AS [°]</th>
<th>K</th>
<th>ρ</th>
</tr>
</thead>
<tbody>
<tr>
<td>A1: indoor office/residential</td>
<td>10^0.64±0.03</td>
<td>10^0.1(1+6γ)</td>
<td>−0.6</td>
</tr>
<tr>
<td>B1: typical urban microcell</td>
<td>10^0.40±0.03</td>
<td>10^0.1(1+6γ)</td>
<td>−0.3</td>
</tr>
<tr>
<td>B3: large indoor hall</td>
<td>10^0.22±0.18</td>
<td>10^0.1(1+6γ)</td>
<td>+0.2</td>
</tr>
<tr>
<td>C1: suburban</td>
<td>10^0.78±0.12</td>
<td>10^0.1(1+6γ)</td>
<td>+0.2</td>
</tr>
<tr>
<td>C2: typical urban macrocell</td>
<td>10^1.00±0.25</td>
<td>10^0.1(1+6γ)</td>
<td>+0.1</td>
</tr>
<tr>
<td>D1: rural macrocell</td>
<td>10^0.70±0.21</td>
<td>10^0.1(1+6γ)</td>
<td>+0.0</td>
</tr>
<tr>
<td>D2a: rural, high-speed</td>
<td>10^0.70±0.31</td>
<td>10^0.1(1+6γ)</td>
<td>+0.0</td>
</tr>
</tbody>
</table>

III. MIMO ZF DETECTION APPROACH AND ANALYSIS

A. ZF Detection for Perfect ICSI

Assuming perfectly-known H, ZF detection for the received-signal from (1) means estimating the symbol transmitted through the kth antenna (i.e., x_k) by mapping the kth element of vector

\[ \sqrt{N_T/E_s} [H^H H]^{-1} H^H r = x + \sqrt{N_T/E_s} [H^H H]^{-1} H^H n, \]

into the closest modulation constellation symbol (i.e., slicing). Note that, although there is no interference among the transmitted symbols in the ideal ZF detector, the symbol-detection noises are correlated across the data streams. Thus, ZF detection is suboptimal, but has been widely studied because it yields low-complexity MIMO receivers [1, p. 153].

B. Conventional ZF Detection Approach for Estimated ICSI

Assuming availability of an estimate G of H, ZF detection is typically undertaken by simply replacing H with G in (6), i.e., with [7] [8]:

\[ \sqrt{N_T/E_s} [G^H G]^{-1} G^H r. \]

This approach is easy to implement but can be difficult to analyze. A ZF detection approach that can be analyzed more easily and that also better accounts for ICSI estimation accuracy is proposed next, based on our work for SIMO systems [15].

C. Distribution of Channel Given ICSI Estimate

Let us assume that the channel matrix H and its estimate G are jointly Gaussian, as is the case for the popular ICSI estimation methods using PSAM and interpolation [15]. Let us denote by h_k^H, with mean h_k^H, and g_k^H, with mean g_k^H, corresponding rows from H and G. Then, the mean and covariance of h given g (both N_T x 1) are [16, p. 562]:

\[ h_m = E\{h|g\} = E\{h\} + R_{hg} R_g^{-1} (g - E\{g\}) \]
\[ R_c = E\{(h - h_m)(h - h_m)^H|g\} = R_h - R_{hg} R_g^{-1} R_{gh} \]

Thus, given G, the channel matrix can be written as

\[ H = H_m + H_c, \]

where H_m has rows given by the vectors h_k^H described by (8), and H_c has rows whose distribution is \( N_c(0, R_c) \) and described by (9). Note that H_m is the MMSE estimate of H given G [17, p. 203] [8, Appendix 1]. Finally, H_m and the required covariance matrices can be computed for various ICSI estimation methods as in [14] [18].

