A Comparative Analysis of Methodologies for the Representation of Nonlinear Oscillations in Dynamic Systems

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Abstract: Nonlinear oscillations in power systems have been extensively studied over the last years, to analyze and to determine their effects on the network. For that purpose, and based on analytical methodologies, normal forms of vector fields and modal series have been developed as a numerical tool, which allows the dynamic analysis of nonlinear systems. In this work, both methods are applied to analyze oscillations in two nonlinear systems. Observed differences between both methods are pointed out mainly those introduced by wrong selection of initial conditions. Time domain responses of a dynamical system used as a benchmark are analyzed in order to show this situation. Besides, a nine bus, three machines system is simulated to compare both nonlinear methodologies with respect to the numerical solution obtained with the direct solution of ordinary differential equations. The main differences observed between both methodologies when the power system is operated under a small perturbation and stress conditions are remarked.

I. INTRODUCTION

Nonlinear oscillations in power systems represent a problem which has been deeply analyzed by working groups around the world. Some tools have been created to facilitate the complexity of analysis and interpretation. For instance, the method of normal forms of vector fields has been utilized in order to analyze oscillations in power systems [1], [2], [3], [4], [5]. The method is based on two transformations: beginning with a linearization process around a stable equilibrium point, obtained by Taylor series expansion. An application of normal forms technique to the linearized system, keeping the nonlinear characteristics given by eigenvalues, allows an approximate solution of the dynamic system in terms of the transformed variable to be obtained. This method has the great advantage of obtaining the simplest form of a set of ordinary differential equations of sequential transformations [1]. However, it has the disadvantage of having a wrong dynamic performance when the resonance condition is presented in the system dependent on eigenvalues [6].

The method of normal forms has been used to study nonlinear modal interactions in power systems as well. Different components in the power system model have been included, which has opened the way to numerous studies of modal interactions. Interpretation of results in normal forms is often a challenging problem [5], [6].

On the other hand, in [7] the modal series method has been proposed as an alternative to the nonlinear analysis of dynamic systems. It is based on transforming a linearized system around a stable equilibrium point; a straightforward linear transformation permits to obtain a linear approximation of a nonlinear system. In the method of modal series, the resonance conditions are not of concern, so the system is solved for any eigenvalue. In this work, both methods are employed to study the oscillations of two different dynamic systems. A comparison of results obtained with both methods is given and analyzed.

II. METHODS FOR THE ANALYSIS OF NONLINEAR OSCILLATIONS

A. NORMAL FORMS

Let us assume the set of autonomous ordinary differential equations represented by,

\[ \dot{x} = f(x) \]  \hspace{1cm} (1)

This system has equilibrium points defined by,

\[ f(x_0) = 0 \]  \hspace{1cm} (2)

where, \( x_0 \) is the vector of state variables evaluated in the stable equilibrium point.

Equation (1) is expanded in Taylor series and evaluated at the equilibrium point to obtain a linear system:

\[ \dot{x} = \frac{1}{2} x^T H x \]  \hspace{1cm} (3)

where \( A \) is the Jacobian matrix and \( H \) is the Hessian matrix, both evaluated at the stable equilibrium point.

The system (3) is a second order approximation of the nonlinear system, which is transformed to a decoupled linear system using the linear transformation expressed in terms of right eigenvectors of \( A \) as,

\[ x = U y \]  \hspace{1cm} (4)

Once the previous transformation is performed, the system in terms of the variable \( y \), is transformed into a linear system through the normal form equation given by [1],

\[ y = z + h_2(z) \]  \hspace{1cm} (5)

The original system is now transformed into the new frame of work defined by the variable \( z \) as,
with the characteristic of analyzing a nonlinear system as a linear model, in the same way as the normal forms does. This formulation has the advantage of representing nonlinear dynamic systems with closed-form solutions, which is a significant advantage in power systems.

Consider the second order nonlinear system given by,

\[ \dot{x}_i(t) = u_i y_{j0} + \sum_{i=1}^{N} h_{kl} z_k z_l \]

where \( h_{kl} \) is defined by (7) and higher order terms are not considered.

