Consensus with Constrained Convergence Rate and Time-Delays

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Abstract. In this paper we discuss consensus problems for networks of dynamic agents with fixed and switching topologies in presence of delay in the communication channels. The study provides sufficient agreement conditions in terms of delay and the second largest eigenvalue of the Perron matrices defining the collective dynamics. We found an exact delay bound assuring the initial network topology preservation. We also present an analysis of the agreement speed when the asymptotic consensus is achieved. Some numerical examples complete the presentation.

1 Introduction

The analysis of multi-agent systems has various applications in many areas encompassing cooperative control of vehicles [3, 5], congestion control in communication networks, flocking [12], distributed sensor networks [2]. In many of these applications, all the agents need to agree with respect to a priori fixed criteria and the agreement might be subject to some speed constraints. Furthermore, the communication channels between the agents are not ideal and may introduce time-delays into dynamics.

The consensus problem for directed graphs with a fixed topology was treated in the case where no convergence speed is imposed (see [11]). We also note that
the consensus problem for undirected graphs with a switching topology generated
by a bounded confidence was considered in [5, 6]. The characterizations given in
these works use the notions of periodically linked together (the agents are peri-
dically linked together) or finally linked together (i.e. the agents are linked in
$\bigcup_{t \geq s} G(t)$, $\forall s \geq 0$, where $G(t)$ is the graph describing the network topology at in-
stant $t$). A frequency domain approach for the consensus problem of a fixed topol-
ogy network of continuous-time integrators agents communicating through delayed
channels was developed in [10]. Precisely when the time-delays in all communica-
tion channels is given by the same constant, the time-delay bound guaranteeing the
consensus is given in terms of the smallest eigenvalue of the matrix defining the
collective dynamics (called the Perron matrix).

The updating rule used in this paper matches in the free of delay case with those
considered by [1] (see also [9]). The time-delay introduced in our model is constant
and is the same for all the communication channels. This delay value can be seen
either as a computational time or as a communication latency, depending on the
application under consideration. Our aim is to find the delay bound that guarantees
the agreement with at least an a priori given speed, in terms of the second largest
eigenvalue of the Perron matrix.

The remainder of this paper is organized as follows. In Section 2 we introduce
some basic notions related to graph theory and we formulate the model. In Section
3 we present the convergence result and derive the exact delay margin assuring
an agreement speed which preserves the network topology. Section 4 provide an
analysis of the agreement speed when the consensus for a switching topology case
is assured. Some numerical examples are presented in Section 5 and Section 6 ends
the paper with some concluding remarks.

2 Preliminaries

2.1 Algebraic Graph Theory Elements

Let $G = (\mathcal{V}, E)$ denotes a directed graph with the set of vertices $\mathcal{V}$ and the set of
edges $E$. Each vertex is labeled by $v_i \in \mathcal{V}$, $i = 1, \ldots, n$ and one says that $(i, j) \in E$
if there exists an edge between $v_i$ and $v_j$. We consider $N_i = \{v_j \in \mathcal{V} \mid (i, j) \in E\}$.
Given two graphs with the same set of vertices, $G_1 = (\mathcal{V}, E_1)$ and $G_2 = (\mathcal{V}, E_2)$ we 
say that $G_1 \subset G_2$ if $E_1 \subset E_2$.

Definition 1. A path in a given graph $G = (\mathcal{V}, E)$ is a union of edges $\bigcup_{k=1}^{p}(i_k, j_k)$
such that $i_{k+1} = j_k, \forall k \in \{1, \ldots, p - 1\}$.

Two nodes $v_i, v_j$ are connected in a graph $G = (\mathcal{V}, E)$ if there exists at least a
path in $G$ joining $v_i$ and $v_j$ (i.e. $i_1 = i$ and $j_p = j$).

A connected graph has all the nodes connected.

The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ of the graph $G$ is the integer matrix with the
$i j$-entry equals to the number of arcs from $i$ to $j$ which is usually 0 or 1. The graph
Laplacian of $G$ is than defined as
\[ L = D - A \]  

where \( D = \text{diag}(d_1, \ldots, d_n) \) is the degree matrix of \( G \) with \( d_i = \sum_{j \neq i} a_{ij}. \) Therefore, \( L \) has the right eigenvector \( 1 = (1, 1, \ldots, 1) \) associated with the eigenvalue 0 (\( L1 = 0 \)).

