Recovery of Sparsely Corrupted Signals

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Abstract

We investigate the recovery of signals exhibiting a sparse representation in a general (i.e., possibly redundant or incomplete) dictionary that are corrupted by additive noise admitting a sparse representation in another general dictionary. This setup covers a wide range of applications, such as image inpainting, super-resolution, signal separation, and recovery of signals that are impaired by, e.g., clipping, impulse noise, or narrowband interference. We present deterministic recovery guarantees based on a novel uncertainty relation for pairs of general dictionaries and we provide corresponding practicable recovery algorithms. The recovery guarantees we find depend on the signal and noise sparsity levels, on the coherence parameters of the involved dictionaries, and on the amount of prior knowledge on the support sets of signal and noise. We finally identify situations under which the recovery guarantees are tight.

I. INTRODUCTION

We consider the problem of identifying the sparse vector $x \in \mathbb{C}^{N_a}$ from $M$ linear and non-adaptive measurements collected in the vector

$$z = Ax + Be$$

where $A \in \mathbb{C}^{M \times N_a}$ and $B \in \mathbb{C}^{M \times N_b}$ are deterministic and general (i.e., not necessarily of the same cardinality and possibly redundant or incomplete) dictionaries, and $e \in \mathbb{C}^{N_b}$ represents the sparse noise vector. The support set of $e$ and the corresponding nonzero entries can be arbitrary; in particular, $e$ may also depend on $x$ and/or the dictionary $A$.

This recovery problem occurs in many applications, some of which are described next:

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• **Clipping:** Non-linearities in (power-)amplifiers or in analog-to-digital converters often cause signal clipping [1]. Specifically, instead of the \( M \)-dimensional signal vector \( \mathbf{y} = \mathbf{A} \mathbf{x} \) of interest, the device in question delivers \( g_a(\mathbf{y}) \), where the function \( g_a(\mathbf{y}) \) realizes entry-wise signal clipping to the interval \([-a, +a]\). Setting \( \mathbf{B} = \mathbf{I}_M \), where \( \mathbf{I}_M \) denotes the \( M \times M \) identity matrix, and rewriting (1) as \( \mathbf{z} = \mathbf{y} + \mathbf{e} \) with \( \mathbf{e} = g_a(\mathbf{y}) - \mathbf{y} \), we see that signal clipping is contained in the model (1). The nonzero entries of \( \mathbf{e} \) are those for which clipping occurred. The vector \( \mathbf{e} \) will therefore be sparse, provided that the clipping level is chosen high enough. Finally, we note that here it is essential that the noise vector \( \mathbf{e} \) may depend on the vector \( \mathbf{x} \) and/or the dictionary \( \mathbf{A} \).

• **Impulse noise:** In numerous applications, one has to deal with the recovery of signals corrupted by impulse noise [2]. Specific applications include, e.g., reading out from unreliable memory [3] or recovery of audio signals impaired by click/pop noise, which typically occurs during playback of old phonograph records. The model in (1) is easily seen to incorporate such impairments. Just set \( \mathbf{B} = \mathbf{I}_M \) and let \( \mathbf{e} \) be the impulse-noise vector. We would like to emphasize the generality of (1) which allows impulse noise that is sparse in general dictionaries \( \mathbf{B} \).

• **Narrowband interference:** In many applications, one is interested in recovering audio, video, or communication signals that are corrupted by narrowband interference. Electric hum, as it may occur in improperly designed audio or video equipment, is a typical example of such an impairment. Electric hum typically exhibits a sparse representation in the Fourier basis as it mainly consists of a tone at some base-frequency and a series of corresponding harmonics, which is captured by setting \( \mathbf{B} = \mathbf{F}_M \) in (1).

• **Super-resolution and inpainting:** Our framework also encompasses super-resolution [4], [5] and inpainting [6] for images, audio, and video signals. In both applications, only a subset of the entries of the (full-resolution) signal vector \( \mathbf{y} = \mathbf{A} \mathbf{x} \) is available. Super-resolution and inpainting now amount to filling in the missing entries of the signal vector such that \( \mathbf{y} = \mathbf{A} \mathbf{x} \). These missing entries are accounted for by choosing the vector \( \mathbf{e} \) such that the entries of \( \mathbf{z} = \mathbf{y} + \mathbf{e} \), corresponding to the missing entries in \( \mathbf{y} \), are set to some (arbitrary) value, e.g., 0. The missing entries of \( \mathbf{y} \) are then filled in by recovering \( \mathbf{x} \) from \( \mathbf{z} \) followed by computation of \( \mathbf{y} = \mathbf{A} \mathbf{x} \). Note that in both applications, the support set \( \mathcal{E} \) is known (i.e., the locations of the missing entries can easily be identified) and the dictionary \( \mathbf{A} \) is typically...
redundant, i.e., $A$ has more dictionary elements (columns) than rows, which demonstrates the need for recovery results that apply to general (i.e., possibly redundant) dictionaries.

- **Signal separation:** Separation of (audio or video) signals into two distinct components also fits into our framework. A prominent example for this task is the separation of texture from cartoon parts in images (see [7], [8] and references therein). In the language of our setup, the dictionaries $A$ and $B$ are chosen such that they allow for sparse representation of the two distinct features; $x$ and $e$ are the corresponding coefficients describing these features (sparsely). Note that here the vector $e$ no longer plays the role of (undesired) noise. Signal separation then amounts to simultaneously extracting the sparse vectors $x$ and $e$ from the observation (e.g., the image) $z = Ax + Be$.

Naturally, it is of significant practical interest to identify fundamental limits on the recovery of $x$ (and $e$, in the case of signal separation) from $z$ in (1). For the noiseless case $z = Ax$ such recovery guarantees are known [9]–[11] and typically set limits on the maximum allowed number of nonzero entries of $x$ or more colloquially on the “sparsity” level of $x$. The corresponding sparsity thresholds are usually expressed in terms of the coherence parameter of the dictionary $A$. For the case of unstructured noise, i.e., $z = Ax + n$ with no constraints imposed on $n$ apart from $\|n\|_2 < \infty$, recovery guarantees were derived in [12]–[15]. The corresponding results, however, do not guarantee perfect recovery of $x$, but only ensure that either the error in the recovered nonzero entries is bounded from above by a function of $\|n\|_2$ or only guarantee perfect recovery of the support set of $x$. Such results are to be expected, as a consequence of the generality of the setup in terms of the assumptions on the noise vector $n$.

A. **Contributions**

In this paper, we consider the following questions: 1) Under which conditions can the vector $x$ (and the vector $e$, in the case of signal separation) be recovered perfectly from the sparsely corrupted observation $z = Ax + Be$, and 2) Can we formulate practical recovery algorithms with corresponding (analytical) performance guarantees? Sparsity of the signal vector $x$ and the error vector $e$ will turn out to be key in answering these questions. More specifically, based on an uncertainty relation for pairs of general (possibly redundant or incomplete) dictionaries, we establish recovery guarantees that depend on the number of nonzero entries in $x$ and $e$, and on the coherence parameters of the dictionaries $A$ and $B$. These recovery guarantees are
obtained for the following different cases: i) The support sets of both \( x \) and \( e \) are known (prior to recovery), ii) the support set of only \( x \) or only \( e \) is known, iii) the number of nonzero entries of only \( x \) or only \( e \) is known, and iv) nothing is known about \( x \) and \( e \). We formulate efficient recovery algorithms and derive corresponding performance guarantees. Finally, we discuss the tightness of our recovery thresholds and we demonstrate the application of our results to image inpainting.

**B. Outline of the Paper**

The remainder of the paper is organized as follows. In Section II, we briefly review relevant previous results, and we put our results into perspective. In Section III, we derive a novel uncertainty relation that lays the foundation for the recovery guarantees reported in Section IV. A discussion of our results is provided in Section V, and numerical results are presented in Section VI. We conclude in Section VII.

**C. Notation**

Lowercase boldface letters stand for column vectors and uppercase boldface letters designate matrices. For the matrix \( M \), we denote its transpose and conjugate transpose by \( M^T \) and \( M^H \), respectively, its (Moore–Penrose) pseudo-inverse by \( M^\dagger = (M^H M)^{-1} M^H \), its \( k \)th column by \( m_k \), and the entry in the \( k \)th row and \( \ell \)th column by \([M]_{k,\ell}\). The \( k \)th entry of the vector \( m \) is \([m]_k\). The space spanned by the columns of \( M \) is denoted by \( \mathcal{R}(M) \). Throughout the paper, we assume that the columns of the dictionaries \( A \) and \( B \) have unit \( \ell_2 \)-norm. The \( M \times M \) identity matrix is denoted by \( I_M \), the \( M \times N \) all zeros matrix by \( 0_{M,N} \), and the all-zeros vector of dimension \( M \) by \( 0_M \). The \( M \times M \) discrete Fourier transform matrix \( \mathbf{F}_M \) is defined as

\[
[F_M]_{k,\ell} = \frac{1}{\sqrt{M}} \exp \left( -\frac{2\pi i (k - 1)(\ell - 1)}{M} \right), \quad k, \ell = 1, \ldots, M
\]

where \( i^2 = -1 \). The Euclidean (or \( \ell_2 \)) norm of the vector \( x \) is denoted by \( \|x\|_2 \), \( \|x\|_1 = \sum_k |[x]_k| \) stands for the \( \ell_1 \)-norm of \( x \), and \( \|x\|_0 \) designates the number of nonzero entries in \( x \). The minimum and maximum eigenvalue of the positive-semidefinite matrix \( M \) is denoted by \( \lambda_{\min}(M) \) and \( \lambda_{\max}(M) \), respectively. The spectral norm of the matrix \( M \) is \( \|M\| = \sqrt{\lambda_{\max}(M^H M)} \).

Sets are designated by upper-case calligraphic letters; the cardinality of the set \( T \) is \( |T| \). The complement of a set \( S \) in some superset \( T \) is denoted by \( S^c \), i.e., \( S \cup S^c = T \) and \( S \cap S^c = \emptyset \).
where $\emptyset$ denotes the empty set. For two sets $S_1$ and $S_2$, $s \in (S_1 + S_2)$ means that $s$ is of the form $s = s_1 + s_2$, where $s_1 \in S_1$ and $s_2 \in S_2$. The set of subsets of $S$ of cardinality less than or equal to $n$ is denoted by $\varphi_n(S)$. The support set of the vector $m$, i.e., the index set corresponding to the nonzero entries of $m$, is designated by supp$(m)$. The matrix $M_T$ is obtained from $M$ by retaining the columns of $M$ with indices in $T$; the vector $m_T$ is obtained analogously from the vector $m$. We define the $N \times N$ diagonal (projection) matrix $P_S$ for the set $S \subseteq \{1, \ldots, N\}$ as follows:

$$
[P_S]_{k,\ell} = \begin{cases} 
1, & \text{if } k = \ell \text{ and } k \in S \\
0, & \text{otherwise}.
\end{cases}
$$

For $x \in \mathbb{R}$, we set $[x]^+ = \max\{x, 0\}$.

II. REVIEW OF RELEVANT PREVIOUS RESULTS

Recovery of the vector $x$ from the sparsely corrupted measurement $z = Ax + Be$ corresponds to a sparse-signal recovery problem subject to \textit{structured} (i.e., sparse) noise. In this section, we review relevant existing results for sparse-signal recovery from noiseless measurements, and we summarize the results available for recovery in the presence of unstructured and structured noise.

