Preservation Theorems for MTL-Chains

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**Definition (Esteva, Godo)**

An **MTL-algebra** is an algebra \( \langle A, \land, \lor, \circ, \to, 1, 0 \rangle \) in which

- \( \langle A, \land, \lor, 1, 0 \rangle \) is a bounded lattice,
- \( \langle A, \circ, 1 \rangle \) is a commutative monoid,
- residuation holds: \( x \circ y \leq z \) iff \( y \leq x \to z \),
- prelinearity holds: \( (x \to y) \lor (y \to x) = 1 \).

The class of all MTL-algebras is a variety, denoted **MTL**.

Alternatively, an MTL-algebra is a representable, commutative, integral residuated lattice with a least element.

**Definition**

An **MTL-chain** is a linearly ordered MTL-algebra

The variety **MTL** is generated by the MTL-chains.
A $t$-norm is a binary operator on $[0, 1]$ that is associative, commutative, order-preserving and has identity 1. A $t$-norm is left-continuous if it is a left-continuous function in the traditional sense.

MTL-chains generalize the notion of a left-continuous $t$-norm from the unit interval $[0, 1]$ to arbitrary bounded chains. The assumption of left-continuity is replaced by the residuation property.

MTL, which stands for Monoidal T-norm Logic, was formulated by Esteva and Godo as the logic of left-continuous $t$-norms.

MTL-algebras are the algebraic semantics of MTL.
PART I: Completions of MTL-chains and preservation of properties by the construction.

PART II: Finite embeddability property for MTL-chains and also preservation of properties by the construction.
Given an MTL-chain $A$, we perform the Dedekind-MacNeille completion on the underlying order of $A$.

This gives us a complete chain $\langle C, \land^C, \lor^C \rangle$ and an embedding $i : A \rightarrow C$ that preserves all meets and joins.

**Problem**

Can we define $\circ^C$ and $\rightarrow^C$ operations on $C$ such that the algebra $C = \langle C, \land^C, \lor^C, \circ^C, \rightarrow^C, 1, 0 \rangle$ is an MTL-chain and such that the embedding $i$ preserves the operations $\circ$ and $\rightarrow$?
Let $A = \langle A, \land, \lor, \circ, \rightarrow, 1, 0 \rangle$ be a fixed MTL-chain.

**Definition**

For each $K \subseteq A$, define

- $K^u = \{ a \in A : (\forall c \in K) \ c \leq a \} = \{ \text{upper bounds of } K \}$
- $K^\ell = \{ a \in A : (\forall c \in K) \ a \leq c \} = \{ \text{lower bounds of } K \}$

**Definition**

A subset $X \subseteq A$ is called **stable** if $X^\ell u = X$.

Note: each stable set is an *upward closed* subset of $A$ (but not every upward closed subset is stable).

Note: For any $K \subseteq A$, the set $K^u$ is stable since $K^u u^\ell u = K^u$. 

The Dedekind-MacNeille Completion

Clint van Alten

Preservation Theorems for MTL-Chains
Dedekind-MacNeille, continued

**Definition**

Let $C$ be the set of all stable subsets of $A$.

**Definition**

For $X, Y \in C$, let

- $X \land^C Y = X \cup Y$,
- $X \lor^C Y = X \cap Y$,
- $1^C = \{1\}$
- $0^C = A$

**Theorem (Dedekind-MacNeille)**

$C = \langle C, \land^C, \lor^C, 1^C, 0^C \rangle$ is a complete chain with order $\leq^C = \supseteq$. The map $i : A \to C$ defined by $i(a) = \{a\}^u$ is an embedding that preserves all meets and joins in $A$. 
Defining $\circ^C$ and $\rightarrow^C$

**Definition**

For $X, Y \in C$, define

- $X \circ^C Y = (X^\ell \circ Y^\ell)^u = \{a \circ b : a \in X^\ell, b \in Y^\ell\}^u$

- $X \rightarrow^C Y = \bigvee^C \{Z \in C : X \circ^C Z \leq^C Y\} = \bigcap\{Z \in C : X \circ^C Z \supseteq Y\}$.

**Definition**

Let $C = \langle C, \wedge^C, \vee^C, \circ^C, \rightarrow^C, 1^C, 0^C \rangle$. 
By construction we know that $C$ is a chain with greatest element $1^C$ and least element $0^C$.

It is easy to see that $\circ^C$ is commutative and has identity $1^C$.

Residuation is immediate by the definition of $\rightarrow^C$.