### 2.2 Consensus Protocol

In the following we consider that each vertex \( v_i \) represents a dynamic agent and the state of the network will be given by \[ x(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot))^\top \in \mathbb{R}^n \] where \( x_i(t) \) is a scalar real value assigned to \( v_i \) at the moment \( t. \) The value \( x_i(t) \) will be called the opinion of the agent \( v_i \) at the moment \( t. \)

In order to motivate the updating rule proposed in the sequel, we consider a synchronized sensors network (i.e. all the sensors clocks are synchronized and each sensor knows at which time was sent a specific information even if it does not receive this information instantaneously). This allows us to ignore the communication delay. On the other hand some encoding-decoding delays are associated to each sensor. One considers that these delays are constant and all are equal \( \tau. \) Therefore, the sensors update their opinion/decision at the time-step \( t + 1 \) considering the information available at time \( t + 1 - \tau \) via the network configuration available at the time-step \( t. \) Thus, the discrete-time collective dynamics is defined by

\[ x(t + 1) = P(t)x(t + 1 - \tau) \]  

where \( P(t) \) is a matrix considered to be doubly stochastic (called the Perron matrix of the system). In the sequel one denotes by \( p_{ij}(t) \) the entries of the Perron matrix \( P(t). \) This model presented above can be also interpreted in terms of interactions between a set of agents and a virtual environment. Considering \( \tau \) the round-trip delay associated to each agent, the actions at instant \( t + 1 \) are determined by the reactions received via the network configuration at time \( t \) of the action done at the time-step \( t + 1 - \tau. \) In other words, any opinion at the instant \( t + 1 \) is a weighted average of the opinion values at the instant \( t + 1 - \tau: \)

\[ x_i(t + 1) = \sum_{j=1}^{n} p_{ij}(t)x_j(t + 1 - \tau), \quad \forall i \in \{1, \ldots, n\}, t \geq 0 \]  

In the sequel we assume that the Perron matrix \( P(t) \) satisfies the following properties:

**Assumption 1** For \( t \in \mathbb{N}, \) the coefficients \( p_{ij}(t) \) satisfy

1. \( p_{ij}(t) \in [0, 1], \) for all \( v_i, v_j \in \mathcal{V}. \)
2. \( \sum_{j=1}^{n} p_{ij}(t) = 1 = \sum_{i=1}^{n} p_{ij}(t), \) for all \( v_i \in \mathcal{V}. \)

Precisely, \( p_{ij}(t) > 0 \) if agent \( j \) communicates at instant \( t \) its current value to agent \( i \) and \( p_{ij}(t) = 0 \) otherwise. Since \( P(t) \) is supposed doubly stochastic we actually consider only balanced graphs. Let us denote by
\[ 1 = \lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t) \geq 0 \]

the eigenvalues of the symmetric positive semi-definite matrix \( P(t)^T P(t) \).

**Remark 1.** An extensively used example of matrix \( P \) satisfying Assumption[1] is \( I - \alpha L \) (see for instance [5, 6, 10]) where \( I \) is the identity matrix and \( L \) is the graph Laplacian matrix associated to the graph \( G \). When \( G \) is undirected \( L \) is symmetric. Moreover \( L \) and \( P \) obtained like this are positive semi-definite and \( \mu_2 = \sqrt{\lambda_2} = 1 - \lambda_2(L) \), where \( \lambda_2(L) \) is the smallest nonzero eigenvalue of \( L \) (one considers \( G \) is connected so the multiplicity of the eigenvalue zero of \( L \) is one). It is noteworthy that \( \lambda_2(L) \) is called the connectivity of \( G \) (see [4] for details on \( \lambda_2(L) \) and graph theory).

In this paper we consider a set of agents that updates their opinions using the algorithm [2] and \( G(t) = (\mathcal{V}, E(t)) \) is the graph representation of the corresponding dynamic network with a switching topology that is time-dependent.

Letting \( E \) the set of edges of the graph \( G(0) \), we consider that the evolution of the network topology is given by

\[ E(t) = \{(i, j) \in E \mid |x_i(t) - x_j(t)| \leq Mp \} \]

(4)

where \( M \in \mathbb{R}_+ \) and \( \rho \in (0, 1) \) are some parameters fixed by the designer. It means that, at each time-step the agents become more confident in their own opinion and they take into account only the neighbors whose opinion approaches enough their own opinion. This procedure lead either to a fast convergent consensus algorithm or to a partition of agents in several groups where the agreement is reached.