A. Recovery in the Noiseless Case

Recovery of $x$ from $y = Ax$ with $A$ being redundant (i.e., $M < N_a$) amounts to solving an underdetermined system of linear equations. Hence, there are infinitely many solutions $x$, in general. However, under the assumption of $x$ being sparse, the situation changes drastically. More specifically, one can recover $x$ from the observation $y = Ax$ by solving

$$
(P_0) \quad \text{minimize } \|x\|_0 \quad \text{subject to } y = Ax.
$$

This results, however, in prohibitive computational complexity, even for small problem sizes. Two of the most popular and computationally tractable alternatives to solving (P0) by an exhaustive search are basis pursuit (BP) [9]–[11], [16]–[18] and orthogonal matching pursuit (OMP) [11], [19], [20]. Two of the most popular and computationally tractable alternatives to solving (P0) by an exhaustive search are basis pursuit (BP) [9]–[11], [16]–[18] and orthogonal matching pursuit (OMP) [11], [19], [20]. BP is essentially a convex relaxation of (P0) and amounts to solving

$$
(BP) \quad \text{minimize } \|x\|_1 \quad \text{subject to } y = Ax.
$$
OMP is a greedy algorithm that iteratively constructs a sparse representation of $y$ by selecting, in each iteration, the column of $A$ that is most “correlated” with the difference between $y$ and its current approximation.

The questions that arise naturally are: Under which conditions does (P0) have a unique solution and when do BP and/or OMP deliver this solution? To formulate the answer to these questions, define $n_x = \|x\|_0$ and the coherence of the dictionary $A$ as

$$\mu_a = \max_{k,\ell,k \neq \ell} |a_k^H a_\ell|.$$  \hspace{1cm} (2)

As shown in [9]–[11], a sufficient condition for $x$ to be the unique solution of (P0) applied to $y = Ax$ and for BP and OMP to deliver this solution is

$$n_x < \frac{1}{2} \left(1 + \mu_a^{-1}\right).$$ \hspace{1cm} (3)

### B. Recovery in the Presence of Unstructured Noise

Recovery guarantees in the presence of unstructured (and deterministic) noise, i.e., for $z = Ax + n$, with no constraints imposed on $n$ apart from $\|n\|_2 < \infty$, were derived in [12]–[15] and references therein. Specifically, it was shown in [12] that a suitably modified version of BP, referred to as BP denoising (BPDN), recovers an estimate $\hat{x}$ satisfying $\|x - \hat{x}\|_2 < C\|n\|_2$ provided that

$$n_x < \frac{1}{4} \left(1 + \mu_a^{-1}\right).$$ \hspace{1cm} (4)

Here, $C > 0$ depends on the coherence $\mu_a$ and on the sparsity level $n_x$ of $x$. Note that the estimate $\hat{x}$ may have errors in the support set. Another result in [12] states that OMP delivers the correct support set (but does not perfectly recover the nonzero entries of $x$) provided that

$$n_x < \frac{1}{2} \left(1 + \mu_a^{-1}\right) - \frac{||n||_2}{\mu_a |x_{\min}|}$$ \hspace{1cm} (5)

where $|x_{\min}|$ denotes the absolute value of the component of $x$ with smallest nonzero magnitude. The recovery condition (5) yields sensible results only if $\|n\|_2/|x_{\min}|$ is small. Results similar to those reported in [12] were obtained in [13], [14]. Recovery guarantees in the case of stochastic noise $n$ can be found in [14], [15]. We finally point out that perfect recovery of $x$ is, in general, impossible in the presence of unstructured noise. In contrast, as we shall see below, perfect recovery is possible under structured noise according to (1).
C. Recovery Guarantees in the Presence of Structured Noise

As outlined in the introduction, many practically relevant signal recovery problems can be formulated as (sparse) signal recovery from sparsely corrupted measurements, a problem that seems to have received comparatively little attention in the literature so far and does not appear to have been developed systematically. Special cases of the general setup (1) were considered in [2], [21]–[26]. Specifically, in [21] it was shown that for \( A = F_M \) and \( B = I_M \), perfect recovery of the \( M \)-dimensional vector \( x \) is possible if

\[
2n_x n_e < M
\]  

where \( n_e = \|e\|_0 \). We emphasize that this result assumes the support set of \( e \) to be known (prior to recovery), an assumption that is often difficult to meet in practice. It is interesting to observe that condition (6) is—in contrast to the recovery guarantee (5)—independent of the \( \ell_2 \)-norm of the noise vector, i.e., \( \|Be\|_2 \) may, in principle, be arbitrarily large.

In [22], [23], probabilistic recovery results (i.e., recovery is guaranteed with high probability) based on the restricted isometry property (RIP) for \( A \) i.i.d. zero-mean Gaussian, \( B \) an orthonormal basis (ONB), and the support set of \( e \) either known or unknown were reported. The case of \( A \) i.i.d. Gaussian with non-zero mean, \( B = I_M \), and the support set of \( e \) unknown was also treated in [25]. In [26] it was shown that for randomly sub-sampled unitary matrices \( A \), given a probabilistic model on both \( x \) and \( e \), one can perfectly recover \( x \) with high probability.

The problem of sparse signal recovery in the presence of impulse noise (i.e., \( B = I_M \)) was considered in [2], where a particular nonlinear measurement process combined with a non-convex program for signal recovery was proposed. In [24], signal recovery based on \( \ell_1 \)-norm minimization in the presence of impulse noise was investigated. The setup in [24], however, differs considerably from the one considered in this paper as \( A \) in [24] needs to be tall (i.e., \( M > N_a \)) and the vector \( x \) to be recovered is not necessarily sparse.

We conclude this overview by noting that the present paper is inspired by [21]. Specifically, we note that the recovery guarantee (6) reported in [21] is obtained from an uncertainty relation that puts limits on how sparse a given signal can simultaneously be in the Fourier basis and in the identity basis. Inspired by this observation, we start by presenting an uncertainty relation, which forms the basis for the recovery guarantees reported in this paper.
III. A GENERAL UNCERTAINTY RELATION FOR $\epsilon$-CONCENTRATED VECTORS

We next present a novel uncertainty relation, which extends the uncertainty relation in [27, Lem. 1] for pairs of general (i.e., possibly redundant or incomplete) dictionaries to vectors that are $\epsilon$-concentrated rather than perfectly sparse. As shown in Section IV, this extension is crucial for the derivation of recovery guarantees for BP.

A. The Uncertainty Relation

Define the mutual coherence between the dictionaries $A$ and $B$ as

$$\mu_m = \max_{k,\ell} |a_k^H b_{\ell}|.$$  

Furthermore, we will need the following definition, which appeared previously in [21].

**Definition 1:** A vector $r \in \mathbb{C}^N$ is said to be $\epsilon_R$-concentrated to the set $R \subseteq \{1, \ldots, N\}$ if $\|P_R r\|_1 \geq (1 - \epsilon_R)\|r\|_1$ where $\epsilon_R \in [0, 1]$. We say that the vector $r$ is perfectly concentrated to the set $R$ and, hence, $|R|$-sparse if $P_R r = r$, i.e., if $\epsilon_R = 0$.

We can now state the following uncertainty relation for pairs of general dictionaries and for $\epsilon$-concentrated vectors.

**Theorem 1:** Let $A \in \mathbb{C}^{M \times N_a}$ be a dictionary with coherence $\mu_a$, $B \in \mathbb{C}^{M \times N_b}$ a dictionary with coherence $\mu_b$, and denote the mutual coherence between $A$ and $B$ by $\mu_m$. Let $s$ be a vector in $\mathbb{C}^M$ that can be represented as a linear combination of columns of $A$ and, equivalently, as a linear combination of columns of $B$. Hence, there exists a pair of vectors $p \in \mathbb{C}^{N_a}$ and $q \in \mathbb{C}^{N_b}$ such that $s = Ap = Bq$ (we exclude the trivial case where both $p = 0_{N_a}$ and $q = 0_{N_b}$). If $p$ is $\epsilon_P$-concentrated to $P$ and $q$ is $\epsilon_Q$-concentrated to $Q$, then the following inequality holds

$$|P| |Q| \geq \frac{\left[(1 + \mu_a)(1 - \epsilon_P) - |P| \mu_a^+ \right] \left[(1 + \mu_b)(1 - \epsilon_Q) - |Q| \mu_b^+ \right]}{\mu_m^2}.$$  

**Proof:** See Appendix A.

In the special case where both dictionaries $A$ and $B$ contain orthonormal columns (i.e., for $\mu_a = \mu_b = 0$), the uncertainty relation in Theorem 1 reduces to

$$|P| |Q| \geq \frac{(1 - \epsilon_P)(1 - \epsilon_Q)}{\mu_m^2}.$$  

1The uncertainty relation continues to hold if either $p = 0_{N_a}$ or $q = 0_{N_b}$, but does not apply to the trivial case $p = 0_{N_a}$ and $q = 0_{N_b}$. In all three cases we have $s = 0_M$. 

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Furthermore, the uncertainty relation in [21, Thm. 1] is a special case of Theorem 1 obtained by setting $A = F_M$ and $B = I_M$, but differs from (8) in two ways:

i) The uncertainty relation in [21] applies only to $A = F_M$ and $B = I_M$. The dependence on $\mu_m$ is not made explicit in [21] as $\mu_m = 1/\sqrt{M}$ for this particular pair of bases. Our formulation (8) covers general pairs of ONBs with mutual coherence $\mu_m$.

ii) Our formulation is based on “$\epsilon$-concentration” for both vectors $p$ and $q$, whereas [21] takes into account “$\epsilon$-concentration” for only one of the vectors.

In the case where both $p$ and $q$ are perfectly concentrated rather than $\epsilon$-concentrated, i.e., for $\epsilon_P = \epsilon_Q = 0$, Theorem 1 corresponds to the uncertainty relation reported in [27, Lem. 1], which we restate here for completeness.

**Corollary 2 ([27, Lem. 1]):** If $P = \text{supp}(p)$ and $Q = \text{supp}(q)$, the following holds:

$$|P||Q| \geq \frac{[1 - \mu_a(|P| - 1)]^+ [1 - \mu_b(|Q| - 1)]^+}{\mu_m^2}.$$  \hspace{1cm} (9)

As detailed in [27], the uncertainty relation in Corollary 2 generalizes the one for two orthonormal bases (ONBs) provided in [18]. Furthermore, it extends the uncertainty relations provided in [28] for pairs of square dictionaries (having the same number of rows and columns) to pairs of general (i.e., possibly redundant or incomplete) dictionaries $A$ and $B$.

**B. Tightness of the Uncertainty Relation**

In certain special cases it is possible to identify signals that saturate the uncertainty relation (7), i.e., signals for which (7) holds with equality. Consider $A = F_M$ and $B = I_M$ so that $\mu_m = 1/\sqrt{M}$, and define the comb signal containing equidistant spikes of unit height as

$$[\delta_{\ell}]_\ell = \begin{cases} 1, & \text{if } (\ell - 1) \mod t = 0 \\ 0, & \text{otherwise} \end{cases}$$

where we shall assume that $t$ divides $M$ without remainder. It can be shown that the vectors $p = \delta_{\sqrt{M}}$ and $q = \delta_{\sqrt{M}}$, both having $\sqrt{M}$ nonzero entries, satisfy $F_M p = I_M q$. If $P = \text{supp}(p)$ and $Q = \text{supp}(q)$, the vectors $p$ and $q$ are perfectly concentrated to $P$ and $Q$, respectively, i.e., $\epsilon_P = \epsilon_Q = 0$. Since $|P| = |Q| = \sqrt{M}$ and $\mu_m = 1/\sqrt{M}$ it follows that $|P||Q| = 1/\mu_m^2 = M$ and hence, $p = q = \delta_{\sqrt{M}}$ saturates (7).