What about associativity?
**Lemma**

If $K, L \subseteq A$, then $K^u$ and $L^u$ are stable sets, and

$$K^u \circ^C L^u = (K^{u\ell} \circ L^{u\ell})^u = (K \circ L)^u.$$ 

**Lemma**

For $X, Y, Z \in C$,

$$(X \circ^C Y) \circ^C Z = (X^{\ell} \circ Y^{\ell})^u \circ^C (Z^{\ell})^u = ((X^{\ell} \circ Y^{\ell}) \circ Z^{\ell})^u = (X^{\ell} \circ Y^{\ell} \circ Z^{\ell})^u = \ldots = X \circ^C (Y \circ^C Z).$$
Recall that \( i : A \rightarrow C \) defined by \( i(a) = \{a\}^u \) is a lattice embedding.

**Lemma**

For all \( a, b \in A \),

- \( i(a \circ b) = i(a) \circ^C i(b) \)
- \( i(a \rightarrow b) = i(a) \rightarrow^C i(b) \).

**Theorem**

Each MTL-chain is embeddable into a complete MTL-chain.
Suppose, for example, that the MTL-chain $A$ satisfies the identity

$$x \land (\neg x) = 0 \quad (1)$$

where $\neg x = x \rightarrow 0$.

Then one can show that for any stable set $X \in C$,

$$X \land^{C} \neg^{C} X = 0^{C},$$

where $\neg^{C} X = X \rightarrow^{C} 0^{C}$. Thus $C$ satisfies (1).

**Lemma**

*If $A$ satisfies (1), then $C$ satisfies (1).*

*We say that the identity (1) is preserved by the construction.*
Problem

- **Which identities** \( s(\vec{x}) = t(\vec{x}) \) **are preserved by the completion?**
- **In other words, if** \( A \) **belongs to a variety** \( \mathcal{V} \), **does** \( C \) **also belong to** \( \mathcal{V} \)?
- **More generally, which inequalities** \( s(\vec{x}) \leq t(\vec{x}) \) **are preserved by the completion?**
The Approximations

**Definition**

For a term $t(x_1, \ldots, x_n)$ in the language $\{\land, \lor, \circ, \rightarrow, \neg, 1, 0\}$, and for $X_1, \ldots, X_n \in C$, define

$$t^*(X_1, \ldots, X_n) = \{ t(a_1, \ldots, a_n) : a_1 \in X_1^\ell, \ldots, a_n \in X_n^\ell \}^u$$

**Definition**

A term $t$ is called

- **stable** if $t^*(X_1, \ldots, X_n) = t^C(X_1, \ldots, X_n)$ for all $X_1, \ldots, X_n \in C$.
- **expanding** if $t^*(X_1, \ldots, X_n) \geq t^C(X_1, \ldots, X_n)$ for all $X_1, \ldots, X_n \in C$.
- **contracting** if $t^*(X_1, \ldots, X_n) \leq t^C(X_1, \ldots, X_n)$ for all $X_1, \ldots, X_n \in C$. 
Theorem

If $s$ and $t$ are stable terms then the identity $s = t$ is preserved. (i.e., if $A$ satisfies $s = t$, then $C$ satisfies $s = t$.)

Proof.

Let $\vec{x} = x_1, \ldots, x_n$ be a list of all the variables occurring in $s$ and $t$ and let $\vec{X} = X_1, \ldots, X_n \in C$. Then

$$s^\ast(\vec{X}) = \{ s(\vec{a}) : \vec{a} \in \vec{X}^\ell \}^u = \{ t(\vec{a}) : \vec{a} \in \vec{X}^\ell \}^u = t^\ast(\vec{X}).$$

Thus, $s^C(\vec{X}) = t^C(\vec{X}).$
Lemma

If $A$ satisfies $s(\vec{x}) \leq t(\vec{x})$ for terms $s, t$, then for all $\vec{X} \in C$,

$$s^*(\vec{X}) \leq^C t^*(\vec{X}).$$

Theorem

If $s$ is contracting and $t$ is expanding, then $s \leq t$ is preserved. (I.e., if $A$ satisfies $s \leq t$, then $C$ satisfies $s \leq t$.)

Proof.

Let $\vec{x}$ be a list of all the variables occurring in $s$ and $t$ and let $\vec{X} \in C$. Using the above lemma,

$$s^C(\vec{X}) \leq^C s^*(\vec{X}) \leq^C t^*(\vec{X}) \leq^C t^C(\vec{X}).$$

hence $s^C(\vec{X}) \leq^C t^C(\vec{c})$.  