D. New ZF Detection Approach for Estimated ICSI

Substituting (10) into (1) recasts the received signal vector as follows:

\[ r = \sqrt{E_s/N_T} H_m x + \sqrt{E_s/N_T} H_c x + n, \]

where \( \nu \) comprises channel estimation error and actual noise and is, therefore, denoted effective noise [7]. Note that the new noise vector \( \nu \) is zero-mean, complex-valued, Gaussian-distributed with correlation matrix:

\[ E\{\nu\nu^H\} = \left[ \frac{E_s}{N_T} \text{tr}(R_e) + N_0 \right] I_{N_T}. \]

Let us assume that beside G, the above covariance matrices are also known. This latter assumption is supported by the fact that the channel statistics fluctuate much more slowly than the channel gains [9] and, thus, can be estimated accurately for low per-symbol complexity [14] [15]. For ICSI estimation methods that employ pilot symbols and interpolation these covariance matrices can be computed as shown in [14] [18].

Then, in the new signal model (11), matrix H_m is known. Therefore, instead of conventional ZF detection as in (7) we can use the following new approach:

\[ \sqrt{N_T/E_s} [H_m^H H_m]^{-1} H_m^H r = x + \sqrt{N_T/E_s} [H_m^H H_m]^{-1} H_m^H \nu. \]

Note that a comparison of the ZF detection approaches from (13) and (7) is beyond the scope of this work, although it is currently under investigation. Based on our previous investigations of similar approaches for SIMO, we anticipate that the new detector will yield better performance than the conventional one in highly-correlated channels, i.e., for predominantly-low AS [14, Fig. 3.13, p. 99].

Now, for the kth stream of the detector from (13), the instantaneous SNR can be expressed as [1] [4] [7]

\[ \gamma_k = \frac{\tilde{\Gamma}_k}{(H_m^H H_m)^{-1})_{k,k}} \]

where \( \tilde{\Gamma}_k \) is given by

\[ \tilde{\Gamma}_k = \frac{E_s}{N_T} \text{tr}(R_e) + N_0. \]

E. ZF Detection AEP Derivation

For Rician fading the N_T x N_T matrix H_m^H H_m has a noncentral Wishart distribution that has been approximated with a central Wishart distribution [4]. Then, the symbol-detection SNR for the kth data stream from (14) has a
Gamma distribution [19, p. 103] described by shape parameter \( \alpha = N_R - N_T + 1 \) and scale parameter \( \beta = \Gamma_k \) [4], where

\[
\Gamma_k = \frac{\bar{\Gamma}_k}{\sum_{j,k} \bar{\Gamma}_{j,k}},
\]

with [4, Section 3]

\[
\Sigma_m = \frac{1}{N_R} \mathbf{H}_d^H \mathbf{H}_d + \mathbf{R}_m.
\]

Above, \( \mathbf{R}_m \) is the covariance matrix of \( \mathbf{h}_m \), which, based on (8), can be written as \( \mathbf{R}_m = \mathbf{R}_{hg} \mathbf{R}_g^{-1} \mathbf{R}_{gh} \).

Hereafter, we discard the stream index \( k \) from our notation, for simplicity.

Since \( \gamma \) from (14) is Gamma distributed with the parameters mentioned above, its probability density function (p.d.f.) is [4, Eqn. (4)]:

\[
p(\gamma) = \frac{1}{(N_R - N_T)!} \frac{\gamma^{N_R - N_T} e^{-\gamma/\Gamma}}{\Gamma^{N_R - N_T + 1}}, \quad \gamma \geq 0,
\]

and its moment generating function (m.g.f.) is [19, p. 106]:

\[
M_{\gamma}(s) = \mathbb{E}\{e^{s\gamma}\} = (1 - \Gamma s)^{-(N_R - N_T + 1)}, \quad s < \frac{1}{\Gamma}.
\]

Given the symbol-detection SNR \( \gamma \), the \( M \)-PSK error probability can be written as [2, Eqn. (8.22)]

\[
P_e(\gamma) = \frac{1}{\pi} \int_0^{M-1} \sin^2 \frac{\pi}{M} \exp \left\{ -\gamma \frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta} \right\} d\theta.
\]

Then, the average error probability (AEP) can be written in terms of the m.g.f. of \( \gamma \) as follows [2, Chapter 9]:

\[
P_e \triangleq \mathbb{E}\{P_e(\gamma)\} = \frac{1}{\pi} \int_0^{M-1} \sin^2 \frac{\pi}{M} M_{\gamma}\left( -\frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta} \right) d\theta.
\]

We are allowed to write (21) because \( s = -\frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta} < 0 < \frac{1}{\Gamma} \).