We will now show that for pairs of general dictionaries $A$ and $B$, identifying signals that saturate the uncertainty relation is NP-hard. For the sake of simplicity, we restrict ourselves to
the case $P = \text{supp}(p)$ and $Q = \text{supp}(q)$, which implies $|P| = \|p\|_0$ and $|Q| = \|q\|_0$. Next, consider the problem

$$\begin{cases}
\text{minimize} & \|p\|_0 \|q\|_0 \\
\text{subject to} & Ap = Bq, \|q\|_0 \geq 1, \|p\|_0 \geq 1.
\end{cases}$$

(U0)

Since we are interested in the minimum of $\|p\|_0 \|q\|_0$ for nonzero vectors $p$ and $q$, we imposed the constraints $\|p\|_0 \geq 1$ and $\|q\|_0 \geq 1$ to exclude the case where $p = 0_{N_a}$ and/or $q = 0_{N_b}$. Now, it follows that for the particular choice $B = y \in \mathbb{C}^M$ and hence $q = q \in \mathbb{C} \setminus \{0\}$ (note that we exclude the case $q = 0$ as a consequence of the requirement $\|q\|_0 \geq 1$) the problem (U0) simplifies to

$$(U0^*) \text{ minimize } \|x\|_0 \text{ subject to } Ax = y$$

where $x = p/q$. However, as $(U0^*)$ is equivalent to (P0), which is known to be NP-hard [29], we can conclude that finding a pair $p$ and $q$ that saturates the uncertainty relation is NP-hard in general.

IV. RECOVERY OF SPARSELY CORRUPTED SIGNALS

Based on the uncertainty relation in Theorem 1, we next derive conditions that guarantee perfect recovery of $x$ from the sparsely corrupted measurement $z = Ax + Be$. Specifically, these conditions depend on the number of nonzero entries of $x$ and $e$, and on the coherence parameters $\mu_a$, $\mu_b$, and $\mu_m$. In contrast to (5), the so obtained conditions will not depend on the $\ell_2$-norm of the noise vector $\|Be\|_2$, which may be arbitrarily large. We consider the following cases: i) the support sets of both $x$ and $e$ are known (prior to recovery), ii) the support set of only $x$ or only $e$ is known, iii) the number of nonzero entries of only $x$ or only $e$ is known, and iv) nothing is known about $x$ and $e$. The uncertainty relation in Theorem 1 plays a key role in the derivation of the recovery guarantees for all scenarios considered. To simplify notation, motivated by the form of the right-hand side (RHS) of (9), we define the function

$$f(u, v) = \frac{[1 - \mu_a(u - 1)]^+ [1 - \mu_b(v - 1)]^+}{\mu_m^2}.$$ 

In the remainder of the paper, $X$ denotes supp($x$) and $E$ stands for supp($e$). We furthermore assume that the dictionaries $A$ and $B$ are known perfectly to the recovery algorithms. Moreover,
we assume that $\mu_m > 0$.

A. Case I: Knowledge of $\mathcal{X}$ and $\mathcal{E}$

We start with the case of both $\mathcal{X}$ and $\mathcal{E}$ known prior to recovery. The values of the nonzero entries of $x$ and $e$ are unknown. This scenario is relevant, for example, in applications requiring recovery of clipped band-limited signals with known spectral support $\mathcal{X}$. Here, we would have $A = F_M$, $B = I_M$, and $\mathcal{E}$ can be determined as follows: Compare the measured signal entries $[z]_i$, $i = 1, \ldots, M$, to the clipping threshold $a$, and if $|[z]_i| = a$ add the corresponding index $i$ to $\mathcal{E}$. Sparse-signal recovery is then performed as follows. We first rewrite the input-output relation in (1) according to

$$z = A_X x_X + B_E e_E = D_{X,E} s_{X,E}$$

with the concatenated dictionary $D_{X,E} = [A_X B_E]$ and the stacked vector $s_{X,E} = [x_X^T e_E^T]^T$. It is now important to realize that we are able to recover the stacked vector $s_{X,E}$ perfectly and hence the nonzero entries of both $x$ and $e$, if the pseudo-inverse $D_{X,E}^\dagger$ exists. In this case, we obtain $s_{X,E}$ according to

$$s_{X,E} = D_{X,E}^\dagger z.$$ (10)

The following theorem states a sufficient condition for $D_{X,E}$ to have full column rank, which then implies existence of the pseudo-inverse $D_{X,E}^\dagger$. This condition depends on the coherence parameters $\mu_a$, $\mu_b$, and $\mu_m$, of the involved dictionaries $A$ and $B$ and on $\mathcal{X}$ and $\mathcal{E}$ only through the cardinalities $|\mathcal{X}|$ and $|\mathcal{E}|$, i.e., the number of nonzero entries in $x$ and $e$, respectively.

**Theorem 3:** Let $z = Ax + Be$ with $\mathcal{X} = \text{supp}(x)$ and $\mathcal{E} = \text{supp}(e)$. Define $n_x = \|x\|_0$ and $n_e = \|e\|_0$. If

$$n_x n_e < f(n_x, n_e)$$ (11)

then the concatenated dictionary $D_{X,E} = [A_X B_E]$ has full column rank.

**Proof:** See Appendix B.

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$^2$If $\mu_m = 0$, the space spanned by the columns of $A$ is orthogonal to the space spanned by the columns of $B$. It is therefore straightforward to separate $Ax$ from $Be$ given $z$ and, in turn, to then recover $x$ from $y = Ax$ using (P0), BP, or OMP, if (3) is satisfied.
For the special case $A = F_M$ and $B = I_M$ (so that $\mu_a = \mu_b = 0$ and $\mu_m = 1/\sqrt{M}$),
the recovery condition (11) reduces to $n_x n_e < M$, a result obtained previously in [21]. It is interesting to observe that Theorem 3 contains further well-known results [9]–[11] as special cases. More specifically, in the noiseless case (i.e., $n_e = 0$) $D_{X,E} = A_X$ and hence, (11) ensures that the $n_x$ columns of $A_X$ are linearly independent if $1 - \mu_a(n_x - 1) > 0$. This result agrees with the fact that no fewer than $1 + 1/\mu_a$ columns of $A$ can constitute a linearly dependent set (see, e.g., [9]–[11]).

Finally, we observe that Theorem 3 yields a sufficient condition on $n_x$ and $n_e$ for any $(M - n_e) \times n_x$-submatrix of $A$ to have full column rank. To see this, consider the special case $B = I_M$ and hence, $D_{X,E} = [A_X I_e]$. Condition (11) characterizes pairs $(n_x, n_e)$, for which all matrices $D_{X,E}$ with $n_x = |X|$ and $n_e = |E|$ are guaranteed to have full column rank. Hence, the submatrix consisting of all rows of $A_X$ with row index in $E^c$ must have full column rank as well. Since the result holds for all support sets $X$ and $E$ with $|X| = n_x$ and $|E| = n_e$, all possible $(M - n_e) \times n_x$-submatrices of $A$ must have full column rank.

B. Case II: Either $X$ or $E$ is Known

Next, we find recovery guarantees for the case where either only $X$ or only $E$ is known prior to recovery.

1) Recovery When $E$ is Known and $X$ is Unknown: A prominent application for this setup is the recovery of clipped band-limited signals [22], [30], where the spectral support of $x$ is unknown, i.e., $X$ is unknown. The support set $E$ can be identified as detailed previously in Section IV-A. Further application examples for this setup include inpainting and super-resolution [4]–[6] of signals that admit a sparse representation in $A$ but with unknown support set. The locations of the missing elements in $y$ are known (and correspond, e.g., to missing paint elements in frescos), i.e., the set $E$ can be determined as described previously in Section I. Inpainting and super-resolution then amount to reconstructing the vector $x$ from the sparsely corrupted measurement $z = Ax + e$ and computing $y = Dx$.

The setting of $E$ known and $X$ unknown was considered previously in [21] for the special case $A = F_M$ and $B = I_M$. The recovery condition (14) in Theorem 4 below extends the result in [21, Thms. 5 and 9] to pairs of general (i.e., possibly redundant or incomplete) dictionaries $A$ and $B$. 

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Theorem 4: Let \( z = Ax + Be \) where \( \mathcal{E} = \text{supp}(e) \) is known. Consider the problem

\[
(P_0, \mathcal{E}) \quad \begin{cases} 
\text{minimize} & \| \tilde{x} \|_0 \\
\text{subject to} & A\tilde{x} \in (\{z\} + \mathcal{R}(B_{\mathcal{E}}))
\end{cases}
\]  

(12)

and the convex program

\[
(BP, \mathcal{E}) \quad \begin{cases} 
\text{minimize} & \| \tilde{x} \|_1 \\
\text{subject to} & A\tilde{x} \in (\{z\} + \mathcal{R}(B_{\mathcal{E}})).
\end{cases}
\]  

(13)

If the number of nonzero entries in \( x \) and \( e \), i.e., \( n_x = \|x\|_0 \) and \( n_e = \|e\|_0 \), satisfy

\[
2n_xn_e < f(2n_x, n_e)
\]  

(14)

then the unique solution of \((P_0, \mathcal{E})\) applied to \( z = Ax + Be \) is given by \( x \) and \((BP, \mathcal{E})\) will deliver this solution.

Proof: See Appendix C.

Solving \((P_0, \mathcal{E})\) requires a combinatorial search, which results in prohibitive computational complexity even for moderate problem sizes. The convex relaxation \((BP, \mathcal{E})\) can, however, often be solved more efficiently. Note that the constraint \( A\tilde{x} \in (\{z\} + \mathcal{R}(B_{\mathcal{E}})) \) defines all possible signals \( \tilde{x} \) that, for the given dictionaries \( A \) and \( B \), and the given error support set \( \mathcal{E} \), are consistent with the measurement outcome \( z \). Among these signals, \((P_0, \mathcal{E})\) finds the sparsest signal \( \tilde{x} \) and \((BP, \mathcal{E})\) finds the signal \( \tilde{x} \) with the smallest \( \ell_1 \)-norm. For \( n_e = 0 \) (i.e., the noiseless case), the set \( \{z\} + \mathcal{R}(B_{\mathcal{E}}) \) reduces to the single-element set \( \{z\} \) so that the constraint in \((P_0, \mathcal{E})\) and \((BP, \mathcal{E})\) becomes \( A\tilde{x} = z \). Hence, \((P_0, \mathcal{E})\) and \((BP, \mathcal{E})\) reduce to the standard problems \((P_0)\) and \( BP \), respectively \([9], [10]\). In this case, the recovery threshold (14) turns into\(^3\) \( n_x < (1 + 1/\mu_a)/2 \), which is the well-known recovery threshold (3) guaranteeing recovery of the sparse vector \( x \) through \((P_0)\) and \( BP \) applied to \( y = Ax \) \([9], [10]\).

