A weighted version of McNaughton’s theorem

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Abstract. We investigate weighted finite state automata with the Büchi-acceptance condition. The costs of computations depend on the state of the computation and on the time when it is executed. We characterize those systems whose behaviour is continuous. Then we introduce weighted Muller-automata and show that their behaviour is even Lipschitz-continuous. Under some not too hard restrictions we show that if the behaviour of a finite weighted Büchi-automaton is Lipschitz-continuous, then it is also Muller-recognizable. This is a weighted version of McNaughton’s classical theorem. Using this result we show that under the same restrictions it is decidable whether the behaviour of two weighted Muller-automata is equal.

1 Introduction

Deciding the monadic second order logics with \( k \) successors can be done using Büchi automata. One of the main steps of the proof is to show that Büchi-recognizable languages are closed with respect to complement. In order to show this, one can use McNaughton’s theorem that states that Büchi-recognizable languages coincide with the Muller-recognizable languages. Then it is easy to see that Muller-recognizable languages are closed with respect to complementation. For the background on automata on infinite words we refer the reader to [10, 9].

In [6], Droste and Kuske introduced weighted Büchi automata. These automata are defined similarly to the classical Büchi-automata with the difference that every transition is equipped with a weight from the real max-plus semiring \( \mathbb{R}_{\text{max}} \) (the \( \mathbb{R}_{\text{max}} \)-semiring is fundamental in max-plus algebra and algebraic optimization, cf. Gaubert and Plus [7], Cuninghame-Green [4]). As a rule the automaton reads an infinite word from and adds the weights of the transitions along the run. However, the later a transition is executed, the smaller gets its impact on the result. This is achieved by introducing a deflation parameter \( d \in [0, 1) \).

In classical \( \mu \)-calculus for transition systems (cf. [8] and [2]) the model checking is often done using automata. In particular, the set of possible paths in the transition system that fulfill a given proposition from \( \mu \)-calculus is an \( \omega \)-recognizable language. Also the decidability of \( \mu \)-calculus can be proved using different kinds of automata that accept infinite words.

In [1] Alfaro, Henzinger and Majumdar define a \( \mu \)-calculus with discounting for probabilistic transition systems. Such discrete dynamic systems occur frequently in systems theory. The mentioned authors considered discounted properties of these systems.

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E.g. a property can encode a possible failure of the system in the future. If such failures happen in the far future, then they are of less importance than those that may appear any moment.

We claim that weighted automata on infinite words can be very useful for doing quantitative analysis of weighted transition systems. For instance a weighted automaton can encode costs that occur in every step of a discrete system. In such a setting, costs that occur in the far future are also of less importance than the ones that appear immediately.

In this paper we consider infinitary formal power series over $\mathbb{R}_{\max}$. First we characterize Büchi-recognizable series that are continuous. Such series are interesting because they encode stable properties (it is e.g. desirable that two processes that are the same in a big initial segment, have the similar running costs). Another reason to consider continuous series is the work of Kulik and Karhumäki in [3] about digital image processing. They use another approach to infinitary formal power-series, but continuous series are of particular interest for them.

In the next step we define a weighted version of Muller-automata with deflation parameter $d$ and study their behaviour. We prove that $d$-Muller-recognizable series are $d$-Büchi-recognizable by generalizing Mullers characterization theorem and afterwards we show that Muller-recognizable series are Lipschitz-continuous. Using this we observe that they are not closed with respect to point-wise maximum.

From then on we restrict to deflation parameters of the form $1/b$ for $b \in \mathbb{N} \setminus \{0,1\}$ and to series that are recognized by weighted automata whose weights are non-negative rational numbers. Our main result in this setting is that all Lipschitz-continuous Büchi-recognizable series are Muller-recognizable. Thus we give a weighted version of McNaughton’s theorem. Eventually we show that the equality of Muller-recognizable series is decidable.

2 Preliminaries

Weighted automata are usual automata where every transition is equipped with an element from given semiring. The weighted automaton reads a word from the left to the right and each step of the computation produces a weight.

In this paper we will intensively make use of the real max-plus semiring which is defined by $\mathbb{R}_{\max} = (\mathbb{R}_{\geq} \cup \{-\infty\}, \max, +, -\infty, 0)$ where $\mathbb{R}_{\geq} = [0, \infty)$ and $-\infty + x = -\infty$ for each $x \in \mathbb{R}_{\max}$. It reflects the intuition that to each computation (represented by a word) is assigned its maximum costs in a nondeterministic machine.

