The Value of Information in Asymmetric All-Pay Auctions

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Abstract This note analyzes a two-player all-pay auction with incomplete information. More precisely, one bidder is uncertain whether his rival has a head start, i.e., an initial advantage in the auction. The game has a unique Bayesian Nash equilibrium outcome whose shape depends on the expected head start. In equilibrium, the potentially stronger player generates an informational rent if and only if her potential head start occurs with high probability or is sufficiently high. This result differs from the case of uncertainty about cost or valuation of the opponent, in which the potentially stronger player always obtains an informational rent.

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1 Introduction

In many contests, one competitor has an initial advantage over her rivals. For instance, an incumbent firm possesses more knowledge about a new task when it is related to their previous work. In a contest for job promotion, a worker with a good reputation has an advantage over her competitors. For an excellent motivation and many other examples of contests in which a player has an initial advantage, see Siegel (2011). In recent literature, these initial advantages are often modeled as deterministic head starts in an all-pay contest (see, e.g., Konrad, 2002, 2004, Kirkegaard, 2011, or Siegel, 2011). In the above mentioned applications, however, a contestant might not be perfectly informed about a potential head start of a rival.

This note aims to provide a deliberately simple contest model, which captures asymmetric information about a potential head start. More precisely, I assume that one player knows whether she has a head start over her rival, while her competitor only knows that a head start occurs with probability $p$.

The main result shows that the competitor with a potential head start can only generate an informational rent if her head start is high enough and occurs with sufficiently high probability. Hence, to exploit her private information, a competitor needs to be sufficiently “strong”. In contrast to this result, I show that for uncertainty about valuations or costs, the “stronger” player always gets an informational rent.

The value of private information about a head start is highest for maximal head start height and moderately high head start probabilities. Intuitively, for a head start probability close to 0 or 1, the additional value of private information is low, since the prior probability of the weaker player contains little uncertainty.

2 The Model

I consider a model with two players $i = 1, 2$ who have a common valuation $v$ for an object; $v$ is henceforth normalized to 1. With probability $p \in (0, 1)$, player 1 enjoys a head start of $s$, where $0 < s < 1$. Player 1 observes whether
she has a head start, while player 2 only knows the probability $p$ that a head start occurs and the height $s$ of the potential head start.

Both players simultaneously submit bids $b_i \in \mathbb{R}_0^+$ and pay costs $c(b_i) = b_i$. A mixed strategy of player 1 assigns a probability distribution over bids for each of her types (head start or no head start). For player 2, a mixed strategy is simply a probability distribution over bids. Unless otherwise stated, the probability distributions in the remainder of the paper are probability distributions over total bids $b_1 + s\mathbf{1}_{\text{headstart}}$ and $b_2$ respectively that are induced by the player’s strategies.

Player 1 wins the object if $b_1 + s\mathbf{1}_{\text{headstart}} \geq b_2$; otherwise player 2 wins.¹ Both players maximize expected payoffs.

## 3 The Equilibrium

As a benchmark, I briefly review the two-player equilibrium in absence of uncertainty, which is discussed as an example in Siegel (2011), Section 3.2.:

**Lemma 1.** Assume player 1 has a head start of $s \in [0, 1)$ (with probability $p=1$). In the unique Nash equilibrium distribution, player 2 places an atom of size $s$ at 0 and player 1 places an atom of size $s$ at $s$. Both players randomize uniformly with density 1 on $(s, 1]$. The expected payoff of player 1 is $s$; the expected payoff of player 2 is 0.

I now turn to discuss how results differ if a potential head start is private information and occurs with probability $p \in (0, 1)$. For this purpose, I derive the unique Bayesian Nash equilibrium of the game depending on the parameter values $s$ and $p$.

**Proposition 1.** Assume $1 - p \geq s$. In the unique Bayesian Nash equilibrium outcome of the game, both players choose a strategy which leads to the uniform distribution on $[0, 1]$ over total bids. Player 1 gets an expected payoff of $s \times p$; the expected payoff of player 2 is 0.

