Sustainability, exemption, and the constrained equal awards rule: a note

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Abstract

We consider the problem of distributing the liquidation value of a bankrupt firm among its creditors. Herrero and Villar [TOP 10 (2002) 261; Mathematical Social Sciences 42 (2001) 307] show that (i) the constrained equal awards rule is the only rule satisfying sustainability and composition down, and that (ii) it is the only rule satisfying exemption, composition down, and bilateral consistency. This note has three results. First, we give a direct and elementary proof of (i). Second, we connect (i) and (ii) by means of the Elevator Lemma, which is introduced by Thomson [Mimeo, University of Rochester, 2000]. Third, by exploiting the Elevator Lemma, we show that the constrained equal awards rule is the only rule satisfying exemption, composition down, and converse consistency. Thus, (ii) can be obtained as a corollary of our new characterization of the rule.

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1. Introduction

We consider the problem of distributing the liquidation value of a bankrupt firm among its creditors. A “division rule” is a function that associates with each situation of this kind, which we call a “bankruptcy problem”, a division of the liquidation value. This division, is called an “awards vector”, and interpreted as the recommendation that a judge could
make. What is the most desirable way of performing the division subject to the condition that each creditor receives neither more than his claim nor a negative award? A well-known rule is the constrained equal awards rule, which assigns equal amounts to all creditors subject to no one receiving more than his claim. This rule satisfies a number of appealing properties. Some of them, which play important roles in our note, are (1) composition down (Moulin, 2000): when the liquidation value decreases, there are two ways to deal with this decrease; we can either cancel the initial division and recalculate the awards for the revised liquidation value, or take the awards calculated on the basis of the initial liquidation value as claims in dividing the revised liquidation value. The requirement is that both procedures should produce the same awards, and (2) consistency: suppose that a rule is chosen. Consider a problem and identify the awards vector chosen by the rule for it. Then, imagine that some creditors leave with their awards. From the viewpoint of the remaining creditors, the new situation is the problem of dividing what is left among them while their claims remain unchanged. The requirement is that the same awards should be chosen for them as initially.

However, many other rules satisfy these properties. Examples are the “proportional” and “constrained equal losses” rules. One may wonder whether the constrained equal awards rule can be singled out from this family.

Herrero and Villar (2002, 2001) provide interesting answers to this question. Because the constrained equal awards rule fully compensates the creditors whose relative values of claims are small, they conjecture that one might be able to base a characterization of the rule on requirements that the rule “gives priority” to smaller claims. The question is how small a claim should be to make this preferential treatment possible. They formulate two standards of smallness: (1) creditor i’s claim $c_i$ is “sustainable” if truncating all claims at $c_i$ results in a situation where there is enough to reimburse everyone, and (2) creditor i’s claim $c_i$ is “exemptive” if it is not greater than equal division. The first property, sustainability, says that if a creditor’s claim is sustainable, he should be fully compensated. The second property, exemption, says that if a creditor’s claim is exemptive, he should be fully compensated. Note that sustainability implies exemption.

Herrero and Villar (2002, 2001) prove that the constrained equal awards rule satisfies sustainability and exemption. Moreover, they show that sustainability and composition down together imply “equal treatment of equals”\(^3\), and that in the two-creditor case, exemption and composition down together imply equal treatment of equals. Using these findings, they prove that: (i) the constrained equal awards rule is the only rule satisfying sustainability and composition down, and that (ii) it is the only rule satisfying exemption, composition down, and “bilateral consistency”\(^4,5\).

This note has three results. First, we give a direct and elementary proof of (i). Second, we connect (i) and (ii) by means of the “Elevator Lemma”, which states that if a rule is

\(^1\) For a comprehensive survey of this literature, initiated by O’Neill (1982), see Thomson (2003).

\(^2\) For a comprehensive survey of the literature on consistency and its converse, see Thomson (2000).

\(^3\) This property says that if two creditors have equal claims, they should receive equal amounts.

\(^4\) It is a weaker version of consistency that restricts attention to two-creditor subgroups.