Rather than solving \((P_0, \mathcal{E})\) or \((BP, \mathcal{E})\), we may attempt to recover the vector \( x \) by exploiting the fact that \( \mathcal{R}(B_{\mathcal{E}}) \) is known (as a consequence of \( B \) and \( \mathcal{E} \) assumed to be known) and projecting the measurement outcome \( z \) onto the orthogonal complement of \( \mathcal{R}(B_{\mathcal{E}}) \). This approach would eliminate the noise and leave us with a standard (but slightly modified, as we shall see below) sparse signal recovery problem for the vector \( x \). We next show that this alternative approach is guaranteed to recover the sparse vector \( x \) provided that condition (14) is satisfied. Let us detail

\[^3\text{For } n_e = 0, (14) \text{ becomes } 0 < f(2n_x, 0), \text{ which then yields } [1 - \mu_a(2n_x - 1)]^+ > 0 \text{ and hence } n_x < (1 + 1/\mu_a)/2.\]
the procedure. If the columns of $B_\epsilon$ are linearly independent, the pseudo-inverse $B_\epsilon^\dagger$ exists, and we can compute the projector onto the orthogonal complement of $\mathcal{R}(B_\epsilon)$ according to:

$$R_\epsilon = I_M - B_\epsilon B_\epsilon^\dagger.$$  

(15)

Applying $R_\epsilon$ to the measurement outcome $z$ yields

$$R_\epsilon z = R_\epsilon (Ax + B_\epsilon e_\epsilon) = R_\epsilon Ax \triangleq \hat{z}$$

where we used the fact that $R_\epsilon B_\epsilon = 0_{M,n_\epsilon}$. We are now left with the standard problem of recovering $x$ from the modified measurement outcome $\hat{z} = R_\epsilon Ax$. What comes to mind first is that computing the standard recovery threshold (3) for the modified dictionary $R_\epsilon A$ should provide us with a recovery threshold for the problem $\hat{z} = R_\epsilon Ax$. It turns out, however, that the columns of $R_\epsilon A$ will, in general, not have unit $\ell_2$-norm, an assumption underlying (3).

What comes to our rescue is that under condition (14) we have (as shown in Theorem 5 below)

$$\|R_\epsilon a_\ell\|_2 > 0 \text{ for } \ell = 1, \ldots, N_a.$$ 

We can, therefore, normalize the modified dictionary $R_\epsilon A$ by rewriting (16) as

$$\hat{z} = R_\epsilon A \Delta \hat{x}$$

(17)

where $\Delta$ is the diagonal matrix with elements

$$[\Delta]_{\ell,\ell} = \frac{1}{\|R_\epsilon a_\ell\|_2}, \quad \ell = 1, \ldots, N_a$$

and $\hat{x} \triangleq \Delta^{-1}x$. Now, $R_\epsilon A \Delta$ plays the role of the dictionary (with normalized columns) and $\hat{x}$ is the unknown sparse vector that we wish to recover. Obviously, supp($\hat{x}$) = supp($x$) and $x$ can be recovered from $\hat{x}$ according to $^4 x = \Delta \hat{x}$. The following theorem shows that (14) is sufficient to guarantee the following: i) The columns of $B_\epsilon$ are linearly independent (guaranteeing the existence of $B_\epsilon^\dagger$), ii) $\|R_\epsilon a_\ell\|_2 > 0$ for $\ell = 1, \ldots, N_a$, and iii) no vector $x' \in \mathbb{C}^{N_a}$ satisfying $\|x'\|_0 \leq 2n_x$ lies in the kernel of $R_\epsilon A$. Hence, (14) enables perfect recovery of $x$ from (17).

**Theorem 5:** If (14) is satisfied, the unique solution of (P0) applied to $\hat{z} = R_\epsilon A \Delta \hat{x}$ is given by $\hat{x}$. Furthermore, BP and OMP applied to $\hat{z} = R_\epsilon A \Delta \hat{x}$ are guaranteed to recover the unique (P0)-solution.

**Proof:** See Appendix D.

$^4$If $\|R_\epsilon a_\ell\|_2 > 0$ for $\ell = 1, \ldots, N_a$, then the matrix $\Delta$ defines a one-to-one mapping.
Since the condition (14) ensures that \([\Delta]_{\ell,\ell} > 0, \ell = 1, \ldots, N_u\), the vector \(x\) can be obtained from \(\hat{x}\) according to \(x = \Delta \hat{x}\). Furthermore, (14) guarantees the existence of \(B_E^\dagger\) and hence, the nonzero entries of \(e\) can be obtained from \(x\) as follows:

\[
e_{E} = B_E^\dagger (z - Ax).
\]

Theorem 5 generalizes the results in [21, Thms. 5 and 9] obtained for the special case \(A = F_M\) and \(B = I_M\) to pairs of general (i.e., possibly redundant or incomplete) dictionaries and additionally shows that OMP delivers the correct solution provided that (14) is satisfied.

Obviously, other well-established sparse-signal recovery algorithms, such as iterative thresholding-based algorithms [31], CoSaMP [32], or message passing algorithms [33], can be applied to (17) to recover \(x\). Finally, we note that for the special case of \(B\) an ONB, the idea of projecting the measurement outcome onto the orthogonal complement of the space spanned by the active columns of \(B\) was put forward in [22] along with recovery thresholds that apply to randomly chosen \(A\) and guarantee recovery with high probability (with respect to the realization of \(A\)).

2) Recovery When \(\mathcal{X}\) is Known and \(\mathcal{E}\) is Unknown: A possible application scenario for this situation is the recovery of spectrally sparse signals with known spectral support that are impaired by impulse noise with unknown impulse locations.

It is evident that this setup is formally equivalent to that discussed above in Section IV-B1, with the roles of \(x\) and \(e\) swapped. In particular, we may apply the projection matrix \(R_\mathcal{X} = I_M - A_\mathcal{X} A_\mathcal{X}^\dagger\) to the corrupted measurement outcome \(z\) to obtain the standard recovery problem \(\hat{z}' = R_\mathcal{X} B \Delta' \hat{e}\), where \(\Delta'\) is a diagonal matrix with diagonal elements \([\Delta']_{\ell,\ell} = 1/\|R_\mathcal{X} b_\ell\|_2\). The corresponding unknown vector is given by \(\hat{e} \triangleq (\Delta')^{-1} e\). The following corollary is a direct consequence of Theorem 5.

**Corollary 6**: Let \(z = Ax + Be\) where \(\mathcal{X} = \text{supp}(x)\) is known. If the number of nonzero entries in \(x\) and \(e\), i.e., \(n_x = \|x\|_0\) and \(n_e = \|e\|_0\), satisfy

\[
2n_x n_e < f(n_x, 2n_e) \quad (18)
\]

\(^5\)To determine whether these recovery algorithms will deliver the unique solution provided that (14) holds, remains an open interesting problem.
then the unique solution of (P0) applied to \( \hat{z}' = R_{X}B\Delta'\hat{e} \) is given by \( \hat{e} = (\Delta')^{-1} e \). Furthermore, BP and OMP applied to \( \hat{z}' = R_{X}B\Delta'\hat{e} \) recover this unique (P0)-solution.

Once we have \( \hat{e} \), the vector \( e \) can be obtained from \( \hat{e} \) according to \( e = \Delta'\hat{e} \) and the nonzero entries of \( x \) are obtained as follows:

\[
x_{X} = A_{X}^\dagger(z - Be).
\]

Note that condition (18) ensures that the columns of \( A_{X} \) are linearly independent and hence \( A_{X}^\dagger \) exists.

C. Case III: Cardinality of \( E \) or \( X \) Known

We next consider the case where neither \( X \) nor \( E \) are known, but knowledge of either \( \|x\|_0 \) or \( \|e\|_0 \) is available (prior to recovery). A possible corresponding application scenario for \( \|x\|_0 \) unknown and \( \|e\|_0 \) known would be the recovery of a sparse pulse-stream with unknown pulse-locations from measurements that are corrupted by electric hum with unknown base-frequency but known number of harmonics (e.g., limited by the acquisition bandwidth). We state our main result for the case \( n_e = \|e\|_0 \) known and \( n_x = \|x\|_0 \) unknown. The case where \( n_x \) is known and \( n_e \) is unknown can be treated similarly.

**Theorem 7**: Let \( z = Ax + Be \), define \( n_x = \|x\|_0 \) and \( n_e = \|e\|_0 \), and assume that \( n_e \) is known. Consider the problem

\[
(P0, n_e) \begin{cases}
\text{minimize} & \|\tilde{x}\|_0 \\
\text{subject to} & A\tilde{x} \in \left( \{z\} + \bigcup_{E' \in \mathcal{P}} \mathcal{R}(B_{E'}) \right)
\end{cases}
\]

where \( \mathcal{P} = \varnothing_{n_e}([1, \ldots, N_b]) \). The unique solution of \( (P0, n_e) \) applied to \( z = Ax + Be \) is given by \( x \) if

\[
4n_xn_e < f(2n_x, 2n_e).
\]

**Proof**: See Appendix E.

We emphasize that the problem \( (P0, n_e) \) has prohibitive computational complexity, in general. Unfortunately, replacing the \( \ell_0 \)-norm of \( \tilde{x} \) in the minimization in (19) by the \( \ell_1 \)-norm does not lead to a computationally tractable alternative either, as the constraint \( A\tilde{x} \in \left( \{z\} + \bigcup_{E' \in \mathcal{P}} \mathcal{R}(B_{E'}) \right) \) specifies a non-convex set, in general.
Nonetheless, the recovery threshold in (20) is interesting as it completes the picture on the impact of knowledge of the support sets of $x$ and $e$ on the recovery thresholds. We refer to Section V-B for a detailed discussion of this matter.

**D. Case IV: No Knowledge of the Support Sets**

Finally, we consider the case of no knowledge (prior to recovery) of the support sets $\mathcal{X}$ and $\mathcal{E}$. A corresponding application scenario would be the restoration of an audio signal (whose spectrum is sparse but $\mathcal{X}$ is unknown) that is corrupted by impulse noise, e.g., click or pop noise, at unknown locations. Another typical application can be found in the realm of signal separation; concretely, e.g., the decomposition of images into two distinct features, i.e., into a part that exhibits a sparse representation in the dictionary $A$ and another part that exhibits a sparse representation in $B$. Decomposition of the image $z$ then amounts to performing sparse-signal recovery based on $z = Ax + Be$ with no knowledge about the support sets $\mathcal{X}$ and $\mathcal{E}$ available prior to recovery. The individual image features are then given by $Ax$ and $Be$.

Recovery guarantees for this case follow from results in [27] for the concatenation of general (possibly redundant or incomplete) dictionaries. This can be seen by writing (1) as follows

$$z = Ax + Be = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

(21)

Now, viewing the concatenation of $A$ and $B$ as a dictionary $D = [A \ B]$ and stacking $x$ and $e$ according to $w = [x^T \ e^T]^T$, we can employ the recovery guarantees in [27] for the vector $w$. In particular, it was shown in [27, Eq. 10] that the unique solution of

$$(P0) \ \ \ \text{minimize} \ |w|_0 \ \ \ \text{subject to} \ \ \ z = Dw$$

is given by $w$ and, furthermore, in [27, Eq. 13] that this solution is delivered by

$$(BP) \ \ \ \text{minimize} \ |\tilde{w}|_1 \ \ \ \text{subject to} \ \ \ z = D\tilde{w}$$

and by OMP applied to $z = Dw$, if the number of nonzero entries of $w$ is less than a corresponding sparsity threshold. These thresholds (i.e., [27, Eq. 10] and [27, Eq. 13]) are more restrictive than those in (11), (14), (18), and (20) (see also Section V-B), which reflects the intuition that additional knowledge about the support sets $\mathcal{X}$ and $\mathcal{E}$ can only improve the recovery guarantees. Obviously, once the vector $w$ has been recovered, we can extract $x$. 

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Note that our recovery guarantees (11), (14), (18), and (20) actually ensure that both $x$ and $e$ can be recovered uniquely from $z$; this is relevant for the signal separation setup as discussed in Section I.

V. DISCUSSION OF THE RECOVERY GUARANTEES

The aim of this section is to interpret the recovery guarantees found in Section IV. Specifically, we investigate the tightness of the recovery thresholds (11), (14), (18), and (20), we discuss the impact of support-set knowledge on these thresholds, and we point out the limitations of our results.