¹The tie-breaking rule is irrelevant for any equilibrium distribution I derive.
Proof. I first show that any strategy which leads to the above distributions is an equilibrium strategy: given the distribution of the rival, the set of best responses for player 1 without a head start and of player 2 is \( b_i \in [0, 1] \), while the best responses for player 1 with a head start are \( b_1 \in [0, 1 - s] \).

An explicit strategy of player 1 which induces the uniform distribution is the following: If no head start occurs, randomize the bid uniformly on \([0, s]\) with density \( \frac{1}{1-p} \) and uniformly on \((s, 1]\) with density \( \frac{1}{1-p} \); if a head start occurs, randomize the bid uniformly on \([0, 1 - s]\) with density \( \frac{1}{1-s} \). The strategy of player 2 is a uniform randomization on \([0, 1]\) with density 1. 

The proof for uniqueness is relegated to the appendix.

Note that in the unique equilibrium outcome of the 2-player all-pay auction without a head start (valuation \( v = 1 \) and costs \( c(b_i) = b_i \)), both players also randomize uniformly on \([0, 1]\). Hence, for the above parameter settings, a random head start does not change the equilibrium distributions.

On the other hand, if the value of a potential head start is higher than \( 1 - p \), there exists no strategy for player 1 which induces the uniform distribution over total bids on \([0, 1]\). Hence, the equilibrium distribution has a different shape, which I characterize in the following proposition:

**Proposition 2.** Assume \( s > 1 - p \). In the unique Bayesian Nash equilibrium outcome, both players randomize uniformly with density 1 on \((0, 1 - p]\) and \((s, 1]\). Player 1 places a mass point of size \( s - (1 - p) \) at \( s \); player 2 places a mass point of size \( s - (1 - p) \) at 0. The expected payoff of player 1 is \( s - (1 - p)^2 \); the expected payoff of player 2 is 0.

Proof. Given the distribution of player 1, any bid in \([0, 1 - p]\) and \((s, 1]\) is a best response for player 2. Without a head start, any bid in \((0, 1 - p]\) is a best response for player 1. In equilibrium, she randomizes uniformly among them. In case of a head start, a best response of player 1 is any bid in \([0, 1 - s]\). To obtain the above distribution, she places a mass point of size \( \frac{s - (1 - p)}{p} \) at zero and randomizes uniformly on \((0, 1 - s]\) with density \( \frac{1}{p} \).

I prove uniqueness of the equilibrium distribution in the appendix.
Hence, if the probability of a head start is high, the shape of the equilibrium distribution is close to that in Lemma 1; in fact, it converges to the distribution in Lemma 1 for $p \to 1$.

Differing from the parameter setting in Proposition 1, the expected payoff of player 1 in Proposition 2 is higher than the expected value of the head start $s \ast p$. Intuitively, player 1 still gets a payoff of $s$ after a head start, but her (additional) payoff is now $s - (1 - p)$ without a head start.

Henceforth, I refer to the (ex-ante) difference in expected payoffs when information about the realization of a possible head start is private compared to the public information case (see Lemma 1) as the informational rent. The main theorem summarizes the previous results:

**Theorem 1.** The player who might have a head start generates an informational rent if and only if she is sufficiently “strong” ex-ante ($s > 1 - p$).

In the following, I derive comparative statics results about the informational rent depending on the parameters. Throughout this analysis, I assume $s > 1 - p$. For a fixed height of head start $s$, the value of the informational rent is non-monotone in $p$:

**Proposition 3.** For any fixed value of $s$, the informational rent is maximal for $p = 1 - \frac{s}{2}$.

**Proof.** Compared to the public information case, the expected payoff of player 1 in the private information setting is higher if and only if she does not have a head start. Hence, the informational rent $r$ equals $r = (1 - p)(s - (1 - p))$. FOC and SOC imply that $p = 1 - \frac{s}{2}$ is the unique maximizer of the informational rent. □

The intuition for an interior solution is simple. For a high value of $p$, the equilibrium is close to the equilibrium in Lemma 1, in which the informational rent is zero. For low values of $p$, the equilibrium is close to the all-pay auction equilibrium without noise (e.g., Hillman and Samet, 1987), in which the informational rent is also zero.