\(^5\) For a study of the implications of sustainability and exemption in conjunction with other desirable properties, see Yeh (2002).
bilateral consistency, and on the subdomain of two-creditor problems, it coincides with some other rule that satisfies a converse of bilateral consistency, "converse consistency", then the two rules coincide in general\(^6\). Indeed, sustainability is equivalent to exemption in the two-creditor case. Thus, (i) implies that the constrained equal awards rule is the only rule satisfying exemption and composition down in that case. Moreover, this uniqueness can be extended to more than two creditors by imposing bilateral consistency. That is the content of (ii). This suggests that instead of proving (i) and (ii) separately, (ii) can be derived from (i) by means of the Elevator Lemma, which is indeed the case because the constrained equal awards rule satisfies converse consistency. Finally, we derive a new characterization of the rule from (i) on the basis of converse consistency, once again by appealing to the Elevator Lemma: the constrained equal awards rule is the only rule satisfying "resource monotonicity". Moreover, as shown by Chun (1999), in the case of bankruptcy problems, bilateral consistency and resource monotonicity together imply converse consistency\(^7\). Thus, (ii) can be obtained as a corollary of our new characterization of the rule.

2. Notation and definitions

There is an infinite set of "potential" creditors, indexed by the set of natural numbers, \(\mathbb{N}\). Let \(\mathcal{N}\) be the class of finite subsets of \(\mathbb{N}\). Given \(N \in \mathcal{N}\), the liquidation value \(E \in \mathbb{R}_+\) of a bankrupt firm has to be divided among its creditors \(N^R\). Let \(c_i\) be creditor \(i\)'s claim. Let \(c=(c_i)_{i \in \mathbb{N}}\) be the claims vector. A bankruptcy problem for \(N\) is a pair \((c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+\) such that \(\sum_{i \in N} c_i \geq E\). Let \(C^N\) be the class of all problems for \(N\). An awards vector for \((c, E) \in C^N\) is a point \(x \in \mathbb{R}_+^N\) such that \(0 \leq x \leq c\) and \(\sum_{i \in \mathbb{N}} x_i \geq E\). Let \(X(c, E)\) be the set of awards vectors of \((c, E)\). A rule is a function defined on \(\bigcup_{N \in \mathcal{N}} C^N\) that associates with each \(N \in \mathcal{N}\) and each \((c, E) \in C^N\) an awards vector of \((c, E)\). Let \(\varphi\) be our generic notation for rules, and \(|N|\) the cardinality of \(N\). For each coalition \(N' \subset N\), we define \(c_{N'}=(c_i)_{i \in N'}\), \(x_{N'}=(x_i)_{i \in N'}\), \(\varphi_{N'}(c, E)\), and so on.

Now, we formally define the constrained equal awards rule\(^{10}\).

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\(^6\) For this expression and a study of the lemma, see Thomson (2000).

\(^7\) Chun (1999) formulates a stronger version of converse consistency, which is called "strong converse consistency", and obtained by requiring that there should be for each problem an awards vector satisfying the hypothesis of converse consistency. He shows that in the case of bankruptcy problems, bilateral consistency and resource monotonicity together imply strong converse consistency. Note that strong converse consistency implies converse consistency. Therefore, bilateral consistency and resource monotonicity together imply converse consistency. However, it is easy to check that converse consistency does not imply bilateral consistency even with resource monotonicity.

\(^8\) By \(\mathbb{R}_+\), we denote the set of real numbers, \(\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}\).

\(^9\) Vector inequalities: \(x \geq y\), \(x \geq y\), and \(x > y\). By \(\mathbb{R}_+^N\) we denote the Cartesian product of \(|N|\) copies of \(\mathbb{R}_+\), indexed by the elements of \(N\).

\(^{10}\) For earlier references to this rule, see Aumann and Maschler (1985) and Dagan (1996).
**Constrained equal awards rule, CEA:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( i \in N \), \( \text{CEA}_i(c, E) = \min\{c_i, \lambda\} \), where \( \lambda \) is chosen such that \( \sum_{i \in N} \text{CEA}_i(c, E) = E \).