The approximate solution for the second order model considered in (17) is,

\[ y_j(t) = f_j^1(t) + f_j^2(t) + \ldots \]

where \( f_j^1(t) \) and \( f_j^2(t) \) are obtained from (16) and (16b) to obtain the time domain solution; this results in,

\[ f_j^1(t) = y_{j0} e^{\lambda_1 t} \]

\[ f_j^2(t) = \frac{1}{\lambda_2 + \lambda_3} \left[ e^{\lambda_2 t} - e^{\lambda_3 t} \right] \]

if the resonance condition is not presented

The right hand of (15) can be solved by finding the following series of differential equations with initial conditions \( f_j^1(0) = [f_1^1(0), f_2^1(0), \ldots, f_N^1(0)] = Y_0 \) and \( f_j^2(0) = 0 \) with \( j = 1,2,\ldots,N \) and \( k > 1 \). The transformed system is limited to consider only at the second order terms of the approximation; that is,

\[ f_j^1(t) = \lambda_j f_j^1 \]

\[ f_j^2(t) = \lambda_j f_j^2 + \sum_{i=1}^{N} C_{kl} f_i^1 f_i^2 \]

Equation (16a) represents the linear term of the approximate solution. Moreover, equation (16b) is referred as the correction terms to the linear approximate solution when the second order terms are considered [6]. The Laplace transform is applied to (16a) and (16b) to obtain the time domain solution; this results in,

\[ f_j^1(t) = y_{j0} e^{\lambda_1 t} \]

\[ f_j^2(t) = \sum_{i=1}^{N} \sum_{i=1}^{N} C_{kl} f_i^1 f_i^2 \]

where,

\[ S_{jk}(t) = \frac{1}{\lambda_2 + \lambda_3} \left[ e^{\lambda_2 t} - e^{\lambda_3 t} \right] \]

if the resonance condition is not presented

The approximate solution for the second order model considered in (17) is,

\[ y_j(t) = f_j^1(t) + f_j^2(t) \]

By using the linear transformation given for equation (4), the complete solution in terms of original variable has the form,

\[ x_i(t) = \sum_{j=1}^{N} \left( u_j y_{j0} - \sum_{k=1}^{N} \sum_{l=1}^{N} u_j h_{kl} y_{k0} y_{l0} \right) e^{\lambda_1 t} + \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} u_j h_{kl} y_{k0} y_{l0} e^{\lambda_2 t} \]

where \( h_{kl} \) is defined by (7) and \( y_{k0}, y_{l0} \) are obtained from \( y_0 = U^{-1} x_0 \).

It should be remarked that the system solution is obtained when the modal analysis is applied at the beginning of the modal series process, therefore (19) depends on the initial eigenvalues.

C. PRINCIPAL DIFFERENCES BETWEEN NORMAL FORMS AND MODAL SERIES METHODS

Some differences between normal forms and modal series methods are of concern. These differences here will be noted through an example based on a second order nonlinear system.

Consider the second order nonlinear system given by,

\[ x_1 = f_1(x_1, x_2) \]

\[ x_2 = f_2(x_1, x_2) \]

with stable equilibrium points defined as,
\begin{align*}
f_1(x_1^0, x_2^0) &= 0 \\
f_2(x_1^0, x_2^0) &= 0
\end{align*}

\text{(21)}

\text{i) Normal Forms Solution}

Based on equation (4) which links the relationship between the transformed variables with Jordan canonical form, the system (20) is linearized around at stable equilibrium point defined by (21), resulting in,

\[ y = Ay + f_1(y) \]

\text{(22)}

where,

\[ f_1(y) = \frac{1}{2} U^{-1} \left[ (Uy)^T H_1^2 Uy \right] = \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} C_{kl} y_k y_l \]

\text{(23)}

where \( C_{kl} \) is defined by (8).

Applying the normal form transformation given by (5) to the special case of second order nonlinear system, with \( z = [z_1, z_2]^T \) and \( h_z(z) \) defined as a complex polynomial vector, results in,

\[ h_z(z) = \sum_{i=1}^{N} \sum_{j=1}^{N} h_{2ij} z_i z_j \]

\text{(24)}

At this step, the calculation of \( z \) values is required by means of the approach described by (12). Finally, the normal forms solution is obtained as,

\[ \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} e^{a_1 t} e^{b_1 t} \\ e^{a_2 t} e^{b_2 t} \end{bmatrix} \]