On the other hand, for a fixed probability $p$ of a head start, the informational rent is increasing in the height of the head start:
Proposition 4. For any fixed value of $p$, the informational rent is increasing in $s$.

Proof. From the previous proof, we know that the informational rent is $r = (1 - p)(s - (1 - p))$. The derivative with respect to $s$, $\frac{dr}{ds} = 1 - p$, is positive. Hence, the informational rent is increasing in $s$.

Intuitively, a higher head start leads player 2 to place more probability mass on 0. This increases the informational rent of player 1 if no head start occurs.

4 Uncertainty About Valuations

In this section, I contrast the main result of this note with the corresponding result for an all-pay auction in which player 1 might have a higher valuation than player 2. More precisely, player 2 has a valuation of $v$ (henceforth normalized to one), while player 1 has a valuation of 1 with probability $1 - p$ and a valuation of $\bar{v} > 1$ with probability $p$. The realization of the valuation of player 1 is her private information. As before, players compete in an all-pay auction with costs $c(b_i) = b_i$. No player has a head start. The following result characterizes the equilibrium for this modified setting:

Proposition 5. In the unique Bayesian Nash equilibrium, player 1 randomizes uniformly with density $\frac{1}{1 - p}$ on $(0, 1]$ and uniformly with density $\frac{1}{\bar{v}}$ on $(1 - p, 1]$. Player 2 places a mass point of size $p - \frac{p}{\bar{v}}$ at 0, randomizes uniformly with density $\frac{1}{1 - p}$ on $(0, 1 - p]$ and uniformly with density $\frac{1}{\bar{v}}$ on $(1 - p, 1]$. The expected payoff of player 1 is $p(\bar{v} - 1) + (1 - p)(p - \frac{p}{\bar{v}})$; the expected profit of player 2 is 0.

Proof. In equilibrium, player 1 randomizes uniformly on $(0, 1 - p]$ with density $\frac{1}{1 - p}$ if her valuation is 1 and uniformly on $(1 - p, 1]$ with density $\frac{1}{p}$ if her

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2See, e.g., Amann and Leininger (1996) and Moldovanu and Sela (2001) for general results on all-pay auctions with incomplete information in which valuations or costs are drawn from continuous distributions.

3To determine their best responses, players only care about the relationship between valuation-cost ratio and winning probability. Hence, a higher valuation of a player is strategically equivalent to a lower (marginal) cost.
valuation is \( v \) (any of these strategies is a best response). For player 2, any bid in \([0, 1]\) is a best response to the strategy of player 1. Hence, the above strategy profiles constitute a Bayesian Nash equilibrium. I prove uniqueness of the equilibrium distributions in the appendix.

Intuitively, player 2 uses a randomization which makes the low valuation type of player 1 indifferent for bids below \( 1 - p \) and the high valuation type indifferent for bids above \( 1 - p \). To make the high type indifferent, player 2 has to use a lower density on the higher values, which entails that he places a mass point at zero. Thereby, the low type of player 1 makes a positive expected profit. Contrary to the result for head starts, the potentially “stronger” player gets an informational rent however small her ex-ante advantage might be.\(^4\) The game-theoretic logic behind this result is that uncertainty about valuations or costs forces the uninformed player to randomize with different densities, while uncertainty about a head start does not.

5 Discussion

The setting has been kept deliberately simple to convey the main intuition. In particular, I have assumed throughout the analysis that the probability distribution over possible head starts is a two-point distribution. I now briefly discuss how the results extend if this assumption is relaxed.