**Remark 2.** It is clear that a smaller \( \rho \) leads to a higher convergence speed. We also note that \( \rho = 1 \), corresponding to the bounded confidence model, imposes no convergence speed constraints.

A natural assumption that is made in this paper states that \( P(t_1) = P(t_2) \) when \( E(t_1) = E(t_2) \) (for similar network configurations the same weighted average is used to determine the opinion values of the next step). It is worth noting that

\[
\sum_{i=1}^{n} x_i(t+1) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}(t) x_j(t-\tau+1) = \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}(t) x_j(t-\tau+1) = \sum_{j=1}^{n} x_j(t-\tau+1), \quad \forall t \leq 0
\]

(5)

Considering an initial condition \( x(t) = x^0, t \in [-\tau, 0] \), one easily obtains that \( S = \sum_{j=1}^{n} x_i(t) \) and \( \text{Ave}(x(t)) = \frac{S}{n} \) are invariant quantities.

**Definition 2 (agreement).** We say nodes \( v_i \) and \( v_j \) asymptotically agree if and only if \( \lim_{t \to \infty} x_i(t) = \lim_{t \to \infty} x_j(t) \). Two nodes asymptotically disagree if \( \lim_{t \to \infty} x_i(t) \neq \lim_{t \to \infty} x_j(t) \). An algorithm guarantees asymptotic consensus if: for every initial condition \( x(t) = x^0, t \in [-\tau, 0] \) and for every sequence \( \{P(t)\} \) allowed by (4) and Assumption[1] all the nodes asymptotically agree.
Remark 3. Since $\text{Ave}(x(t))$ is an invariant quantity, the algorithm (2) may guaranty only the asymptotic average-consensus ($\lim_{t\to\infty} x_i(t) = \text{Ave}(x(t)), \forall i$).

3 Consensus Problem for Networks with Fixed Topology

Let us consider the positive function
\[ V(t) = ||x(t) - x^*|| \]
where $x^* = \text{Ave}(x(t))\mathbb{1}$ where $\mathbb{1}$ represents the column vector whose elements are all equal to one. For all $(i, j) \in E$ the following holds:
\[
V(0)^2 \geq |x_i(0) - x_i^*|^2 + |x_j(0) - x_j^*|^2 \geq \frac{1}{2}(|x_i(0) - x_i^*| + |x_j(0) - x_j^*|)^2
\]
\[
\geq \frac{1}{2}|x_i(0) - x_j(0)|^2
\]
Therefore, in the sequel we set $M = \sqrt{2}V(0)$ in order to guaranty that all the possible initial transmission lines appear in the initial configuration of the network.

3.1 Convergence Result

In this paragraph we prove that the algorithm (2) assures the asymptotic average-consensus when the network topology does not evolve according to (4) but is fixed. Obviously, in order to reach the consensus we have to assume that the network topology is given by a strongly connected graph (the opinion of each agent can be accessed after a given delay by all the other agents). Precisely, the discrete-time collective dynamics is defined by
\[ x(t) = Px(t - \tau) \]
where $P$ is a doubly stochastic matrix. The eigenvalues of the symmetric stochastic matrix $P^\top P$ are denoted by
\[ 0 \leq \lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_2 \leq \lambda_1 = 1 \]
Since the graph is connected, 1 is a simple eigenvalue so $\lambda_2 < 1$ and the following result holds.

Proposition 1. The algorithm (6) guaranties the asymptotic average-consensus. Moreover if $\mu_2 = \sqrt{\lambda_2}$ one has
\[ V(t + k\tau) \leq \mu_2^k V(0), \forall t \in [-\tau, 0], k \in \mathbb{N} \]
Proof. Let us recall that a doubly stochastic matrix $P$ (and $P^\top$) has always the left and right eigenvector $\mathbb{1}$ corresponding to the eigenvalue $\lambda_1 = 1$. If $\{u_i\}$ is the set of orthonormal eigenvectors of the symmetric stochastic matrix $P^\top P$, then
\[ x(t) = \sum_{i=1}^{n} a_i(t)u_i, \quad a_i(t) \in \mathbb{R}, \forall i, \forall t \geq -\tau \]

and using the linearity of \( P \) we get

\[ P^\top x(t) = P^\top Px(t - \tau) = \sum_{i=1}^{n} \lambda_i a_i(t - \tau)u_i \]