\text{(25)}

\[ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \sum_{k=1}^{N} \sum_{l=1}^{N} h_{2ij}^k z_i^k z_j^k e^{(a_i + b_j) t} \]

\text{(26)}

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \Delta \delta(t) \\ \Delta \sigma(t) \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \]

\text{(27)}

\text{ii) Modal Series Solution}

In order to solve the nonlinear system by modal series method, the Jordan canonical form transformed system is applied, e.g. from (12) we obtain,

\[ y_j(t) = f_j(t) + f_j^1(t) \]

\text{(28)}

The full solution obtained for the second order nonlinear system is given by,

\[ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} + \begin{bmatrix} f_1^1(t) \\ f_2^1(t) \end{bmatrix} \]

\text{(29)}

where this time domain solution is presented as a function of the Jordan variables; that is,

\[ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y_1^0 e^{a_1 t} + \sum_{k=1}^{N} \sum_{l=1}^{N} h_{2ij}^k y_i^k e^{(a_i + b_j) t} \\ y_2^0 e^{a_2 t} + \sum_{k=1}^{N} \sum_{l=1}^{N} h_{2ij}^k y_i^k e^{(a_i + b_j) t} \end{bmatrix} \]

\[ + \sum_{k=1}^{N} \sum_{l=1}^{N} h_{2ij}^k y_i^k e^{(a_i + b_j) t} \]

\text{(30)}

\[ \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \Delta \delta(t) \\ \Delta \sigma(t) \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \]

\text{(31)}

\text{iii) Resume of main differences between Normal Forms and Modal Series methods}

- Both methods are based on analyzing a nonlinear system as a linear one, through Taylor series expansion around a stable equilibrium point.
- The normal forms method needs a pair of transformations: from original variables \( x \), applying (4) the coordinates \( y \) are obtained, and then a nonlinear transformation given by (5) generates the uncoupled and minimal system, in terms of \( z \) variables. The modal series method needs only the transformation of \( x \) variables onto the \( y \) variables. This step reduces considerably the computational effort.
- The modal series method has an alternative of solving the system even when a modal resonance is presented in the system [7].
- The approach of obtaining a solution of the \( z \) variables (12) is not necessary in the modal series method. This is a significant reduction on numerical calculations since a set of nonlinear algebraic solutions is usually a hard problem to be solved.

**III CASE STUDIES**

3.1 Benchmark Test System

The test system proposed in [6] is used to carry-out a linear iterative study of the results obtained by the application of both methods. The system consists of four nonlinear differential equations, which have the nonlinear condition handled by the \( \varepsilon \) constant.

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ \mu & -1 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & \mu & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \frac{1}{2} \varepsilon \lambda_1^2 \]

\text{(32)}

- Hessian Matrix, \( H_{ij} = 0 \) except \( H_{11} = \varepsilon \)
- Eigenvalues, \( \lambda_1 = -1 + \sqrt{\mu + i} \), \( \lambda_2 = -1 - \sqrt{\mu + i} \), \( \lambda_3 = -1 + \sqrt{\mu - i} \), \( \lambda_4 = -1 - \sqrt{\mu - i} \)

The eigenvalues have a strong resonance in \(-1 + i\) when \( \mu = 0 \) [6]. Therefore, the parameter \( \mu \) is assumed \( \mu \neq 0 \) to diagonalize the matrix \( A \) of (32).

The system is simulated with the initial condition situated in \( x_0 = [0.9, 0.9, 0.9, 0.9] \) [10]. According to the approaches defined previously for the normal form and modal series methods, initial conditions for \( y_0 \) and \( z_0 \) variables are required. Considering the initial values of \( x_0 \) and using (4), the initial conditions of \( y_0 \) are

\[ y_0 = \begin{bmatrix} 0.09823784313103 + 0.09823784313104i \\ 0.09823784313103 - 0.09823784313104i \\ 0.9157043873736 - 0.9157043873736i \\ 0.9157043873736 + 0.9157043873736i \end{bmatrix} \]