Functions to $\mathbb{R}_{\max}$ can be interpreted as partial functions to $\mathbb{R}_{\geq}$ in the evident way. For a function $f$ from some set $X$ to $\mathbb{R}_{\max}$ we define $\text{supp}(f)$ to be the set of all elements from $X$ that are not mapped to $-\infty$.

An $\mathbb{R}_{\max}$-weighted transition system $T$ over $\Sigma$ is a pair $(Q, T)$ such that (i) $Q$ is a finite non-empty set of states and (ii) $T \subseteq Q \times \Sigma \times (\mathbb{R}_{\max} \setminus \{-\infty\}) \times Q$ is a finite set of transitions\footnote{Note that in contrast to other authors (e.g.[5]) we do not allow $-\infty$ as costs of a transition. However, this has no influence on the expressive power.}. We say that $T$ is deterministic if $(p, a, x_1, q_1) \in T$ and $(p, a, x_2, q_2) \in T$ implies that $x_1 = x_2$ and $q_1 = q_2$. $T$ is called complete if for every $p \in Q$ and
\(a \in \Sigma\) there is \(x \in \mathbb{R}_{\text{max}}\) and \(p' \in Q\) such that \((p, a, x, p') \in T\). A finite path \(P\) in \(T\) is a finite word on \(T\) of the form \((q_1, a_{i+1}, x_{i+1}, q_{i+1})_{i \in \{0, \ldots, n-1\}}\) for some positive integer \(n\). The length of the path \(P\) is the length of \(P\) considered as a word. We call \(q_0\) and \(q_n\) domain and codomain of \(P\) and we denote them by \(\text{dom}(P)\) and \(\text{cod}(P)\), respectively. The label of \(P\) is the finite word \(w := a_1a_2\ldots a_n\). We also say that \(P\) is a \(w\)-labeled path from \(q_0\) to \(q_n\). Analogously, an infinite path \(P\) is an infinite word on \(T\) of the form \((p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}\). The domain of \(P\) is \(p_0\), denoted by \(\text{dom}(P)\). The label of \(P\) is the infinite word \(\alpha := a_1a_2a_3\ldots\). We also say that \(P\) is an \(\alpha\)-labeled path. Let \(k \in \mathbb{N}^+\). The initial segment of length \(k\) of \(P\), denoted by \(P|_{k}\), is the finite path \((p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \{0, \ldots, k-1\}}\). Moreover, for an infinite path \(P\), we define the infinity set of \(P\), denoted by \(\text{Infty}(P)\), as the set of all states of \(T\) which occur infinitely often in \(P\).

To every finite path \(P = (p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \{0, \ldots, n-1\}}\) we can assign its running cost in \(A\), denoted by \(\text{rcost}_{\mathcal{A}}(P)\):

\[
\text{rcost}_{\mathcal{A}}(P) := x_1 + d \cdot x_2 + d^2 \cdot x_3 + \cdots + d^{n-1} \cdot x_n
\]

and \(\text{rcost}_{\mathcal{A}}(P) := 0\) if \(n = 0\). Analogously we define the running cost of an infinite path \(P = (p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}\) according to

\[
\text{rcost}_{\mathcal{A}}(P) := \sum_{i=0}^{\infty} x_{i+1}d^i.
\]

Note that this series only converges if \(|d| < 1\).

An \(\mathbb{R}_{\text{max}}\)-weighted automaton \(A\) over \(\Sigma\) is a tuple \((Q, T, \text{in}, \text{out})\) where \((Q, T)\) is an \(\mathbb{R}_{\text{max}}\)-weighted transition system and \(\text{in}, \text{out} : Q \to \mathbb{R}_{\text{max}}\) are cost functions for entering and leaving each state, respectively. We call \(A\) deterministic if \((Q, T)\) is deterministic and there is precisely one \(q_0 \in Q\) such that \(\text{in}(q_0) \geq 0\). A finite path in \(A\) is called initial if its domain is an element of \(\text{supp}(\text{in})\) and successful if it is initial and if its codomain is an element of \(\text{supp}(\text{out})\). Let \(P\) be a path in \(A\). Then the cost of \(P\), denoted by \(\text{cost}_{\mathcal{A}}^*(P)\), is defined by:

\[
\text{cost}_{\mathcal{A}}^*(P) := \text{in}(p_0) + \text{rcost}_{\mathcal{A}}(P) + d^n \cdot \text{out}(p_n).
\]

The behavior of \(A\), denoted by \(\|A\|_d^*\), is the function \(\|A\|_d^* : \Sigma^* \to \mathbb{R}_{\text{max}}\) defined by

\[
(\|A\|_d^*, w) := \max\{\text{cost}_{\mathcal{A}}^*(P) \mid P \text{ is successful } w\text{-labeled path in } A\}
\]

where \(w \in \Sigma^*\), with the convention that \(\max\emptyset = -\infty\).