A. Tightness of the Recovery Thresholds in the Fourier-Identity Case

We show that in the special case $A = F_M$ and $B = I_M$ the recovery thresholds (11), (14), (18), and (20) are tight. The general approach taken is to construct pairs of vectors $x$ and $e$ that saturate the recovery threshold, i.e., to find vectors for which the corresponding recovery guarantee “just fails”, and to demonstrate that for these pairs the vector $x$ cannot be recovered uniquely. Note that for Case IV discussed in Section IV-D, it has already been established that the threshold guaranteeing (P0)-uniqueness [17, Thm. I.1] is tight. For the remaining cases, discussed in Sections IV-A, IV-B, and IV-C, the insights developed below appear to be new.

The recovery-threshold saturating signals we shall use are all variations of comb signals. The comb signal has the following key property, which is crucial in the context of establishing tightness for the Fourier-identity pair case:

$$F_M \delta_t = \sqrt{M} \delta_{M/t}. \quad (22)$$

Note that for $A = F_M$ and $B = I_M$, we have $\mu_a = \mu_b = 0$ and $\mu_m = 1/\sqrt{M}$.

- Case I: Knowledge of $X$ and $E$. Consider the vectors $x = \lambda \delta_{\sqrt{M}}$ and $e = (1 - \lambda) \delta_{\sqrt{M}}$ with $\lambda \in (0, 1)$. Note that here $n_x = n_e = \sqrt{M}$ and hence $n_x n_e = M = f(n_x, n_e)$. Therefore, $x$ and $e$ saturate condition (11). Now, consider the alternative vectors $x' = \lambda' \delta_{\sqrt{M}}$ and $e' = (1 - \lambda') \delta_{\sqrt{M}}$ where $\lambda' \in (0, 1)$ with $\lambda' \neq \lambda$. Note that $x$ and $x'$ are supported on the same set $X$, and $e$ and $e'$ are supported on the same set $E$. Furthermore, $(x, e)$ and $(x', e')$ result in the same measurement outcome, i.e.,

$$F_M x + e = F_M x' + e' = \delta_{\sqrt{M}}.$$
Based on the measurement outcome, we can therefore not decide between x and x' \neq x.

- **Case II: Only \mathcal{E} is known.** Consider x = \delta_{2\sqrt{M}} - \delta_{\sqrt{M}} with n_x = \sqrt{M}/2 and e = \delta_{\sqrt{M}} with n_e = \sqrt{M}. This pair (x, e) saturates (14) as 2n_xn_e = M = f(2n_x, n_e). Now, consider the alternative vector x' = \delta_{2\sqrt{M}}. We now compare the pairs (x, e) and (x', e). Note that for both pairs the error support set is given by \mathcal{E} = \text{supp}(e). Both x and x' are in the admissible set of vectors satisfying the constraint in (P0, \mathcal{E}) and (BP, \mathcal{E}) in (12) and (13), respectively. In particular, one can verify that

\[
(F_M x) \in \{z\} + \mathcal{R}((I_M)_{\mathcal{E}})
\]

and

\[
(F_M x') \in \{z\} + \mathcal{R}((I_M)_{\mathcal{E}})
\]

where z = F_M x + e = \frac{1}{2} \delta_{\sqrt{M}}/2. Thus, we identified two pairs (x, e) and (x', e), for which the constraints in (12) and (13) are satisfied. Furthermore, x and x' \neq x satisfy \|x\|_0 = \|x\|_0 and \|x'\|_1 = \|x\|_1. Therefore, (P0, \mathcal{E}) and (BP, \mathcal{E}) cannot distinguish between x and x' and hence cannot uniquely recover x.

- **Case III: Cardinality of \mathcal{E} known.** Consider the vectors x = \delta_{2\sqrt{M}} and e = -\delta_{2\sqrt{M}}. Since n_x = n_e = \sqrt{M}/2, we have 4n_xn_e = M = f(2n_x, 2n_e). The pair (x, e) therefore saturates condition (20). Now, consider the alternative pair of vectors x' = \delta_{2\sqrt{M}} - \delta_{\sqrt{M}} and e' = -\delta_{2\sqrt{M}} + \delta_{\sqrt{M}}. Note that for both pairs the error support-set cardinality is given by n_e = \sqrt{M}/2. Both x and x' are in the admissible set in (P0, n_e) in (12), i.e., one can verify that

\[
(F_M x) \in \{z\} + \bigcup_{\mathcal{E}' \in \mathcal{P}} \mathcal{R}((I_M)_{\mathcal{E}'})
\]

and

\[
(F_M x') \in \{z\} + \bigcup_{\mathcal{E}' \in \mathcal{P}} \mathcal{R}((I_M)_{\mathcal{E}'})
\]

where z = F_M x + e = \frac{1}{2} \delta_{\sqrt{M}}/2 - \delta_{2\sqrt{M}} and \mathcal{P} = \emptyset_{n_e}(\{1, \ldots, M\}). Thus, we identified two pairs (x, e) and (x', e'), for which the constraint in (19) is satisfied. Furthermore, x and x' \neq x satisfy \|x'\|_0 = \|x\|_0. Therefore, (P0, n_e) cannot uniquely recover x.

- **Case IV: No knowledge of the support sets \mathcal{X} and \mathcal{E}.** We recall that recovery thresholds for this case were already reported in [27] for general A and B. For the particular case A = F_M
and $B = I_M$, the condition guaranteeing uniqueness of the (P0)-solution derived in [17, Thm. I.1] is given by $n_x + n_e < \sqrt{M}$. Tightness of this threshold was demonstrated in [17]. Consider $x = \delta_{\sqrt{M}}$ with $n_x = \sqrt{M}$ and $e = 0_M$ with $n_e = 0$. This pair $(x, e)$ saturates the recovery condition stated above as $n_x + n_e = \sqrt{M}$. Now, consider the alternative pair of vectors $x' = 0_M$ and $e' = \delta_{\sqrt{M}}$ and note that $(x, e)$ and $(x', e')$ yield the same measurement outcome $z = F_M x + e = F_M x' + e' = \delta_{\sqrt{M}}$. Furthermore, we have $\|x\|_0 + \|e\|_0 = \|x'\|_0 + \|e'\|_0 = \sqrt{M}$. Based on the measurement outcome $z$, (P0) can therefore not uniquely recover $(x, e)$.

For the case of $A$ and $B$ arbitrary ONBs, the threshold guaranteeing that the unique (P0)-solution $(x, e)$ be recovered through BP is given by $n_x + n_e < (\sqrt{2} - 0.5)/\mu_m$ [18, Thm. 3].

In [34, Prop. 4] an explicit design of ONBs was provided for which this threshold is tight.

We finally note that establishing tightness of the recovery thresholds (11), (14), (18), and (20) for general dictionaries $A$ and $B$ seems difficult and is closely related to establishing tightness of the underlying uncertainty relation (7).

B. Factor of Two in the Recovery Thresholds

Comparing the recovery thresholds (11), (14), (18), and (20) (Cases I-III), we observe that the price to be paid for not knowing the support set $\mathcal{X}$ or $\mathcal{E}$ is a reduction of the recovery threshold by a factor of two (note that in Case III, both $\mathcal{X}$ and $\mathcal{E}$ are unknown, however, the cardinality of either $\mathcal{X}$ or $\mathcal{E}$ is known). For example, consider the recovery thresholds (11) and (14). For given $n_e \in [0, 1 + 1/\mu_b]$, solving (11) for $n_x$ yields

$$n_x < \frac{(1 + \mu_a)(1 - \mu_b(n_e - 1))}{n_e(\mu_m^2 - \mu_a \mu_b) + \mu_a(1 + \mu_b)}.$$

Similarly, solving (14) for $n_x$, we get

$$n_x < \frac{1}{2} \frac{(1 + \mu_a)(1 - \mu_b(n_e - 1))}{n_e(\mu_m^2 - \mu_a \mu_b) + \mu_a(1 + \mu_b)}.$$

Hence, knowledge of $\mathcal{X}$ prior to recovery allows for twice as many nonzero entries in $x$ compared to the case where $\mathcal{X}$ is not known. This factor-of-two penalty has the same roots as the well-known factor-of-two penalty in spectrum-blind sampling [35]–[37].

\[The same result also follows as a special case of [27, Eq. 10].]
We illustrate the factor-of-two penalty in Figs. 1 and 2, where the recovery thresholds (11), (14), (18), (20), and (for completeness) [27, Eq. 13] (which guarantees that $x$ is recovered through BP and OMP in Case IV without knowledge of $\mathcal{X}$ or $\mathcal{E}$) are shown. In Fig. 1, we consider the case $\mu_a = \mu_b = 0$ and $\mu_m = 1/\sqrt{64}$. We can see that for $\mathcal{X}$ and $\mathcal{E}$ known the threshold evaluates to $n_x n_e < 64$, when only $\mathcal{X}$ or $\mathcal{E}$ is known we have $n_x n_e < 32$, and finally in the case where only $n_e$ is known we get $n_x n_e < 16$. Note furthermore that in Case IV, where no knowledge of the support sets is available, the recovery threshold is worse than in the case where $n_e$ is known.
C. The Square-Root Bottleneck

The recovery thresholds presented in Section IV hold for all signal and noise realizations $x$ and $e$ and for all dictionary pairs (with given coherence parameters). However, as is well-known in the sparse signal recovery literature, deterministic recovery guarantees are fundamentally limited by the so-called square-root bottleneck [38]. More specifically, in the noiseless case (i.e., for $e = 0_{N_b}$), the threshold (3) states that recovery can be guaranteed only for up to $\sqrt{M}$ nonzero entries in $x$. Put differently, for a fixed number of nonzero entries $n_x$ in $x$, i.e., for a fixed sparsity level, the number of measurements $M$ required to recover $x$ through (P0), BP, or OMP is on the order of $n_x^2$. In addition, in our setting, we have a tradeoff between the sparsity levels of $x$ and $e$. This tradeoff is illustrated through the following two examples.

Fig. 2. Recovery thresholds (11), (14), (18), (20), and [27, Eq. 13] for $\mu_a = 0.1258$, $\mu_b = 0.1319$, and $\mu_m = 0.1321$. 

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Example 1: For $M = p^k$, with $p$ prime and $k \in \mathbb{N}^+$, we can construct dictionaries $A$ and $B$ consisting of the concatenation of $A$ and $B$ ONBs (where $A + B \leq M + 1$), respectively, such that $\mu_a = \mu_b = \mu_m = 1/\sqrt{M}$ [10], [39]. For the resulting dictionary pair, we then evaluate the threshold (14) relevant for the case of $\mathcal{X}$ unknown and $\mathcal{E}$ known. Assuming that $n_e = \alpha\sqrt{M}$ with $\alpha \in [0, 1]$ and taking $M \gg 1$, we obtain a sparsity threshold on the number of allowed nonzero entries in $\mathbf{x}$ of roughly $(1 - \alpha)\sqrt{M}/2$ (ignoring terms of order lower than $\sqrt{M}$). This shows that there is a tradeoff between the sparsity levels of $\mathbf{x}$ and $\mathbf{e}$ (through the parameter $\alpha$) and that both sparsity levels scale as $\sqrt{M}$.

Example 2: Let $A = F_M$ and $B = I_M$ (therefore, $\mu_a = \mu_b = 0$ and $\mu_m = 1/\sqrt{M}$) and consider condition (14) guaranteeing perfect recovery when $\mathcal{X}$ is unknown and $\mathcal{E}$ is known. Assume that $n_e$ is linear in $M$ rather than proportional to $\sqrt{M}$ as in Example 1; we take $n_e = \alpha M$ with $\alpha \in (0, 1]$. Evaluating (14) then yields $n_x < 1/(2\alpha)$, which means that allowing the number of errors to grow proportional to $M$ results in a sparsity threshold on $\mathbf{x}$ that is constant and does not scale in $M$ at all.