Clint van Alten

Preservation Theorems for MTL-Chains
Which terms are stable/contracting/expanding?

Problem

Which terms are stable/contracting/expanding?

Note that **MTL** satisfies

\[ x \circ (y \lor z) = (x \circ y) \lor (x \circ z) \]
\[ x \circ (y \land z) = (x \circ y) \land (x \circ z) \]
\[ x \rightarrow (y \lor z) = (x \rightarrow y) \lor (x \rightarrow z) \]
\[ x \rightarrow (y \land z) = (x \rightarrow y) \land (x \rightarrow z) \]
\[ (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z) \]
\[ (x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z) \]
\[ x \land (y \lor z) = (x \land y) \lor (x \land z) \]

Thus every **MTL**-term can be written as

\[ \bigvee \bigwedge_{i,j} s_{ij} \]

where each \( s_{ij} \) is a \{\circ, \rightarrow, \neg, 1, 0\}-term.
We use the following abbreviations:

- $P$ is the set of all $\{\circ\}$-terms and 0 and 1.
  (i.e., $P = \{1, 0, x, y, x \circ y, x \circ x, x \circ y \circ z, \ldots\}$)
- $\neg P = \{-s : s \in P\}$
- $P(\neg P)$ is the $\{\circ\}$-closure of $\neg P$
- $\neg P(\neg P) = \{-s : s \in P(\neg P)\}$. 
Proposition
The following terms are **stable**: \( \bigvee_i s_i, \) where \( s_i \in P \).

Recall that “stable = stable” is preserved.

Proposition
The following terms are **contracting**: \( \bigvee_i s_i, \) where

(i) \( s_i \in P, \)

(ii) \( s_i \in P(\neg P) \) and meets of these,

(iii) \( s_i \) is a product of (i) and (ii) that do not share a variable.

Proposition
The following terms are **expanding**: \( \bigvee_i \bigwedge_j s_{ij}, \) where

(i) \( s_{ij} \in P \cup \neg P(\neg P), \)

(ii) \( s_{ij} \in \{ u \rightarrow v : u \in P(\neg P), v \in P \}. \)

Recall that “contracting \( \leq \) expanding” is preserved.
A class $\mathcal{K}$ of algebras has the finite embeddability property (FEP, for short) if,
for any $A \in \mathcal{K}$ and any finite $B \subseteq A$,
there exists a finite $D \in \mathcal{K}$
and an embedding $i : B \rightarrow D$ such that all existing
operations in $B$ are preserved.

If $(\forall \overline{x})(s(\overline{x}) = t(\overline{x}))$ is an identity that fails in a class $\mathcal{K}$, then for
some $A \in \mathcal{K}$ and some $\overline{a} \in A$, $s(\overline{a}) \neq t(\overline{a})$.
The evaluation of $s(\overline{a})$ and $t(\overline{a})$ in $A$ uses finitely many elements
that form a finite set $B \subseteq A$.
If $\mathcal{K}$ has the FEP, then $B$ can be embedded into a finite $D \in \mathcal{K}$
and $(\forall \overline{x})(s(\overline{x}) = t(\overline{x}))$ fails in $D$ since all existing operations in $B$
are preserved.
The finite embeddability property for a class $\mathcal{K}$ implies that $\mathcal{K}$ has the finite model property in the sense that any identity that fails in $\mathcal{K}$ will fail in a finite algebra in $\mathcal{K}$. (In fact, the strong finite model property.)

If $\mathcal{K}$ is a variety with the FEP then $\mathcal{K}$ is generated by its finite members.

**MTL** is known to have the FEP.

**Problem**

*Which subvarieties of MTL have the FEP?*

*I.e., which identities are preserved by the construction?*

Note: Since each subvariety of MTL is generated by its MTL-chains, it is enough for us to show that the class of MTL-chains in a given variety has the FEP.
We start with an MTL-chain $\mathcal{A} = \langle A, \land, \lor, \circ, \rightarrow, 1, 0 \rangle$ and a finite subset $B \subseteq A$. We may assume that $0, 1 \in B$. We must construct a finite MTL-chain $\mathcal{D}$ into which we can embed $B$.

**Definition**

- Let $M$ be the closure of $B$ under $\circ$.
- Let $M^* = \{ a \rightarrow b : a \in M \text{ and } b \in B \}$.

**Definition**

For $c \in M^*$ and $a \in M$, let

- $c^\ell = \max\{ b \in M : b \leq c \}$
- $a^u = \min\{ d \in M^* : a \leq d \}$
Definition
An element \( c \in M^* \) is called \textbf{stable} if \( c^{\ell u} = c \).