Functions from \(\Sigma^*\) to \(\mathbb{R}_{\text{max}}\) are called formal power series\(^2\). A series \(S\) is called \(d\)-recognizable if it is the behavior of a finite \(\mathbb{R}_{\text{max}}\)-weighted automaton. By \(\text{Rec}_d(\Sigma^*)\) we will denote the set of all \(d\)-recognizable formal power series. In case that \(S\) can be recognized by a deterministic \(\mathbb{R}_{\text{max}}\)-weighted automaton, we call \(S\) \(d\)-subsequential.

\(^2\) The term formal power series usually will be abbreviated by series or FPS.
3 \(\mathbb{R}_{\text{max}}\)-weighted Büchi-Automata

In this section, we consider non-terminating executions of finite \(\mathbb{R}_{\text{max}}\)-weighted automata, as done in [5]. From now on, we restrict the deflation parameter \(d\) to the values satisfying \(0 \leq d < 1\).

Let \(\mathcal{A} := (Q, T, \text{in}, \text{out})\) be an \(\mathbb{R}_{\text{max}}\)-weighted automaton. Then \(\mathcal{A}\) can also be used to read infinite words. Roughly speaking, the cost of an infinite word can be obtained from the costs of its finite prefixes. We say that an infinite path of \(\mathcal{A}\) is initial if its domain is an element of \(\text{supp}(\text{in})\) and Büchi-successful if it is initial and if the intersection of its infinity set with \(\text{supp}(\text{out})\) is not empty. Let \(P := (p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}\) be an infinite path in \(\mathcal{A}\). Then the cost of \(P\) in \(\mathcal{A}\), denoted by \(\text{cost}_A(P)\), is defined by:

\[
\text{cost}_A^B(P) := \limsup_{n \in \mathbb{N}} \{\text{in}(p_0) + \text{rcost}_A(P|_n) + d^n \cdot \text{out}(p_n)\}.
\]

The Büchi behavior of \(\mathcal{A}\), denoted by \(\|\mathcal{A}\|_A^B\), is the function \(\|\mathcal{A}\|_A^B : \Sigma^\omega \rightarrow \mathbb{R}_{\text{max}}\) defined by:

\[
(\|\mathcal{A}\|_A^B, \alpha) := \sup\{\text{cost}_A^B(P) \mid P \text{ is Büchi-successful } \alpha\text{-labeled path in } \mathcal{A}\}
\]

for \(\alpha \in \Sigma^\omega\), with the convention that \(\sup \emptyset = -\infty\).

Such functions we will call infinitary formal power series.

In order to distinguish \(\mathbb{R}_{\text{max}}\)-weighted automata on infinite words from those on finite words, we call them \(\mathbb{R}_{\text{max}}\)-weighted Büchi automata.

For an infinitary FPS \(S\) we define its support to be the set \(\text{supp}(S) := \{\alpha \in \Sigma^\omega \mid (S, \alpha) \neq -\infty\}\). Let \(R, S : \Sigma^\omega \rightarrow \mathbb{R}_{\text{max}}\) be two infinitary formal power series. We define \(R \oplus S\) and \(\max(R, S)\) to be the pointwise sum\(^\dagger\) and the pointwise maximum of the two series, respectively. Let \(L\) be an \(\omega\)-language. Then the characteristic function of \(L\), is defined as \(1_L : \Sigma^\omega \rightarrow \mathbb{R}_{\text{max}}\) with \((1_L, \alpha) := 0\) if \(\alpha \in L\) and \((1_L, \alpha) := -\infty\), otherwise.

An infinitary FPS \(S\) is called \(d\)-Büchi recognizable if it is the behavior of a finite \(\mathbb{R}_{\text{max}}\)-weighted Büchi-automaton. In case that \(S\) can be recognized by a deterministic \(\mathbb{R}_{\text{max}}\)-weighted Büchi automaton, we call \(S\) \(d\)-Büchi subsequential. We denote the set of all \(d\)-Büchi recognizable infinitary formal power series by \(\text{Rec}_d^B(\Sigma^\omega)\).