As in the classical sparse-signal recovery literature, the square-root bottleneck can be broken by performing a probabilistic analysis (see, e.g., [27], [38], [40]). This line of work is, albeit interesting, outside the scope of this paper and is left for future work.

VI. NUMERICAL RESULTS

In this section, we first report simulation results and compare them to the corresponding analytical results in the paper. We will find that even though our thresholds are pessimistic in general, they do reflect the recovery behavior correctly. In particular, we will see that the factor-of-two penalty can also be observed in the numerical results. We then demonstrate, through a simple inpainting example, that perfect signal recovery in the presence of sparse errors is possible even if the corruptions are significant.

A. Impact of Support-Set Knowledge on Recovery Thresholds

We first compare simulation results to the recovery thresholds (11), (14), (18), and (for completeness) [27, Eq. 13].

Throughout this subsection, for a given pair of dictionaries $A$ and $B$ we generate signal vectors $\mathbf{x}$ and error vectors $\mathbf{e}$ as follows: $n_x$ and $n_e$ are given and the support sets of the $n_x$-sparse vector
and the $n_e$-sparse vector $e$ are chosen uniformly at random among all possible support sets of cardinality $n_x$ and $n_e$, respectively. Once the support sets have been chosen, we generate the nonzero entries of $x$ and $e$ by drawing from i.i.d. zero mean, unit variance Gaussian random variables. For each pair of support-set cardinalities $n_x$ and $n_e$, we perform 1,000 Monte-Carlo trials and declare success of recovery whenever the estimate $\hat{x}$ satisfies

$$
\| \hat{x} - x \|_2 < 10^{-3} \| x \|_2.
$$

(23)

We plot the 50% success-rate contour, i.e., the border between the region of pairs $(n_x, n_e)$ for which (23) is satisfied in at least 50% of the trials and the region where (23) is satisfied in less than 50% of the trials. The estimate $\hat{x}$ is obtained as follows:

- **Case I:** When $\mathcal{X}$ and $\mathcal{E}$ are both known, we perform recovery according to (10).
- **Case II:** When either only $\mathcal{E}$ or only $\mathcal{X}$ is known, we apply BP and OMP using the modified dictionary as detailed in Theorem 5 and Corollary 6, respectively.
- **Case IV:** When neither $\mathcal{X}$ nor $\mathcal{E}$ is known, we apply BP and OMP using the concatenated dictionary in (21) as described in [27].

Note that for **Case III**, i.e., the case where the cardinality $n_e$ of $\mathcal{E}$ is known only, as pointed out in Section IV-C, we do not have an efficient recovery algorithm (making use of the knowledge of $n_e$). This case is hence not considered here.

1) **Recovery Performance for the Hadamard-Identity Pair using BP and OMP:** We take $M = 64$, let $A$ be the Hadamard ONB [41] and set $B = I_M$, which results in $\mu_a = \mu_b = 0$ and $\mu_m = 1/\sqrt{M}$. Fig. 3 shows the 50% success-rate contour for four different scenarios of support-set knowledge. For perfect knowledge of $\mathcal{X}$ and $\mathcal{E}$, we observe that the 50% success-rate contour is at about $n_x + n_e \approx M$, which is significantly better than the sufficient condition $n_x n_e < M$ (guaranteeing perfect recovery) provided in (11).\(^7\) When either only $\mathcal{X}$ or only $\mathcal{E}$ is known, the recovery performance is essentially independent of whether $\mathcal{X}$ or $\mathcal{E}$ is known. This is also reflected by the analytical thresholds (14) and (18) when evaluated for $\mu_a = \mu_b = 0$ (see also Fig. 1). Furthermore, OMP is seen to outperform BP. When neither $\mathcal{X}$ nor $\mathcal{E}$ is known, OMP again outperforms BP.

\(^7\)For $A = F_M$ and $B = I_M$ it was proven in [42] that a set of randomly chosen columns from both $A$ and $B$ is linearly independent (with high probability) given that the total number of chosen columns, i.e., $n_x + n_e$ here, does not exceed a constant proportion of $M$. 

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As discussed in Section V-B the factor-of-two penalty is observed among the Cases I-III. It is interesting to see that the factor-of-two behavior discussed in Section V-B is reflected in Fig. 3 for $n_x = n_e$ between the Cases I and II. In particular, we can observe that for full support-set knowledge (Case I) the 50% success-rate is achieved at $n_x = n_e \approx 31$. If either $\mathcal{X}$ or $\mathcal{E}$ only is known (Case II), OMP achieves 50% success-rate at $n_x = n_e \approx 23$, demonstrating a factor-of-two penalty according to $31 \cdot 31 \approx 23 \cdot 23 \cdot 2$. Note that for BP the factor-of-two behavior cannot be inferred from our simulations. For lack of an efficient recovery algorithm (making use of the knowledge of $n_e$) we do not show numerical results for Case III.

2) Impact of $\mu_a, \mu_b > 0$: We take $M = 64$ and generate the dictionaries $A$ and $B$ as follows. Using the alternating projection method described in [43], we generate an approximate...
equiangular tight frame (ETF) for $\mathbb{R}^M$ consisting of 160 columns. We split this frame into two sets of 80 elements (columns) each and organize them in the matrices $A$ and $B$ such that the corresponding coherence parameters are given by $\mu_a \approx 0.1258$, $\mu_b \approx 0.1319$, and $\mu_m \approx 0.1321$. Fig. 4 shows the 50% success-rate contour for four different scenarios of support-set knowledge. In the case where either only $X$ or only $E$ is known and in the case where $X$ and $E$ are unknown, we use OMP for recovery. It is interesting to observe a performance asymmetry between the case where only $X$ is known and the case where only $E$ is known. This difference is due to $\mu_a, \mu_b > 0$ and is also reflected by the analytical thresholds (14) and (18) (see also Fig. 2).

We finally note that in all cases considered above, the numerical results show that recovery
is possible for significantly higher sparsity-levels $n_x$ and $n_e$ than indicated by the corresponding analytic thresholds (11), (14), (18), and [27, Eq. 13] (see also Figs. 1 and 2). The underlying reasons are i) the deterministic nature of our results, i.e., the recovery guarantees in (11), (14), (18), and [27, Eq. 13] are valid for all dictionary pairs (with given coherence parameters) and all signal and noise realizations (with given sparsity level), and ii) we plot the 50% success-rate contour, whereas our analytical results guarantee perfect recovery in 100% of the cases.

B. Inpainting Example

In transform coding, one is typically interested in maximally sparsifying the signals to be encoded [44]. In our setting, this would mean that the dictionary $A$ should be chosen such that it leads to maximally sparse representations of signals from within the considered family of signals. We next demonstrate, however, that in the presence of structured noise, the signal dictionary $A$ should additionally be incoherent to the noise dictionary $B$. This extra requirement can lead to very different criteria for designing transform bases (frames).

To illustrate this point, and to show that perfect recovery can be guaranteed even when the $\ell_2$-norm of the noise term $Be$ is arbitrarily large, we consider the recovery of a sparsely corrupted $512 \times 512$-pixel grayscale image of the main building of ETH Zurich. The dictionary $A$ is taken to be either the two-dimensional discrete cosine transform (DCT) or the Haar wavelet decomposed on three octaves [45]. We first “sparsify” the image by retaining only 15% of the largest entries of the image’s representation $x$ in $A$. We then corrupt (by overwriting with text) 18.8% of the pixels in the sparsified image by setting them to the brightest grayscale value, i.e., the errors are sparse in $B = I_M$. This means that the noise is structured but may take on large $\ell_2$-norm. Image recovery is performed according to (10) if $X$ and $E$ are known (note that knowledge of $X$ is usually not available in inpainting applications). We use BP when only $E$ is known and when neither $X$ nor $E$ are known. The recovery results are evaluated by computing the mean-squared error (MSE) between the sparsified image and its recovered version. Figs. 5 and 6 show the corresponding results. As expected, the MSE increases with decreasing knowledge on the support set. More interestingly, we note that even though Haar wavelets often yield a smaller approximation error when sparsely representing images compared to the DCT (for the same sparsity level), in the present case the wavelet decomposition performs worse than the DCT. This is due to the fact that in the presence of structured errors sparsification alone is not
the only factor determining the performance of a transform coding basis (frame). Rather the
mutual coherence between the dictionary used to represent the signal and that used to represent
the structured noise becomes highly relevant. Specifically, in the example at hand, we have
\(\mu_m = 0.5\) for the Haar-wavelet and the identity basis, and \(\mu_m \approx 0.004\) for the DCT and the
identity basis. Together with the analytical thresholds (11), (14), (18), and (20) depending on
the mutual coherence according to \(1/\mu_m^2\), this explains the performance difference between the
Haar wavelet decomposition and the DCT basis. We therefore conclude that the choice of the
transform basis for a sparsely corrupted signal should not only aim at sparsifying the signal as
much as possible but should also take into account the mutual coherence between the transform
basis (frame) and the noise sparsity basis (frame).

VII. Conclusion

The setup considered in this paper, in its generality, appears to be new and a number of
interesting extensions are possible, such as deriving recovery guarantees for the more general
setting of compressible signals or a probabilistic analysis leading to sparsity thresholds that
guarantee recovery with high probability. Note that probabilistic recovery guarantees for the
case where nothing is known about the signal and noise support sets readily follow from the
results in [27]. Furthermore, a restricted isometry property (RIP)-based formulation of the results
in this paper would be of interest.

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Appendix A

Proof of Theorem 1

Note that for the special case \(\epsilon_P = \epsilon_Q = 0\), Theorem 1 was already proven in [27, Lem. 1].
Our proof for the general case \(\epsilon_P, \epsilon_Q \neq 0\) provided here, follows the proof of [27, Lem. 1] in its
main arguments. Assume that \(s \in \mathbb{C}^M\) can be represented as a linear combination of columns
of \(A\) and, equivalently, as a linear combination of columns of \(B\), i.e., we have

\[ s = Ap = Bq \quad (24) \]
with \( p \in \mathbb{C}^{N_a} \) and \( q \in \mathbb{C}^{N_b} \). We exclude the trivial case \( p = 0_{N_a} \) and \( q = 0_{N_b} \). As will be shown, the uncertainty relation continues to hold if either \( p = 0_{N_a} \) or \( q = 0_{N_b} \).