The CEA rule satisfies the following properties.

**Sustainability:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( i \in N \), if \( \sum_{j \in N} \min\{c_j, c_i\} \leq E \), then \( \varphi_i(c, E) = c_i \).

**Exemption:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( i \in N \), if \( c_i \geq \frac{E}{|N|} \), then \( \varphi_i(c, E) = c_i \).

**Composition down:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( E' \leq E \), we have \( \varphi(c, E') = \varphi(c, E) \).

**Bilateral consistency:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( N' \subseteq N \) with \(|N'| = 2\), if \( x \equiv \varphi(c, E) \), then \( x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i) \).

Next is a converse of **bilateral consistency:** given a problem, it allows one to determine the desirability of a proposed awards vector by assessing its desirability when restricted to two-creditor subgroups for the reduced problem that results from the departure of the complementary subgroup with their awards.

**Converse consistency:** For each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( N' \subseteq N \) with \(|N'| = 2\), if \( x \equiv \varphi(c, E) \), then \( x_{N'} = \varphi(c_{N'}, \sum_{i \in N'} x_i) \).

**Remark:** Notice that composition down implies resource monotonicity, which in turn implies “resource continuity”\(^{11,12}\).

3. The results

Herrero and Villar (2002) show that the constrained equal awards rule is the only rule satisfying sustainability and composition down by making use of a lemma which

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\(^{11}\) Resource monotonicity says that if the liquidation value increases, no one should receive less than he did initially. Resource continuity says that small changes in the liquidation value should not lead to large changes in the awards vector.

\(^{12}\) In the remark, we claim that composition down implies resource monotonicity. To see this, let \( \varphi \) be a rule satisfying composition down. Then, it follows that for each \( N \subseteq \mathcal{N} \), each \((c, E) \in \mathcal{C}^N\), and each \( E' \geq E \) with \( E' \leq \sum_{i \in N} c_i \), we have \( \varphi(c, E) = \varphi(c, E') \). Since \( \varphi(c, E) \) is an awards vector, it follows that \( \varphi(c, E) \leq \varphi(c, E') \). Thus, \( \varphi \) satisfies resource monotonicity.
demonstrates that these two properties imply equal treatment of equals. Here we offer a more direct and simpler proof without using the lemma.

**Theorem 1.** *(Herrero and Villar, 2002, Theorem 2.1)* The constrained equal awards rule is the only rule satisfying sustainability and composition down.

**Proof.** Clearly, the constrained equal awards rule satisfies the two properties. Conversely, let \( \varphi \) be a rule satisfying the two properties. Let \( N \subseteq N, (c, E) \in C^N \), and \( y = \varphi(c, E) \). Without loss of generality, we assume that \( c_1 \leq c_2 \leq \ldots \leq c_{|N|} \). Let \( N_{\text{sus}}(c, E) = \{ i \in N | \sum_{j \in N} \min \{ c_i, c_j \} \leq E \} \) and \( x = \text{CEA}(c, E) \). We show that \( y = x \).

Suppose, by contradiction, that \( y \neq x \). By sustainability, for each \( i \in N_{\text{sus}}(c, E) \), \( x_i = y_i = c_i \). Since \( \sum_{j \in N} x_j = \sum_{j \in N} y_j = E \), there exists \( i \in N \setminus N_{\text{sus}}(c, E) \) such that \( y_i \leq \lambda \leq c_i \), where \( \lambda = x_i \) for each \( i \in N \setminus N_{\text{sus}}(c, E) \). Let \( j \in N \setminus N_{\text{sus}}(c, E) \) be such that \( y_j \leq \lambda = c_j \) for each \( i \in N \setminus N_{\text{sus}}(c, E) \). Since \( y_j = \min_{i \in N \setminus N_{\text{sus}}(c, E)} c_i + |N_{\text{sus}}(c, E)|y_j \), \( y_j = y_{j} \). Clearly, \( \sum_{i \in N} y_i \geq E' \). Thus, \( (y, E') \) is well-defined. By composition down, \( \varphi(y, E') = \varphi(y, E') \). By sustainability, \( \varphi(y, E') = y_j \). By a similar argument, we can show that for each \( E^* \in \{ E', E \} \), \( \varphi(c, E^*) = y_j \).