The following lemma shows that the cost of Büchi-successful paths in \(\mathbb{R}_{\text{max}}\)-weighted Büchi automata are a constant added to a convergent infinite sum.

**Lemma 1.** Let \(\mathcal{A} := (Q, T, \text{in}, \text{out})\) be an \(\mathbb{R}_{\text{max}}\)-weighted Büchi automaton and let \(P := (p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}\) be a Büchi-successful path in \(\mathcal{A}\). Then

\[
\text{cost}_A^B(P) = \text{in}(p_0) + \text{rcost}_A(P) = \text{in}(p_0) + \sum_{i=0}^{\infty} x_{i+1} \cdot d^i.
\]

**Proof.** Let \(a_n := \text{in}(p_0) + \text{rcost}_A(P|_n) + d^n \cdot \text{out}(p_n)\), for every \(n \in \mathbb{N}\). Since \(P\) is Büchi-successful, \(\limsup_{n \in \mathbb{N}} \{a_n\} \geq 0\). This means that there is a

\(^\dagger\) observe that this is the so called Hadamard-product of the series.
subsequence \( \{a_{n_k}\}_{k \in \mathbb{N}} \) of \( \{a_n\}_{n \in \mathbb{N}} \) such that \( \limsup_{n \in \mathbb{N}} \{a_n\} = \lim_{k \to \infty} a_{n_k} \). We have

\[
\lim_{k \to \infty} a_{n_k} = \lim_{k \to \infty} (\text{rcost}_A(P|_{n_k}) + d^{nk} \cdot \text{out}(p_{n_k})) = \lim_{k \to \infty} \text{rcost}_A(P|_{n_k}).
\]

Define now the sequence \( b_n = \text{in}(p_0) + \text{rcost}_A(P|_n) \). Then the sequence \( (b_n)_{n \in \mathbb{N}} \) is monotonously growing and bounded from above. Therefore it is convergent. Hence with the above we get that

\[
\lim_{k \to \infty} a_{n_k} = \lim_{n \to \infty} b_n = \text{in}(p_0) + \text{rcost}_A(P).
\]

As a consequence of the previous proposition we get that the costs of a Büchi-successful path do not depend on the actual values of the function \( \text{out} \).

We call \( A \) normalized if \( \text{in}(p), \text{out}(p) \in \{0, -\infty\} \), for all \( p \in Q \) and if \( \text{supp}(\text{in}) \) consists of precisely one state. It is not hard to see that every \( d \)-Büchi recognizable function can be computed by a complete and normalized \( \mathbb{R}_{\max} \)-weighted Büchi automaton. For normalized weighted Büchi-automata we will use a shortened representation \( A = (Q, T, q_0, F) \) where \( F \subseteq Q \) and where \( q_0 \in Q \). This is called the normalized representation of \( A \). For the rest of this paper, we only use normalized representations for \( d \)-Büchi recognizable infinitary formal power series.

We continue by giving some simple properties of \( d \)-Büchi recognizable functions.

**Lemma 2.** [5, Lemma 6.9] An \( \omega \)-language is Büchi-recognizable if and only if its characteristic function is \( d \)-Büchi recognizable.

**Lemma 3.** Let \( S_1, S_2 \) be a \( d \)-Büchi recognizable functions and let \( b \in \mathbb{R}_{\geq 0} \). Then

1. \( \text{supp}(S_1) \) is a Büchi recognizable \( \omega \)-language,
2. \( b \cdot S \) is \( d \)-Büchi recognizable,
3. \( S_1 \oplus S_2 \) is \( d \)-Büchi recognizable,
4. \( \max(S_1, S_2) \) is \( d \)-Büchi recognizable. \( \square \)

### 4 Continuous \( d \)-Büchi-recognizable series

On \( \Sigma^\omega \) we define a metric \( \text{dist}_d \) according to

\[
\text{dist}_d(\alpha, \beta) := \begin{cases} 
  d^n & \alpha \neq \beta, \alpha|_n = \beta|_n, \alpha|_{n+1} \neq \beta|_{n+1} \\
  0 & \text{else.}
\end{cases}
\]