Since \( \lambda_1 = 1 \) and \( u_1 = 1 \) one obtains that

\[ P^\top x(t + k\tau) = P^\top Px(t + (k - 1)\tau) = a_1(0) \cdot 1 \]

\[ + \sum_{i=2}^{n} \lambda_i^k a_i(0)u_i, \forall t \in [-\tau, 0] \]

which leads to

\[ P^\top x(t) \xrightarrow{t \to \infty} a_1(0) \cdot 1 \]

Multiplying by \( 1^\top \) at the left one arrives at

\[ \sum_{j=1}^{n} x_i(t) = 1^\top x(t) \xrightarrow{t \to \infty} na_1(0) \]

so using the invariance of \( S \) (see (5)) one has \( x^* = a_1(0) \cdot 1 \) which means

\[ x(t) \xrightarrow{t \to \infty} a_1(0) \cdot 1 \quad (7) \]

Let \( t \in [-\tau, 0] \) and \( k \in \mathbb{N} \), it is straightforward that

\[ V^2(t + k\tau) = (x(t + k\tau) - x^*)^\top (x(t + k\tau) - x^*) \]

\[ = (x(t + (k - 1)\tau) - x^*)^\top P^\top P(x(t + (k - 1)\tau) - x^*) \]

\[ = \left( \sum_{i=2}^{n} a_i(t + (k - 1)\tau)u_i \right)^\top \left( \sum_{i=2}^{n} \lambda_i a_i(t + (k - 1)\tau)u_i \right) \]

\[ = \sum_{i=2}^{n} \lambda_i a_i(t + (k - 1)\tau)^2 \leq \lambda_2 V^2(t + (k - 1)\tau) \]

Repeating the procedure \( k \) times and taking into account that \( V(t) = V(0), \forall t \in [-\tau, 0] \) the proof is finished. \( \Box \)

**Remark 4.** A version of Proposition 1 in the free of delays case can be found in [10]. Even if the presence of delay is not crucial, for the sake of completeness we
preferred to provide a proof in the delayed case. Moreover, our result emphasizes both the convergence speed of the consensus protocol (6) and the consensus value.

### 3.2 Delay Margin Assuring a Fixed Network Topology

Let us consider now the network topology evolution (4) and derive the relation between $\rho$ (in (3)), $\lambda_2(0)$ and $\tau$ assuring that $E(t) = E, \forall t \geq 0$. In other words, given an initial configuration (which determines $P(0)$ and consequently $\lambda_2(0)$) we derive a delay margin that assures an agreement speed at least equal with the convergence speed of the sequence $\{\rho^i\}$.

**Proposition 2.** The algorithm (2) guarantees the asymptotic average-consensus of a network of agents whose topology evolution is given by (4) and $E(t) = E, \forall t \geq 0$ if $\tau \leq \log_\rho \mu_2(0)$.

**Proof.** Let us consider again $t \in [-\tau, 0]$ and denote $P = P(0)$, $\lambda_2 = \lambda_2(0)$. Like in the proof of Proposition 1 we deduce that

$$V(1) = ||P(x(1 - \tau) - x^*)|| \leq \mu_2 V(0) \quad (8)$$

For $\rho \in (0, 1)$ one has $\tau \leq \log_\rho \mu_2 \Leftrightarrow \mu_2 \leq \rho^\tau \leq \rho$. On the other hand $|x_i(1) - x_j(1)| \leq \sqrt{2}V(1)$. Therefore, (8) leads to

$$|x_i(1) - x_j(1)| \leq \sqrt{2}V(0)\rho = M\rho$$

So, if $(i, j) \in E$ one gets $(i, j) \in E(1)$ which means $E(1) = E$ (see (4)).

Supposing that $E(T) = E, \forall T \leq \tau - 1$ (and consequently $P(T) = P, \forall T \leq \tau - 1$) one obtains

$$V(T + 1) = ||P(x(T - \tau + 1) - x^*)|| \leq \mu_2 V(0)$$

Therefore

$$V(T + 1) \leq \rho^\tau V(0) \leq \rho^{T+1} V(0)$$

and as above

$$|x_i(T + 1) - x_j(T + 1)| \leq \sqrt{2}V(0)\rho^{T+1} = M\rho^{T+1}$$

implying $E(T + 1) = E$. Resuming, we have proved that

$$V(T) \leq \rho^T V(0) \quad \& \quad E(T) = E, \forall T \in [0, \tau] \quad (9)$$