Notice that \( \varphi(c, \sum_{i \in N} c_i) = c_j \), and that composition down implies both resource monotonicity and resource continuity. Therefore, there exists \( \tilde{E} < \sum_{i \in N} c_i \) such that \( y_j < \varphi(\tilde{E}, \tilde{E}) \) and \( E'' < \tilde{E} \) where \( \tilde{y} = \varphi(c, \tilde{E}) \) and \( E'' = \sum_{i \in N_{\text{sus}}(c, E)} c_i + |N_{\text{sus}}(c, E)|\tilde{y} \). Since \( \sum_{i \in N} \tilde{y}_i = \tilde{E} \) and \( \tilde{E} > E'' \), then \( (\tilde{y}, E'') \) is well-defined. By composition down, \( \varphi(c, E'') = \varphi(\tilde{y}, E'') \). By sustainability, \( \varphi(\tilde{y}, E'') = \tilde{y}_j \). Thus, \( \varphi(c, E'') = \tilde{y}_j \). Notice that \( y_j < \tilde{y}_j \) so that \( E' < E'' \). Since \( \tilde{y}_j < \lambda \) and \( E'' = \sum_{i \in N_{\text{sus}}(c, E)} c_i + |N_{\text{sus}}(c, E)|\lambda \), then \( E'' < E \). Therefore, \( E' < E' < E \). By the previous argument, \( \varphi(c, E'') = y_j \). This contradicts \( y_j < \tilde{y}_j \). \( \square \)

Next, we introduce the Elevator Lemma\(^\text{13}\).

**Elevator Lemma.** *(Thomson, 2000)*. If a rule \( \varphi \) is bilaterally consistent and coincides with a conversely consistent rule \( \varphi' \) in the two-creditor case, then \( \varphi \) coincides with \( \varphi' \) in general.

**Proof.** See Thomson (2000). \( \square \)

With the help of the Elevator Lemma and Theorem 1, we derive our new characterization of the constrained equal awards rule, and Theorem 2 in Herrero and Villar (2001).

**Theorem 2.** The constrained equal awards rule is the only rule satisfying exemption, composition down, and converse consistency.

\(^{13}\)This lemma is introduced by Thomson (2000). It asserts that if a (possibly “multi-valued”) rule is consistent and is a subrule of a conversely consistent rule in the two-agent case, then the inclusion holds for any number of agents. The lemma stated here is the special case for “single-valued” rules.
**Proof.** Clearly, the constrained equal awards rule satisfies the three properties. Conversely, let \( \varphi \) be a rule satisfying the three properties. Notice that *exemption* is equivalent to *sustainability* in the two-creditor case. By Theorem 1, \( \varphi = \text{CEA} \) in the two-creditor case. Notice that \( \varphi \) is *conversely consistent*, and that CEA is *bilaterally consistent*. By the Elevator Lemma, \( \varphi = \text{CEA} \). □

**Corollary 1.** (Herrero and Villar, 2001, Theorem 2) The constrained equal awards rule is the only rule satisfying exemption, composition down, and bilateral consistency.

The following examples show that the properties in Theorem 2 are independent.

**Example 1.** A rule that satisfies *composition down* and *converse consistency*, but not *exemption*.

The proportional rule is an example.