These are the initial conditions considered in the modal series method. On other side, executing the process mainly defined
by (12), the solution of \( z_0 \) variables are obtained, these are valued as,

\[
\begin{bmatrix}
0.17065465334736 + 0.10344067799111i \\
-0.09303766959970 + 0.06112783382073i \\
1.53952248078033 + 0.15359747338677i \\
-0.25407825782765 + 1.31697916445931i
\end{bmatrix}
\]

which are the values introduced in (11) for the final solution of (32). The Figure 1(a) shows the evolution of the state variables obtained by the Normal Forms method, modal series method and the direct numerical solution method of (32), respectively. The last is obtained by direct numerical integration of (32). It is observed that the solution has differences in both methods, with respect to the real solution. This difference is due principally to the initial condition. This affirmation is confirmed with Figure 1(b), which shows the similarity between the three solutions when the application of both methods is delayed 5 seconds respect to the initial time \( t = 0 \). In this case, the initial conditions for \( x_0, y_0 \) and \( z_0 \) variables are,

\[
\begin{align*}
x_0 &= [-0.7704051 - 0.0503111 - 0.1830765 - 0.6193213] \\
y_0 &= [-0.26572038483132 + 0.32169503681242i \\
-0.2657203843132 - 0.32169503681242i \\
-0.37806014533163 + 0.43200787384690i \\
-0.37806014533163 - 0.43200787384690i \\
-0.29252947246535 + 0.39548915009020i \\
-0.29252947246565 - 0.39548915009025i \\
-0.24009613635374 + 0.33685671076341i \\
-0.24009613635360 - 0.33685671076342i]
z_0 &=
\end{align*}
\]

It is observed also that the best response is reproduced with the modal series method, since is much closer to the direct numerical solution.
angles and speed rotors is introduced into the system. This perturbation is,

\[ \Delta x = [0.45e-2; 0.5e-2; -0.35e-3; 0.2e-2; 0.25e-5; -0.9e-4] \]

Figure 3 shows the oscillations observed as a rotor angle difference \( \delta_{13} \) and speed rotor in the synchronous machine 3 \( \omega_{3} \). In this case, the approximation by normal forms and modal series is very close to direct numerical solution. It is demonstrated that small perturbations have a good reproduction, even when a linear approximation is applied. The eigenvalues obtained in this case study are:

\[ \lambda_{1,2} = -0.37292070357288 \pm 13.35317436904827i \]
\[ \lambda_{3,4} = -0.17320817584593 \pm 8.68681585332089i \]
\[ \lambda_{5,6} = -0.00000000000002 \]

Now, a stress condition is applied to the power system model through an increase of 0.1 \( p.u. \) in the mechanical power of the synchronous machine 3. This situation is analyzed in the oscillations observed in Figure 4, again for the rotor angle difference \( \delta_{12} \) and the speed rotor \( \omega_{3} \).

It can be noticed from this Figure 4 that the waveforms obtained by normal forms and modal series methods do not present any numerical difference between them, but when these responses are compared against to the direct numerical solution, a numerical difference is observed. This difference is due principally to the linear approximation; however, the degree of accuracy is a function of level stress and initial perturbation in the system [9]. The normal forms and modal series methods both are more specific for analyzing small perturbation and lower stress levels introduced to the power system. The authors consider in this case that it may be necessary to include some higher order terms in both methods. Some nonlinear information is missing in the approximation, which implies the lack of dynamical reproduction in the waveforms showed in Figure 4. Also, it is possible that different manifolds are followed by the numerical solution and approximations of the both methods, although the same steady state point could be reached for the whole methods.

\[
\begin{align*}
\lambda_{1,2} &= -0.37292070357288 \pm 13.35317436904827i \\
\lambda_{3,4} &= -0.17320817584593 \pm 8.68681585332089i \\
\lambda_{5,6} &= -0.00000000000002
\end{align*}
\]

\[
\begin{align*}
\text{Numerical Full Solution} & \quad \text{Normal Form} \\
\text{Modal Series}
\end{align*}
\]

Figure 4. Time response comparison of NF and MS methods respect of Numerical solution when a small perturbation of rotor angle, speed rotor and a stress condition in generator 3 are presented

IV CONCLUSIONS

An application of normal forms and modal series methods to dynamic systems has been illustrated. When the solution of both methods is compared with respect to the direct numerical integration of differential equations, it is shown that it strongly depends on the initial conditions and parameters of the original system.

The main contribution of the analysis is to demonstrate the viability of the application of the modal series method as an easier and less complex alternative to analyze dynamic systems, rather than normal forms, and to show how the initialization of the system also modifies the solution.
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VII BIOGRAPHIES
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