Since \( 0 < d < 1 \), this defines indeed a metric on \( \Sigma^\omega \). We say that an infinitary formal power series is continuous in \( \alpha \) if \( (S, \alpha) \geq 0 \) and for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( \beta \in \text{supp}(S) \) with \( \text{dist}_d(\alpha, \beta) < \delta \) we have that \( |(S, \alpha) - (S, \beta)| < \varepsilon \). \( S \) is called continuous if it is continuous on each word from its support.
In order to characterize the continuous $d$-Büchi-recognizable series, we have to introduce some additional notions: Recall that given a language $L \subseteq \Sigma^*$, the language $\overline{L} \subseteq \Sigma^\omega$ contains all $\omega$-words $\alpha$ such that infinitely many finite segments of $\alpha$ are words of $L$. Let us generalize this construction to formal power series:

Let $R : \Sigma^* \rightarrow \mathbb{R}_{\text{max}}$ be a $d$-recognizable series. Let us suppose that for all $\alpha \in \Sigma^\omega$, the sequence $\{(R, \alpha |_{n+1})\}_{n \in \mathbb{N}}$ is bounded from above. Then we define $\overline{R} : \Sigma^\omega \rightarrow \mathbb{R}_{\text{max}}$ by

$$(\overline{R}, \alpha) := \limsup_{n \in \mathbb{N}} \{(R, \alpha |_{n+1})\}.$$ 

**Lemma 4.** Let $A = (Q, T, \text{in}, \text{out})$ be a weighted automaton with $\text{supp(out)} = Q$ and let $S_A$ be the behavior of $A$. Then $\overline{S}_A$ is $d$-Büchi recognizable and continuous.

**Proof.** Let $S_A^B$ be the Büchi-behavior of $A$. Let $\alpha \in \text{supp}(\overline{S}_A)$. Consider all maximal paths through $A$ for all finite prefixes of $\alpha$, or by the prefix-order. Then this is a finitely branching infinite tree. Hence, by König's lemma, there is an $\alpha$-labeled infinite path $P = (p_i, a_{i+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}$ in $A$, for which every prefix is maximal. Hence

$$\limsup_{n \in \mathbb{N}} (\overline{S}_A, \alpha |_n) = \limsup_{n \in \mathbb{N}} (\text{in}(p_0) + r \cdot \text{cost}_A(P |_n) + d^n \cdot \text{out}(p_n))$$

$$= \text{cost}_A^B(P).$$

$P$ is Büchi-successful and maximal since every prefix of $P$ is maximal. Therefore $\text{cost}_A^B(P) = (S_A^B, \alpha)$.

Let now $\alpha \in \text{supp}(S_A^B)$. Then there is a Büchi-successful $\alpha$-labeled path in $A$. Hence there are infinitely many prefixes of $\alpha$ that are in the support of $S_A$. Hence $\alpha \in \text{supp}(S_A)$.

Now let $\alpha \in \text{supp}(S_A^B)$ and let $\{\beta_n\}$ be a sequence of $\omega$-words in $\text{supp}(S_A^B)$ that converges to $\alpha$. As we saw above, there is a Büchi successful path for $\alpha$ with the costs $(S_A^B, \alpha)$. Moreover, all prefixes of this path are maximal paths for the corresponding prefixes of $\alpha$. Let $\text{dist}_A(\beta_n, \alpha) = d^n$ then $|(S_A^B, \beta_n) - (S_A^B, \alpha)| \leq d^n \cdot \frac{M}{1 - d}$ where $M$ is the maximal weight appearing in $A$. Hence $(S_A^B, \beta_n)$ converges to $(S_A^B, \alpha)$ and $S_A^B$ is continuous.

**Theorem 5.** Let $S$ be a $d$-Büchi-recognizable series. Then $S$ is continuous if and only if there is a weighted automaton $A' = (Q, T, \text{in}, \text{out})$ with $\text{supp(out)} = Q$ and with finitary behavior $R$ such that

$$S = \overline{R} \oplus 1_{\text{supp}(S)}.$$ 

**Proof.** Let $S$ be continuous and let $A = (Q, T, q_0, F)$ be a Büchi-automaton that recognizes $S$. Without loss of generality, we can assume that $A$ is trim, that is that every state of $A$ lies on a Büchi-successful path. We define $A' = (Q, T, \text{in}, \text{out})$ according to $\text{in}(q) = -\infty$ for $q \neq q_0$ and $\text{in}(q_0) := 0$. Moreover, $\text{out}(q) := 0$ for all $q \in Q$.