Now, substituting (19) into (21) yields the new ZF AEP expression

\[
P_e = \frac{1}{\pi} \int_0^{M-1} \left( \frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta + \Gamma \sin^2 \frac{\pi}{M}} \right)^{(N_R - N_T + 1)} d\theta,
\]

which relates MIMO ZF symbol-detection performance to the constellation size, antenna number, \( K \)-factor, \( \mathbf{H}_d \), spatial correlation (i.e., AS), and ICSI estimation accuracy. Using (22), it can readily be confirmed that the MIMO ZF per-stream diversity order is \( (N_R - N_T + 1) \) [1, p. 153].

IV. NUMERICAL RESULTS

Let us first validate, by computing the AEP with (22) and the average error rate (AER) by simulation, the approximation of the noncentral Wishart distribution with the corresponding central Wishart distribution, assuming perfectly known CSI. Simulation results are shown for 150,000 \( \mathbf{H}_w \) samples and for AS and \( K \) equal to their averages for scenario A1. The AEP and AER, averaged over all \( N_T \) streams, are shown vs. the bit-SNR \( \gamma_b = \frac{E_s}{N_0} \log_2 K \).

![Fig. 1. AEP from (22) and AER from simulation for ZF MIMO detection.](image)

Simulation results are shown for 150,000 \( \mathbf{H}_w \) samples, for Rayleigh fading (i.e., \( K = 0 \)) as well as for Rician fading with rank(\( \mathbf{H}_{d,n} \)) = 1. Obviously, since the approximation has no effect for \( K = 0 \), the simulation and analytical results match perfectly for Rayleigh fading. The match is also very good for Rician fading, confirming that the discussed Wishart approximation is indeed highly accurate, for rank(\( \mathbf{H}_d \)) = 1. Usefully, AEP computation with (22) requires 5 – 6 orders of magnitude shorter time than AER computation by simulation. Finally, note that, in order to achieve AEP = 10^-2, the ZF MIMO scheme requires 5.5 dB higher SNR for Rician fading with rank(\( \mathbf{H}_d \)) = 1, than for Rayleigh fading.

However, other numerical results, which we show in [18] and in the journal version of this paper (to be submitted soon), indicate that the approximation of the noncentral Wishart distribution with the central Wishart distribution is accurate only when the deterministic channel matrix component has non-full rank. Thus, the AEP in (22) should be employed only in the low-rank case (which, fortunately, is the predominant case for practical single-user MIMO). This point has not been made in previous work that has employed this Wishart distribution approximation to derive performance measures for Rician fading [4] [5] [6].

Next, we show the MIMO ZF AEP computed with (22) and then averaged over all \( N_T \) transmitted streams and over 1000 samples of AS with scenario-dependent distribution (as in Table I), for QPSK and least-squares ICSI estimation as described in [20]. The \( K \)-factor is set to:

- Zero (i.e., Rayleigh fading; unrealistic)
- The mean of the distribution from Table I (unrealistic)
- Random value, with distribution from Table I (realistic)

Fig. 2 shows that ZF detection performance is poor for all
Finally, these results reveal that also the AS–K because of a different slope of the AEP vs. SNR curves.

detection approaches.

numerical results comparing the conventional and the new ZF correlation should be accounted for in link-budget design.

is somewhat worse than the performance for average

K

optimistic. Furthermore, note that

K

randomness has a non-negligible effect on performance.

V. CONCLUSIONS

An AEP expression has been derived herein for a new MIMO ZF detection technique that accounts for antenna correlation and for ICSI estimation accuracy. This AEP expression has been sought in order to help evaluate the realistic performance of MIMO communications systems in scenarios with random AS and K much quicker than by simulation. The AEP derivation is based on an approximation of the noncentral Wishart distribution with a central Wishart distribution.

We have validated that our AEP expression yields accurate results for practical conditions. On the other hand, unlike previous work, we have also clearly indicated the cases in which the Wishart distribution approximation is inaccurate, and made clear that then the derived AEP expression should not be applied. Using this expression, we have found that the performance of MIMO ZF detection can be very poor for actual Rician fading compared to theoretical Rayleigh fading. Furthermore, we found that the AS and K randomness and correlation have a non-negligible effect on performance.

REFERENCES