The no informational rent result from Proposition 1 remains valid if the distribution over head starts is stochastically dominated by the uniform distribution on \([0, 1]\). Intuitively, for these parameter settings, there exists a strategy for player 1 which leads to the uniform distribution on \([0, 1]\). In this case, both players randomize uniformly on \([0, 1]\) in equilibrium and player 1 does not get an informational rent.

From an economic point of view, the model suggests that an incumbent needs to be perceived as sufficiently “strong” to generate an informational

\(^4\)In the public information case, the expected payoff of player 1 is \( p(v - 1) \); see, e.g., Hillman and Samet, 1987, for the equilibrium construction with different or equal valuations.
rent. In particular, a small potential lead does not even change the behavior of the uninformed player. In case of a potentially lower cost or higher valuation, however, the “stronger” player always induces her rival to bid less aggressively.

Hence, despite a seemingly small difference in modeling, the informational rent result is very sensitive to the source of private information. If the “stronger” player can influence her ex-ante perception, she would prefer to do so in terms of potentially lower marginal costs or higher valuations rather than claiming a higher potential head start.

6 Appendix

In this appendix, I prove uniqueness of the equilibrium distributions derived in the paper. The following auxiliary result, which holds true for all settings discussed in this paper, is now standard in static game theory with a continuous state space (see, e.g., Burdett and Judd, 1983). Hence, the proof is omitted.

Lemma 2. In any (Bayesian) Nash equilibrium, no player type places a mass point for any bid above 0.

The following proof for uniqueness proceeds in three steps: First, I characterize location and size of potential mass points. Then, I show that player 2 makes zero profits. The last step establishes uniqueness.

Proof for Uniqueness of the Equilibrium Distribution in Proposition 1 and 2: Step 1 (Mass Points): By Lemma 2, player 1 might only place a mass point at 0 or $s$, while player 2 might only place a mass point at 0. Since both players have to be indifferent on each interval on which they randomize, they have to randomize uniformly with density 1 on the same intervals. Thus, potential mass points of both players have the same size and can only be at $s$ for player 1 and at 0 for player 2.

Step 2 (Zero Profits for Player 2): Consider the following case distinction:
(i) If player 2 places a mass point at 0, which entails zero profits, he gets at most zero profits for any bid by optimality.

(ii) If player 2 does not place a mass point at 0, nobody places a mass point and both players randomize uniformly with density 1 on the same intervals by Step 1. Hence, the distributions have to be uniform on $[0,1]$ to ensure non-negative profits of player 2. In this case, player 2 also makes zero profits.

**Step 3 (Uniqueness):** Step 2 implies that the support of the equilibrium cdf $F$ of each player reaches 1. Note the support of $F$ on $[s,1]$ has to be connected and has to start at $s$: If no player bids with positive probability on an interval $I = [a,b] \subset [s,1]$, a player who bids $y = \inf\{x : F(x) > F(a)\}$ (or an $\epsilon$-bid above $y$) would strictly prefer to bid $a$ instead, which contradicts optimality.

Connectedness of the support and uniform randomization with density 1 uniquely fix the size of a potential mass point at $s$. Moreover, as the support also has to be connected on $[0,s]$ starting at 0, the distributions are uniquely fixed as described in Proposition 1 and 2.

**Proof for Uniqueness in Proposition 5:** By Lemma 2, no player places a mass point above 0. By the same argument as in the previous proof, the support of both players cdf’s has to be connected on $(0,1)$. To satisfy indifference of player 2 for any bid in $(0,1)$, player 1 has to randomize uniformly on $(0,1]$. To keep both types of player 1 indifferent between a subset of strategies in $(0,1]$, player 2 has to randomize with density 1 against the low valuation type and with density $\frac{1}{v}$ against the high valuation type. The randomization against the high type has to be on higher values $(x,1]$ than against the low type (on $(0,x]$) to ensure that all bids for which player 2 randomizes with density $\frac{1}{v}$ are best responses for the high type. The value of $x$ and the size of the mass point at 0 are uniquely determined by the probability that player 1 is a high type.
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References