Let us now suppose that $E(T) = E, \forall T \leq T^*$ (and consequently $P(T) = P, \forall T \leq T^*$) and consider $T^* = t^* + k\tau, k \in \mathbb{N}, t^* \in [0, \tau - 1]$. The procedure used in the proof of Proposition 1 and (9) yield

$$V(T^* + 1) = V(t^* + k\tau + 1) \leq \mu_2^k V(t^* + 1) \leq \rho^{k\tau} \rho^{t^*+1} V(0)$$
which immediately leads to
\[ |x_i(T^* + 1) - x_j(T^* + 1)| \leq \sqrt{2}V(0)\rho^{T^* + 1} = M\rho^{T^* + 1} \]

implies \( E(T^* + 1) = E \). In other words we have proved the statement by induction. \( \square \)

**Definition 3.** Considering \( u_i(0), i = 1, \ldots, n \) the orthonormal eigenvectors of the symmetric matrix \( P(0)^\top P(0) \), any initial condition can be written as \( x(0) = \sum_{i=1}^{n} a_i(0) u_i(0) \). The set of admissible initial conditions is then defined by \( H = \{ x(0) \in \mathbb{R}^n \mid a_2(0) \neq 0 \} \).

The Lebesgue measure of the set \( \mathbb{R}^n \setminus H \) is zero and this set will be neglected in the further development. In other words we always consider that \( x(0) \in H \).

The next result shows that for all admissible initial conditions \( x(0) \in H \), \( \tau \leq \log_\rho \mu_2(0) \) is a minimal requirement assuring that the network topology is fixed.

**Proposition 3.** Let consider a network of agents whose topology evolution is given by (4) and the discrete-time collective dynamics given by (2). If \( \tau > \log_\rho \mu_2(0) \) then for all \( x(0) \in H \) there exists \( t > 0 \) such that \( E(t) \subsetneq E \).

**Proof.** Let us suppose that the statement is false. Thus \( E(t) = E \), \( \forall t \geq 0 \) which implies \( P(t) = P, \lambda_i(t) = \lambda_i, u_i(t) = u_i, \forall i, \forall t \geq 0 \) reducing (2) to (6). Since the network topology evolution is given by (4) we deduce that
\[ |x_i(t) - x_j(t)| \leq M\rho^t, \quad \forall (i, j) \in E, t \geq 0 \]  
and as in the proof of Proposition [2]
\[ V(t) \leq \rho^t V(0), \quad \forall t \geq 0 \]  
(11)

On the other hand (7) leads to \( x(t) - x^* = \sum_{i=2}^{n} a_i(t) u_i, \quad \forall t \). Therefore, taking into account the orthogonality of \( \{ u_i \} \) one obtains
\[ V(t + k\tau)^2 = (x(t + k\tau) - x^*)^\top (x(t + k\tau) - x^*) \]
\[ = \sum_{i=2}^{n} \lambda_i a_i(0)^2 \leq \lambda_2 a_2(0)^2, \quad \forall t \in [-\tau, 0] \]  
(12)

Combining (11) and (12) one arrives at
\[ \lambda_2 a_2(0)^2 \leq V(0)^2 \rho^{2k\tau} \rho^{2t} = \left( \frac{\rho^t}{\mu_2} \right)^k \geq \frac{|a_2(0)|}{V(0)^t}, \quad \forall t \in [-\tau, 0], \forall k \in \mathbb{N} \]

But \( 0 < \rho^\tau < \mu_2 \) means \( \left( \frac{\rho^t}{\mu_2} \right)^t \xrightarrow{t \to \infty} 0 \) and \( x(0) \in H \) means \( \frac{|a_2(0)|}{V(0)^t} > 0 \) which leads to a contradiction. \( \square \)
4 Agreement Speed in Networks with Dynamic Topology

In the sequel we consider a network of agents with the discrete-time collective dynamics given by (2) whose topology evolution is given by (4). In the previous sections we have shown that $\tau \leq \log_\rho \mu_2(0)$ guarantees a fixed network topology, the consensus and an agreement speed higher than the convergence speed of $(\rho^t)_{t \geq 0}$. We have also noticed that $\tau > \log_\rho \mu_2(0)$ is equivalent with a time-dependent network topology. In this section we analyze the agreement speed for $\tau > \log_\rho \mu_2(0)$. It is worth noting that under this hypothesis the global consensus may not be achieved and the agents organize themselves into several communities where a local agreement is reached (see Section 5). However in this paper we analyze only the case when the global consensus can be achieved, thus the agents are at least finally linked together.