- The stable elements are elements in \( M^* \) that ‘cover’ an element in \( M \).
- Note that every \( b \in B \) is stable since \( B \subseteq M \cap M^* \), so \( b^{\ell} = b \) and \( b^{u} = b \), hence \( b^{\ell u} = b \).
- Note that there are only finitely many stable elements. This follows from the fact that \( \langle M, \leq \rangle \) has no ascending chains and \( \langle M^*, \leq \rangle \) has no descending chains.
Constructing $D$

**Definition**

Let $D$ be the set of all stable elements.

The lattice operations on $D$ are the same as those on $A$. Both 1 and 0 are stable and are the bounds of $D$.

**Definition**

\[
\begin{align*}
    a \circ_D b &= (a^\ell \circ b^\ell)^u \\
    a \rightarrow_D b &= a^\ell \rightarrow b
\end{align*}
\]

**Definition**

Let $D = \langle D, \land, \lor, \circ_D, \rightarrow_D, 1, 0 \rangle$. 

Clint van Alten
Preservation Theorems for MTL-Chains
$D$ is an MTL-chain

- Associativity of $\circ^D$ follows as in the completion case.
- Residuation can be shown.
- The other properties of MTL are straightforward.
- Thus $D$ is a finite MTL-chain.

**Lemma**

*If $a, b \in B$ and $a \circ b \in B$, then $a \circ^D b = a \circ b$.*

*If $a, b \in B$ and $a \rightarrow b \in B$, then $a \rightarrow^D b = a \rightarrow b$.*

- Thus the map $i : B \rightarrow D$ defined by $i(b) = b$ is an embedding of $B$ into $D$ that preserves all existing operations in $B$. 
The Approximations

Definition
For a term $t(x_1, \ldots, x_n)$ in the language $\{\land, \lor, \circ, \to, \neg, 1, 0\}$, and for $c_1, \ldots, c_n \in D$, define

$$t^*(c_1, \ldots, c_n) = [t(c_1^\ell, \ldots, c_n^\ell)]^u$$

Definition
A term $t$ is called
- stable if $t^*(c_1, \ldots, c_n) = t^D(c_1, \ldots, c_n)$ for all $c_1, \ldots, c_n \in D$.
- expanding if $t^*(c_1, \ldots, c_n) \leq t^D(c_1, \ldots, c_n)$ for all $c_1, \ldots, c_n \in D$.
- contracting if $t^*(c_1, \ldots, c_n) \geq t^D(c_1, \ldots, c_n)$ for all $c_1, \ldots, c_n \in D$. 
Main Theorems

Theorem

If $s$ and $t$ are stable terms, then $s = t$ is preserved by the construction. (i.e., if $A$ satisfies $s = t$, where $s$ and $t$ are stable, then $D$ satisfies $s = t$.)

Theorem

If $s$ is contracting and $t$ is expanding, then $s \leq t$ is preserved by the construction.

Theorem

If $\mathcal{V}$ is a variety of MTL-algebras whose defining identities are preserved by the above construction, then $\mathcal{V}$ has the FEP.
Recall the following abbreviations:

- $P$ is the set of all $\{\circ\}$-terms.
  (i.e., $P = \{1, x, y, x \circ y, x \circ x, x \circ y \circ z, \ldots\}$)
- $\neg P = \{\neg s : s \in P\}$.
- $P(\neg P)$ is the $\{\circ\}$-closure of $\neg P$.
- $\neg P(\neg P) = \{\neg s : s \in P(\neg P)\}$.

Also $P(P \cup \neg P)$ is the $\{\circ\}$-closure of $P \cup \neg P$ and $\neg P(P \cup \neg P) = \{\neg s : s \in P \cup \neg P\}$.
Which terms are stable/contracting/expanding?

Proposition

The following terms are stable: \( \bigvee_i \bigwedge_j s_{ij}, \) where \( s_{ij} \in P \cup \neg P. \)

Recall that “stable = stable” is preserved.

Proposition

The following terms are contracting: \( \bigvee_i \bigwedge s_{ij}, \) where \( s_{ij} \in P(P \cup \neg P). \)

Proposition

The following terms are expanding: \( \bigvee_i \bigwedge_j s_{ij}, \) where

(i) \( s_{ij} \in P \cup \neg P(P \cup \neg P), \)

(ii) \( s_{ij} \in \{ u \rightarrow v : u \in P(P \cup \neg P), v \in P \}. \)

Recall that “contracting \( \leq \) expanding” is preserved.