Left-multiplication in (24) by \( A^H \) yields

\[
A^H Ap = A^H Bq. \tag{25}
\]

Next, we lower-bound the absolute value of the \( r \)th entry \((r = 1, \ldots, N_a)\) of the left-hand side (LHS) of (25) according to

\[
\left| \left[ A^H Ap \right]_r \right| = \left| \left[ p \right]_r + \sum_{\ell \neq r} \left[ A^H A \right]_{r,\ell} \left[ p \right]_\ell \right| \\
\geq \left| \left[ p \right]_r \right| - \mu_a \sum_{\ell \neq r} \left| \left[ p \right]_\ell \right| \\
= (1 + \mu_a) \left| \left[ p \right]_r \right| - \mu_a \left\| p \right\|_1 \tag{26}
\]

where the inequality follows by applying the reverse triangle inequality and using the fact that the absolute values of the off-diagonal entries of \( A^H A \) are upper-bounded by the coherence \( \mu_a \) of \( A \). Next, we provide an upper bound for the absolute value on the \( r \)th entry of the RHS of (25) according to:

\[
\left| \left[ A^H Bq \right]_r \right| = \left| A^H b_r q \right| \\
= \left| \sum_{\ell} a^H_r b_\ell q_\ell \right| \\
\leq \sum_{\ell} \mu_m \left| q_\ell \right| \tag{27}
\]

where the inequality follows from the triangle inequality and the fact that the absolute value of \( a^H_r b_\ell \) is upper-bounded by \( \mu_m \) for all \( r \) and \( \ell \). Combining the lower and upper bounds in (26) and (28) yields

\[
(1 + \mu_a) \left| \left[ p \right]_r \right| - \mu_a \left\| p \right\|_1 \leq \mu_m \left\| q \right\|_1. \tag{29}
\]

Summing both sides of (29) over all \( |P| \) indices \( r \) in \( P \) and using \( \left\| P_{\mathcal{P}} p \right\|_1 \geq (1 - \epsilon_{\mathcal{P}}) \left\| p \right\|_1 \), we obtain

\[
(1 + \mu_a)(1 - \epsilon_{\mathcal{P}}) \left\| p \right\|_1 - \left| P \right| \mu_a \left\| p \right\|_1 \leq \left| P \right| \mu_m \left\| q \right\|_1. \tag{30}
\]
Since $|\mathcal{P}| \mu_m \|q\|_1 \geq 0$, we can replace the LHS of (30) by the tighter bound
\[
[(1 + \mu_a)(1 - \epsilon_P) - |\mathcal{P}| \mu_a^+] \|p\|_1 \leq |\mathcal{P}| \mu_m \|q\|_1.
\] (31)

Left-multiplying both sides of (24) by $B^H$, and performing steps similar to those used to arrive at (31) yields
\[
[(1 + \mu_b)(1 - \epsilon_Q) - |\mathcal{Q}| \mu_b^+] \|q\|_1 \leq |\mathcal{Q}| \mu_m \|p\|_1.
\] (32)

If $\|p\|_1 > 0$ and $\|q\|_1 > 0$, we can multiply both sides of the inequalities in (31) and (32) and divide the result by $\|p\|_1 \|q\|_1$ to arrive at the uncertainty relation
\[
|\mathcal{P}| \|Q| \geq \frac{[(1 + \mu_a)(1 - \epsilon_P) - |\mathcal{P}| \mu_a^+] [(1 + \mu_b)(1 - \epsilon_Q) - |\mathcal{Q}| \mu_b^+]}{\mu_m^2}.
\] (33)

If $\|p\|_1 = 0$ and $\|q\|_1 > 0$, we obtain from (32) that
\[
|Q| \geq (1 + \mu_b^{-1})(1 - \epsilon_Q).
\] (34)

Similarly, if $\|q\|_1 = 0$ and $\|p\|_1 > 0$, we obtain from (31) that
\[
|\mathcal{P}| \geq (1 + \mu_a^{-1})(1 - \epsilon_P).
\] (35)

Both (34) and (35) are contained in (33) as special cases. This can be seen as follows. Consider the case where $\|p\|_1 = 0$ and $\|q\|_1 > 0$. Here, we have $p = 0_{N_a}$ and, hence, the vector $p$ is $\epsilon_P$-concentrated to the empty set $\mathcal{P} = \emptyset$ for every concentration parameter $\epsilon_P$. Evaluating (33) for $|\mathcal{P}| = 0$ and $\epsilon_P = 0$ yields the condition
\[
[(1 + \mu_b)(1 - \epsilon_Q) - |\mathcal{Q}| \mu_b^+] \leq 0
\]
which is equivalent to (34). A similar line of reasoning applies to the case $\|q\|_1 = 0$ and $\|p\|_1 > 0$.

**Appendix B**

**Proof of Theorem 3**

We prove the full column-rank property of $D_{\mathcal{X},\mathcal{E}}$ by showing that under (11) there is a unique pair $(x, e)$ with $\text{supp}(x) = \mathcal{X}$ and $\text{supp}(e) = \mathcal{E}$ satisfying $z = Ax + Be$. Assume that there exists an alternative pair $(x', e')$ such that $z = Ax' + Be'$ with $\text{supp}(x') \subseteq \mathcal{X}$ and $\text{supp}(e') \subseteq \mathcal{E}$.
(i.e., the support sets of $x'$ and $e'$ are contained in $\mathcal{X}$ and $\mathcal{E}$, respectively). This would then imply that

$$Ax + Be = Ax' + Be'$$

and thus

$$A(x - x') = B(e' - e).$$

Since both $x$ and $x'$ are supported in $\mathcal{X}$ it follows that $x - x'$ is also supported in $\mathcal{X}$, which implies $\|x - x'\|_0 \leq n_x$. Similarly, we get $\|e' - e\|_0 \leq n_e$. Defining $p = x - x'$ and $P = \text{supp}(x - x') \subseteq \mathcal{X}$, and, similarly, $q = e' - e$ and $Q = \text{supp}(e' - e) \subseteq \mathcal{E}$, we obtain the following chain of inequalities:

$$n_x n_e \geq \|p\|_0 \|q\|_0 = |P| |Q|$$

$$\geq \frac{[1 - \mu_a(|P| - 1)]^+ [1 - \mu_b(|Q| - 1)]^+}{\mu_m^2}$$

$$\geq \frac{[1 - \mu_a(n_x - 1)]^+ [1 - \mu_b(n_e - 1)]^+}{\mu_m^2} = f(n_x, n_e) \quad (36)$$

where (36) follows by applying the uncertainty relation in Theorem 1 (with $\epsilon_P = \epsilon_Q = 0$ since both $p$ and $q$ are perfectly concentrated to $P$ and $Q$, respectively) and (37) is a consequence of $|P| \leq n_x$ and $|Q| \leq n_e$. Obviously, (37) contradicts the assumption in (11), which completes the proof.

**APPENDIX C**

**PROOF OF THEOREM 4**

We begin by proving that $x$ is the unique solution of $(P0, \mathcal{E})$ applied to $z = Ax + Be$. Assume that there exists an alternative vector $x'$ that satisfies $Ax' \in (\{z\} + \mathcal{R}(B_e))$ with $\|x'\|_0 \leq n_x$. This would imply the existence of a vector $e'$ with $\text{supp}(e') \subseteq \mathcal{E}$, such that

$$Ax + Be = Ax' + Be'$$

and hence

$$A(x - x') = B(e' - e).$$
Since \( \text{supp}(e) = \mathcal{E} \) and \( \text{supp}(e') \subseteq \mathcal{E} \), we have \( \text{supp}(e' - e) \subseteq \mathcal{E} \) and hence \( \|e' - e\|_0 \leq n_e \). Furthermore, since both \( x \) and \( x' \) have at most \( n_x \) nonzero entries (at possibly different positions), we have \( \|x - x'\|_0 \leq 2n_x \). Defining \( p = x - x' \) and \( \mathcal{P} = \text{supp}(x - x') \), and, similarly, \( q = e' - e \) and \( \mathcal{Q} = \text{supp}(e' - e) \subseteq \mathcal{E} \), we obtain the following chain of inequalities

\[
2n_x n_e \geq \|p\|_0 \|q\|_0 = |\mathcal{P}| \cdot |\mathcal{Q}|
\]

\[
\geq \frac{[1 - \mu_a(|\mathcal{P}| - 1)]^+ [1 - \mu_b(|\mathcal{Q}| - 1)]^+}{\mu_m^2} \geq \frac{[1 - \mu_a(2n_x - 1)]^+ [1 - \mu_b(n_e - 1)]^+}{\mu_m^2} = f(2n_x, n_e)
\]

(38)

where (38) follows by applying the uncertainty relation in Theorem 1 (with \( \epsilon_P = \epsilon_Q = 0 \) since both \( p \) and \( q \) are perfectly concentrated to \( \mathcal{P} \) and \( \mathcal{Q} \), respectively) and (39) is a consequence of \( |\mathcal{P}| \leq 2n_x \) and \( |\mathcal{Q}| \leq n_e \). Obviously, (39) contradicts the assumption in (14), which concludes the first part of the proof.

We next prove that \( x \) is also the unique solution of \( (BP, \mathcal{E}) \) applied to \( z = Ax + Be \). Assume that there exists an alternative vector \( x' \) that satisfies \( Ax' \in (\{z\} + \mathcal{R}(B\mathcal{E})) \) with \( \|x'\|_1 \leq \|x\|_1 \). This would imply the existence of a vector \( e' \) with \( \text{supp}(e') \subseteq \mathcal{E} \), such that

\[
Ax + Be = Ax' + Be'
\]

and hence

\[
A(x - x') = B(e' - e).
\]

Defining \( p = x - x' \), we obtain the following lower bound for the \( \ell_1 \)-norm of \( x' \)

\[
\|x'\|_1 = \|x - p\|_1 = \|P_{\mathcal{X}}(x - p)\|_1 + \|P_{\mathcal{X}'}p\|_1
\]

\[
\geq \|P_{\mathcal{X}}x\|_1 - \|P_{\mathcal{X}}p\|_1 + \|P_{\mathcal{X}'}p\|_1
\]

\[
= \|x\|_1 - \|P_{\mathcal{X}}p\|_1 + \|P_{\mathcal{X}'}p\|_1
\]

(40)

where (40) is a consequence of the reverse triangle inequality. Now, the \( \ell_1 \)-norm of \( x' \) can be smaller than or equal to that of \( x \) only if \( \|P_{\mathcal{X}}p\|_1 \geq \|P_{\mathcal{X}'}p\|_1 \). This would then imply that the difference vector \( p \) needs to be at least 50%-concentrated to the set \( \mathcal{P} = \mathcal{X} \) (of cardinality \( n_x \)), i.e., we require that \( \epsilon_P \leq 0.5 \). Defining \( q = e' - e \) and \( \mathcal{Q} = \text{supp}(e' - e) \), and noting that
supp(e) = ℋ and supp(e′) ⊆ ℋ, it follows that |Q| ≤ ne. This leads to the following chain of inequalities:

\[ n_x n_e \geq |P| |Q| \]
\[ \geq \frac{[(1 + \mu_a)(1 - \epsilon_P) - |P| \mu_a] + [1 - \mu_b (|Q| - 1)]}{\mu^2_m} \]
\[ \geq \frac{1}{2} \frac{[1 - \mu_a (2n_x - 1)]^+ [1 - \mu_b (n_e - 1)]^+}{\mu^2_m} \]  (41)
\[ \geq \frac{1}{2} \frac{[1 - \mu_a (2n_x - 1)]^+ [1 - \mu_b (n_e - 1)]^+}{\mu^2_m} \]  (42)

where (41) follows from the uncertainty relation in Theorem 1 applied to the difference vectors p and q (with \( \epsilon_P \leq 0.5 \) since p is at least 50%-concentrated to P and \( \epsilon_Q = 0 \) since q is perfectly concentrated to Q) and (42) is a consequence of \( |P| = n_x \) and \( |Q| \leq n_e \). Rewriting (42), we obtain

\[ 2n_x n_e \geq \frac{[1 - \mu_a (2n_x - 1)]^+ [1 - \mu_b (n_e - 1)]^+}{\mu^2_m} = f(2n_x, n_e). \]  (43)

Since (43) contradicts the assumption in (14), this proves that x is the unique solution of (BP, ℋ) applied to \( z = Ax + Be \).

APPENDIX D

PROOF OF THEOREM 5

We first show that condition (14) ensures that the columns of \( B_\mathcal{H} \) are linearly independent. Then, we establish that \( \| R_\mathcal{H} a_\ell \|_2 > 0 \) for \( \ell = 1, \ldots, N_a \). Finally, we show that the unique solution of (P0), BP, and OMP applied to \( \hat{z} = R_\mathcal{H} A \Delta \hat{x} \) is given by \( \hat{x} = \Delta^{-1} x \).