**Assumption 2** There exist $M^* > 0$, $\gamma < \rho$ such that $|x_i(t) - x_i^t| \leq M^*\gamma^t$, $\forall i, t$.

**Proposition 4.** Consider a network of agents with the discrete-time collective dynamics given by (2) whose topology evolution is given by (4). Suppose the agents asymptotically agree and Assumption 2 holds, then

$$\exists T^* > 0 \text{ such that } |x_i(t) - x_j(t)| \leq M\rho^t, \forall t \geq T^*$$

(13)

Therefore, the network topology is fix for all $t \geq T^*$ i.e. $E(t) = E(T^*) = E(0)$, $P(t) = P(T^*) = P(0)$, $\forall t \geq T^*$. Moreover $\tau < \log_\rho \mu_2(T^*)$.

**Proof.** Assumption 2 yields

$$|x_i(t) - x_j(t)| \leq |x_i(t) - x_i^t| + |x_j(t) - x_j^t| \leq 2M^*\gamma^t$$

On the other hand, it is clear that $\gamma < \rho$ implies the existence of $T^* > 0$ such that $2M^*\gamma^t \leq M\rho^t$, $\forall t \geq T^*$. Thus, (13) holds and $E(t) = E(T^*) = E(0)$, $P(t) = P(T^*) = P(0)$, $\forall t \geq T^*$.

The last part of the statement will be proven by contradiction. Let us consider that $\mu_2(T^*) > \rho^T$. It is noteworthy that we can suppose $x(T^*) \in H$ (see [8] for details). Then Proposition 3 implies the existence of a time step $t > T^*$ such that $E(t) \neq E(T^*)$ contradicting the first part of the statement. In other words the following sequence of inequality must hold $\mu_2(T^*) \leq \rho^T$ and the proof is finished.

Since $E(0)$ has a finite number of elements and $E(t) \subset E(0)$ we deduce that $E(t)$ belongs to a finite set of possible configurations. Therefore, $P(t)$ belongs to a finite set of doubly stochastic matrices $\mathcal{M} = \{P_1, \ldots, P_m\}$. The subspace of dimension $n - 1$ orthogonal to $\text{span}\{1\}$ will be denoted by $\mathcal{F}$.

**Lemma 1.** The subspace $\mathcal{F}$ of $\mathbb{R}^n$ is an $P_k$ - invariant subspace for all $k \in 1, \ldots, m$.

**Proof.** Let $x \in \mathcal{F}$ and $P \in \mathcal{M}$. From the definition of $\mathcal{F}$ one has that $\mathbb{1}^\top x = 0$. Since $P$ is doubly stochastic it follows that $\mathbb{1}^\top Px = \mathbb{1}^\top x = 0$. Thus, $Px \perp \mathbb{1}$ which means $Px \in \mathcal{F}$. □
In the sequel we consider $P_k : \mathcal{F} \mapsto \mathcal{F}$. Thus $P_k \in \mathbb{R}^{(n-1) \times (n-1)}$ and the spectrum of the initial $P_k$ is obtained by adding $\{1\}$ to the spectrum of the new $P_k$ (see [5] for details). In the proof of Proposition 1 we have noticed that $x(0) - x^* \in \mathcal{F}$, so, $x(t) - x^* \in \mathcal{F}, \forall t \geq 0$.

**Definition 4.** The joint spectral radius of a set of matrices $\mathcal{M} = \{P_1, \ldots, P_m\}$ is defined by:

$$\delta(\mathcal{M}) = \limsup_{k \to \infty} \delta_k(\mathcal{M})$$

where

$$\delta_k(\mathcal{M}) = \max_{s(i) \in \{1, \ldots, m\}, \forall i \in 1, \ldots, k} ||P_{s(k)} \cdots P_{s(1)}||^{1/k}, \forall k \geq 1$$

**Proposition 5.** Consider a network of agents with the discrete-time collective dynamics given by (2) whose topology evolution is given by (3).

a) If $\tau > \log \rho \mu_2(0)$ the consensus cannot be achieved faster than $O(\rho^t)$. 