**Example 2.** A rule that satisfies *exemption* and *converse consistency*, but not *composition down*.

Let \( \varphi^1 \) be defined as follows\(^{14} \). Let \( N \subseteq N^c, (c, E) \in C^N \), and \( N_0 = \emptyset \). Given \( k \in \mathbb{N} \) such that \( 1 \leq k \leq |N| \), let \( N_k(c) = \{ i \in N | c_i = \min_{j \in N \cup \{s\}} N_s(c) c_j \} \) and \( y_k = \min_{j \in N \cup \{s\}} N_s(c) c_j \). Now, for each \( i \in N_k(c) \),

\[
\varphi^1(c, E) = \begin{cases} 
0 & \text{if } 0 \leq E \leq \sum_{s<k} N_s(c) y^s, \\
\frac{E - \sum_{s<k} N_s(c) y^s}{N_k(c)} & \text{if } \sum_{s<k} N_s(c) y^s < E \leq \sum_{s \leq k} N_s(c) y^s, \\
c_i & \text{otherwise.}
\end{cases}
\]

**Example 3.** A rule that satisfies *exemption* and *composition down*, but not *converse consistency*.

Let \( \varphi^2 \) be defined as follows. Let \( N \subseteq N^c \) and \( (c, E) \in C^N \). Without loss of generality, we assume that \( c_1 \leq c_2 \leq \ldots \leq c_{|N|} \). Then,

\[
\varphi^2(c, E) = \begin{cases} 
\varphi^*(c, E) & \text{if } |N| = 3 \text{ and for each pair } \{i, j\} \subseteq N, c_i \neq c_j, \\
\text{CEA}(c, E) & \text{otherwise.}
\end{cases}
\]

\(^{14}\)This rule is due to Herrero and Villar (2002).
where $\varphi^*$ is defined as follows: let $N = \{i, j, k\}$ and $c_i < c_j < c_k$,

$$\varphi^*(c, E) = \begin{cases} 
\left( \frac{E}{3}, \frac{E}{3}, \frac{E}{3} \right) & \text{if } \frac{E}{3} \leq c_i, \\
\left( c_i, \frac{E}{3} + \frac{1}{3} \left( \frac{E}{3} - c_i \right), \frac{E}{3} + \frac{2}{3} \left( \frac{E}{3} - c_i \right) \right) & \text{if } c_i < \frac{E}{3} \leq \frac{c_i + 3c_j}{4}, \\
(c_i, c_j, E - c_i - c_j) & \text{otherwise.}
\end{cases}$$

The independence of properties in Theorem 1 is established by Herrero and Villar (2002). However, the independence of properties in Corollary 1 is absent from Herrero and Villar (2001). Note that Examples 1 and 2 are bilaterally consistent, and also that Example 3 is not bilaterally consistent. Therefore, the same examples can be used to show that the properties in Corollary 1 are also independent.

4. Dual results

In the literature, the counterpart of each characterization can be derived by utilizing dual relations between rules, and properties of rules. Given a rule $\varphi$, its dual $\varphi^d$ is defined as follows: for each $N \subseteq N$ and each $(c, E) \in C^N$, $\varphi^d(c, E) = c - \varphi(c, \sum_{i \in N} c_i - E)$. Clearly, the constrained equal awards and constrained equal losses15 rules are dual of each other (Herrero and Villar, 2001). Similarly, two properties are dual if whenever a rule satisfies one of them, its dual satisfies the other16. The dual of sustainability is “independence of residual claims” (Herrero and Villar, 2002). The dual of exemption is “exclusion” (Herrero and Villar, 2001). The property, which is the dual of composition down, is “composition up”17 (Young, 1988; Moulin, 2000). The dual of bilateral consistency is itself (Herrero and Villar, 2001). The dual of converse consistency is also itself (Thomson and Yeh, 2001).

By exploiting the above dual relations between rules, and properties of rules, we obtain the following results. Firstly, the counterpart of Theorem 1 is that the constrained equal losses rule is the only rule satisfying independence of residual claims and composition up (Herrero and Villar, 2002). Secondly, the counterpart of Corollary 1 is that the rule is the only one satisfying exclusion, composition up, and bilateral consistency (Herrero and Villar, 2001). Lastly, the counterpart of Theorem 2 is that it is the only rule satisfying exclusion, composition up, and converse consistency.

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15 This rule assigns amounts such that the losses experienced by all creditors are equal subject to no one receiving a negative award. For an earlier reference to this rule, see Dagan (1996).
16 For a study of dual relations between rules, and properties of rules, see Thomson and Yeh (2001).
17 This property is introduced by Young (1988).
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References