A. The columns of \( B_\mathcal{H} \) are linearly independent

Condition (14) can only be satisfied if \( [1 - \mu_b (n_e - 1)]^+ > 0 \), which implies that \( n_e < 1 + 1/\mu_b \). It was shown in [9]–[11] that for a dictionary B with coherence \( \mu_b \) no fewer than \( 1 + 1/\mu_b \) columns of B can be linearly dependent. Hence, the \( n_e \) columns of \( B_\mathcal{H} \) must be linearly independent.
B. $\|R_e a_\ell\|_2 > 0$ for $\ell = 1, \ldots, N_a$

We have to verify that condition (14) implies $\|R_e a_\ell\|_2 > 0$ for $\ell = 1, \ldots, N_a$. Since $R_e$ is a projector and, therefore, Hermitian and idempotent, it follows that

$$\|R_e a_\ell\|_2^2 = a_\ell^H R_e a_\ell$$

$$= |a_\ell^H R_e a_\ell|$$

$$\geq 1 - \left| a_\ell^H B_e (B_e^H B_e)^{-1} B_e^H a_\ell \right|$$

$$\geq C_1$$

(44)

where (44) is a consequence of $R_e^H R_e = R_e$, and (45) follows from the reverse triangle inequality and $\|a_\ell\|_2 = 1$, $\ell = 1, \ldots, N_a$. Next, we derive an upper bound on $C_1$ according to

$$C_1 \leq \lambda_{\max}\left((B_e^H B_e)^{-1}\right) \|B_e^H a_\ell\|_2^2$$

$$\leq \lambda_{\min}^{-1}(B_e^H B_e) n_e \mu_m^2$$

(46)

(47)

where (46) follows from the Rayleigh-Ritz theorem [46, Thm. 4.2.2] and (47) results from

$$\|B_e^H a_\ell\|_2^2 = \sum_{i \in \mathcal{I}} |b_i^H a_\ell|^2 \leq n_e \mu_m^2.$$

Next, applying Geršgorin’s disc theorem [46, Theorem 6.1.1], we arrive at

$$\lambda_{\min}(B_e^H B_e) \geq [1 - \mu_b(n_e - 1)]^+. $$

(48)

Combining (45), (47), and (48) leads to the following lower bound on $\|R_e a_\ell\|_2^2$,

$$\|R_e a_\ell\|_2^2 \geq 1 - \frac{n_e \mu_m^2}{[1 - \mu_b(n_e - 1)]^+}.$$  

(49)

Note that if condition (14) holds for $n_x \geq 1$, it follows that $n_e \mu_m^2 < [1 - \mu_b(n_e - 1)]^+$ and hence the RHS of (49) is strictly positive. This ensures that $\Delta$ defines a one-to-one mapping. Furthermore, it is important to note that this means that every column of $A$ has a nonzero component that is orthogonal to $\mathcal{R}(B_e)$. We next show that, moreover, condition (14) ensures that for every vector $x' \in \mathbb{C}^{N_a}$ satisfying $\|x'\|_0 \leq 2n_x$, $Ax'$ has a nonzero component that is orthogonal to $\mathcal{R}(B_e)$.

$^8$The case $n_x = 0$ is not interesting, as $n_x = 0$ corresponds to $x = 0_{N_a}$ and hence recovery of $x = 0_{N_a}$ only could be guaranteed.
C. Unique recovery through (P0), BP, and OMP

We now need to verify that (P0), BP, and OMP (applied to $\hat{z} = R_\varepsilon A\Delta \hat{x}$) recover the vector $\hat{x} = \Delta^{-1}x$ provided that (14) is satisfied. This will be accomplished by deriving an upper bound on the coherence $\mu(R_\varepsilon A\Delta)$ of the modified dictionary $R_\varepsilon A\Delta$, which—via the well-known coherence-based recovery guarantee $n_x < (1 + 1/\mu(R_\varepsilon A\Delta))/2$—leads to a recovery threshold guaranteeing perfect recovery of $\hat{x}$. This threshold is then shown to be equivalent to (14). More specifically, the well-known sparsity threshold in (3) guarantees that the unique solution of (P0) applied to $\hat{z} = R_\varepsilon A\Delta \hat{x}$ is given by $\hat{x}$, and, furthermore, that this unique solution can be obtained through BP and OMP if $[9]–[11]$

$$n_x < \frac{1}{2} \left(1 + \frac{1}{\mu(R_\varepsilon A\Delta)}\right). \quad (50)$$

It is important to note that $\|\hat{x}\|_0 = \|x\|_0 = n_x$. With

$$[\Delta]_{\ell,\ell} = \frac{1}{\|R_\varepsilon a_\ell\|_2}, \quad \ell = 1, \ldots, N_a$$

we obtain

$$\mu(R_\varepsilon A\Delta) = \max_{r,\ell,\ell \neq r} \frac{|a_r^H R_\varepsilon^H R_\varepsilon a_\ell|}{\|R_\varepsilon a_r\|_2 \|R_\varepsilon a_\ell\|_2}. \quad (51)$$

Next, we upper-bound the RHS of (51) by upper-bounding the numerator and lower-bounding the denominator. For the numerator we have

$$|a_r^H R_\varepsilon^H R_\varepsilon a_\ell| = |a_r^H R_\varepsilon a_\ell| \quad (52)$$

$$\leq |a_r^H a_\ell| + |a_r^H B_\varepsilon B_\varepsilon^\dagger a_\ell| \quad (53)$$

$$\leq \mu_a + \left|a_r^H B_\varepsilon (B_\varepsilon^H B_\varepsilon)^{-1} B_\varepsilon^H a_\ell\right| \quad (54)$$

where (52) follows from $R_\varepsilon^H R_\varepsilon = R_\varepsilon$, (53) is obtained through the triangle inequality, and (54) follows from $|a_r^H a_\ell| \leq \mu_a$. Next, we derive an upper bound on $C_2$ according to

$$C_2 \leq \|B_\varepsilon^H a_r\|_2 \left\| (B_\varepsilon^H B_\varepsilon)^{-1} B_\varepsilon^H a_\ell\right\|_2 \quad (55)$$

$$\leq \|B_\varepsilon^H a_r\|_2 \left\| (B_\varepsilon^H B_\varepsilon)^{-1}\right\| \|B_\varepsilon^H a_\ell\|_2 \quad (56)$$
where (55) follows from the Cauchy-Schwarz inequality and (56) from the Rayleigh-Ritz theorem [46, Thm. 4.2.2]. Defining \( i = \arg \max_r \| B^H \varepsilon a_r \|_2 \), we further have

\[
C_2 \leq \left\| \left( B^H \varepsilon B \right)^{-1} \left\| B^H \varepsilon a_i \right\|_2 \right. \\
= \lambda_{\text{max}} \left( \left( B^H \varepsilon B \right)^{-1} \right) \left\| B^H \varepsilon a_i \right\|_2^2.
\]

We obtain an upper bound on \( C_2 \) using the same steps that were used to bound \( C_1 \) in (46) – (48):

\[
C_2 \leq \frac{n_e \mu^2_m}{C_b},
\]

where \( C_b = [1 - \mu_b(n_e - 1)]^+ \). Combining (54) and (57) leads to the following upper bound

\[
\| a^H \varepsilon^H R^H \varepsilon a_r \| \leq \mu_a + \frac{n_e \mu^2_m}{C_b}.
\]

Next, we derive a lower bound on the denominator on the RHS of (51). To this end, we set \( j = \arg \min_r \| R^H \varepsilon a_r \|_2 \) and note that

\[
\| R^H \varepsilon a_r \|_2 \| R^H \varepsilon a_j \|_2 \geq \| R^H \varepsilon a_j \|_2^2 \\
\geq 1 - \frac{n_e \mu^2_m}{C_b}
\]

where (59) follows from (49). Finally, combining (58) and (59) we arrive at

\[
\mu(R^H \varepsilon A \Delta) \leq \frac{\mu_a C_b + n_e \mu^2_m}{C_b - n_e \mu^2_m}.
\]

Inserting (60) into the recovery threshold in (50), we obtain the following threshold guaranteeing recovery of \( \hat{x} \) from \( \hat{z} = R^H \varepsilon A \Delta \hat{x} \) through (P0), BP, and OMP:

\[
n_x < \frac{1}{2} \left( \frac{C_b(1 + \mu_a)}{\mu_a C_b + n_e \mu^2_m} \right).
\]

Since \( 2n_x n_e \mu^2_m \geq 0 \), we can transform (61) into

\[
2n_x n_e \mu^2_m < C_b[1 - \mu_a(2n_x - 1)]^+ \\
= [1 - \mu_b(n_e - 1)]^+ [1 - \mu_a(2n_x - 1)]^+.
\]

Rearranging terms in (62) finally yields

\[
2n_x n_e < f(2n_x, n_e)
\]

which proves that (14) guarantees recovery of the vector \( \hat{x} \) (and thus also of \( x = \Delta \hat{x} \)) through (P0), BP, and OMP.
APPENDIX E
PROOF OF THEOREM 7

Assume that there exists an alternative vector \( x' \) that satisfies \( Ax' \in (\{z\} + \bigcup_{E \in \mathcal{P}} \mathcal{R}(B_E)) \) (where \( \mathcal{P} = \mathcal{P}_{n_e}(\{1, \ldots, N_b\}) \)) with \( \|x'\|_0 \leq n_x \). This implies the existence of a vector \( e' \) with \( \|e'\|_0 \leq n_e \) such that

\[
Ax + Be = Ax' + Be'
\]

and therefore

\[
A(x - x') = B(e' - e).
\]

From \( \|x\|_0 = n_x \) and \( \|x'\|_0 \leq n_x \) it follows that \( \|x - x'\|_0 \leq 2n_x \). Similarly, \( \|e\|_0 = n_e \) and \( \|e'\|_0 \leq n_e \) imply \( \|e' - e\|_0 \leq 2n_e \). Defining \( p = x - x' \) and \( \mathcal{P} = \text{supp}(x - x') \), and, similarly, \( q = e' - e \) and \( \mathcal{Q} = \text{supp}(e' - e) \), we arrive at

\[
4n_xn_e \geq \|p\|_0 \|q\|_0 = |\mathcal{P}| |\mathcal{Q}|
\]

\[
\geq \frac{[1 - \mu_a(|\mathcal{P}| - 1)]^+ [1 - \mu_b(|\mathcal{Q}| - 1)]^+}{\mu_n^2}
\]

\[
\geq \frac{[1 - \mu_a(2n_x - 1)]^+ [1 - \mu_b(2n_e - 1)]^+}{\mu_n^2} = f(2n_x, 2n_e)
\]

(63)

(64)

where (63) follows from the uncertainty relation in Theorem 1 applied to the difference vectors \( p \) and \( q \) (with \( \epsilon_{\mathcal{P}} = \epsilon_{\mathcal{Q}} = 0 \) since both \( p \) and \( q \) are perfectly concentrated to \( \mathcal{P} \) and \( \mathcal{Q} \), respectively) and (64) is a consequence of \( |\mathcal{P}| \leq 2n_x \) and \( |\mathcal{Q}| \leq 2n_e \). Obviously, (64) is in contradiction to (20), which concludes the proof.

REFERENCES


Fig. 5. Recovery results using the DCT basis for the signal dictionary and the identity basis for the noise dictionary, for the cases where (b) $\mathcal{X}$ and $\mathcal{E}$ are known, (c) only $\mathcal{E}$ is known, and (d) no support-set knowledge is available. (Picture origin: ETH Zürich/Esther Ramseier).
Fig. 6. Recovery results for the signal dictionary given by the Haar wavelet decomposition and the noise dictionary given by the identity basis, for the cases where (b) both support sets $\mathcal{X}$ and $\mathcal{E}$ are known, (c) only $\mathcal{E}$ is known, and (d) no support-set knowledge is available. (Picture origin: ETH Zürich/Esther Ramseier).