b) If the agreement speed is not faster than $O(\rho^t)$ then 

$$\tau \geq \log \rho \delta(\mathcal{M}).$$

**Proof.** a) If the consensus is achieved with an agreement speed higher than the convergence speed of $(\rho^t)_{t \geq 0}$, Proposition 4 assures us that $\tau \leq \log \rho \mu_2(0)$. Thus, when $\tau > \log \rho \mu_2(0)$ the consensus cannot be achieved with an agreement speed higher than the convergence speed of $(\rho^t)_{t \geq 0}$.

b) The agreement speed is at most the convergence speed of $(\rho^t)_{t \geq 0}$ if there exists $M^* > 0$ and a subsequence $(x(t_p))_{p \geq 0}$ such that 

$$M^* \rho^p \leq ||x(t_p) - x^*||, \forall p \geq 0$$

For $t = t^* + k\tau, t^* \in [-\tau + 1, 0], k \geq 1$ one has

$$||x(t) - x^*|| = ||A_k \cdots A_1 (x(t) - x^*)|| \leq ||A_k \cdots A_1|| \cdot V(0)$$

where $A_i \in \mathcal{M}, \forall i = 1, \ldots, k$. Therefore,

$$M^* \rho^{p+ kp^\tau} \leq ||A_k \cdots A_1|| \cdot V(0), \forall t_p^* \in [-\tau + 1, 0], p \geq 0 \tag{14}$$

where $t_p^*$ and $k_p$ are chosen such that $t_p^* + k_p \tau = t_p, \forall p \geq 0$. Relation (14) can be further rewritten as

$$M^* \leq M^* \rho^{t_p^*} \leq \left( \frac{||A_k \cdots A_1||^{1/k_p}}{\rho^{t_p^*}} \right)^{k_p} \cdot V(0)$$

$$\leq \left( \frac{\delta(\mathcal{M})}{\rho^{t_p^*}} \right)^{k_p} \cdot V(0), \forall t_p^* \in [-\tau + 1, 0], p \geq 0 \tag{15}$$

Inequality (15) may hold only if $\frac{\delta(\mathcal{M})}{\rho^{t_p^*}} > 1$. Thus, we conclude that $\tau \geq \log \rho \delta(\mathcal{M})$. \qed
5 Numerical Examples

We illustrate our results using the karate club network initially studied by Zachary in [13]. This is a social network with 34 agents shown in Figure 1. The Perron matrix defining the collective dynamics is given by $P(t) = I - \alpha L(t)$ where $I$ is the identity matrix, $L(t)$ is the graph-Laplacian matrix at the moment $t$, and $\alpha$ is set to 0.05. The parameters of the model where chosen as follows: $M = \sqrt{2}V(0)$ and $\rho = 0.99$. The model was simulated for an initial condition chosen randomly in $[0, 1]^{34}$. The computations show that $\mu_2(0) = 0.9766$ and $\log_\rho \mu_2(0) = 2.356$. Simulations were performed as long as enabled by floating point arithmetics for $\tau = 2$, $\tau = 3$ and $\tau = 5$ (see (2)). The experimental results have proven that the network topology remains fix for $\tau = 2 < \log_\rho \mu_2(0)$ (Figure 1 left). For $\tau = 3 > \log_\rho \mu_2(0)$ the topology changes. Furthermore, only local consensus can be reached in this case as can be seen in Figures 1 right.

Fig. 1 Graph of the Zachary karate club network. Left: For $\tau = 2$ the initial network configuration is kept constant and the agents asymptotically reach the consensus. Right: For $\tau = 3$ some links are canceled during the simulation leading to two disconnected communities inside the graph.

Fig. 2 The Lyapunov functions using a logarithmic scale for $\tau \in \{1, 2, 3, 4, 5\}$. Dashed line represents $\sqrt{\rhoMp}$ using the same logarithmic scale.
The study of the agreement speed is summarized in Figure 2. The agreement speed decreases when \( \tau \) increases. The agents try to reach a global agreement as far as the Lyapunov function \( V(t) \leq \sqrt{nM}w^t \). When this condition is violated the agents organize themselves in communities where a local agreement is reached. It is noteworthy that the condition holds inside each group replacing \( n \) by the number of agents in the corresponding community.

6 Conclusions

In this paper we have analyzed a model of multi-agent system with decaying confidence and time-invariant delay in the communication channels. The study provide an exact delay bound that assure the preservation of the network topology. We have also shown that increasing the delay, the agreement speed decreases and the global consensus may not be reached. A sufficient condition for the global agreement of the agents in our model can be expressed in terms of the eigenvalues of the matrix defining the collective dynamics. Numerical simulations show us the necessity of this condition.

References