Let $R$ be the finitary behavior of $A'$ and let $\alpha \in \text{supp}(S)$. We show that $(\overline{R}, \alpha) = (S, \alpha)$. For this we define

$$m_r(\alpha) := \max \{\text{cost}^*_A(P |_r) \mid P \text{ is } \alpha \text{-labeled, Büchi-successful path}\},$$

$$M_r(\alpha) := \max \{\text{cost}^*_A(P) \mid P \text{ is } \alpha |_r \text{-labeled initial path}\}.$$
Then $M_r(\alpha) = (R, \alpha \mid _\tau)$. Clearly we have that $M_r(\alpha) \geq m_r(\alpha)$. However, take any maximal initial path in $A'$ for $\alpha \mid _\tau$. Then this path can be completed to a Büchi-successful path $P_r$ for a word $\beta_r$. Clearly, we have that $|m_r(\alpha) - (S, \alpha)| \leq d^r \cdot \frac{M}{1-d}$, and $|M_r(\alpha) - (S, \beta_r)| \leq d^r \cdot \frac{M}{1-d}$, where in both cases $M$ is the maximal weight in $A$. Hence, by the triangle-inequality, we get that

$$|m_r(\alpha) - M_r(\alpha)| \leq 2d^r \cdot \frac{M}{1-d} + |(S, \alpha) - (S, \beta_r)|.$$  

But the right hand side of this inequality converges to 0 when $r$ goes to infinity. So we get that

$$(\overline{R}, \alpha) = \lim_{r \to \infty} M_r(\alpha) = \lim_{r \to \infty} m_r(\alpha) = (S, \alpha).$$

The other direction of the claim follows from Lemma 4 and from the fact that continuity is not spoiled when adding a characteristic series to a continuous series. \qed

5 $\mathbb{R}_{\text{max}}$-weighted Muller Automata

In this section, we introduce $\mathbb{R}_{\text{max}}$-weighted Muller automata, this is, finite deterministic automata with costs from $\mathbb{R}_{\text{max}}$, that read infinite words and that are equipped with a powerful acceptance condition.

An $\mathbb{R}_{\text{max}}$-weighted Muller automaton $\mathcal{A}$ is a tuple $(Q, T, q_0, \text{in}, \{\text{out}_i\}_{i \in \mathcal{I}})$ where $(Q, T)$ is a deterministic $\mathbb{R}_{\text{max}}$-weighted transition system, $q_0 \in Q$ is called initial state, in : $Q \to \mathbb{R}_{\text{max}}$ is the input function for states and is such that $\text{supp}(\text{in}) = \{q_0\}$, and $\mathcal{I}$ is a finite set and for each $i \in \mathcal{I}$, we have $\text{out}_i : Q \to \mathbb{R}_{\text{max}}$. We call the set $\{\text{out}_i\}_{i \in \mathcal{I}}$ the set of accepting functions.

Let $\mathcal{A} := (Q, T, q_0, \text{in}, \{\text{out}_i\}_{i \in \mathcal{I}})$ be an $\mathbb{R}_{\text{max}}$-weighted Muller automaton. Let $P$ be an infinite path in $\mathcal{A}$. We say that $P$ is initial in $\mathcal{A}$ if $\text{dom}(P) = q_0$ and is Muller-successful if it is Muller-initial and if there is an $i \in \mathcal{I}$ such that $\text{Infy}(P) = \text{supp}(\text{out}_i)$. In this case, we say that $P$ is Muller-successful with respect to $i$. For $i \in \mathcal{I}$, the cost of $P$ with respect to $i$, denoted by $\text{cost}^M_{\mathcal{A}}(P)$, is defined as follows:

$$\text{cost}^M_{\mathcal{A}}(P) := \lim_{n \to \infty} \text{sup} \{\text{in}(p_0) + \text{rcost}_{\mathcal{A}}(P) + d^n \cdot \text{out}_i(p_n)\}.$$  

Let $\alpha \in \Sigma^\omega$. We define the behavior $(\|A\|_d^M, \alpha)$ of $\mathcal{A}$ on $\alpha$, as being $\max_{j \in \mathcal{J}} \{\text{cost}^M_{\mathcal{A}}(P_\alpha)\}$ where $\mathcal{J}$ is the subset of $\mathcal{I}$ such that $j \in \mathcal{J}$ if and only if there is an $\alpha$-labeled path which is Muller-successful with respect to $j$. The function $\|A\|_d^M$ is called the behavior of $\mathcal{A}$. An infinitary formal power series is called $d$-Muller-recognizable if it is the behavior of some finite Muller-automaton. We denote the set of all $d$-Muller-recognizable infinitary formal power series by $\text{Rec}^M_d(\Sigma^\omega)$. It is not hard to see that every $d$-Muller-recognizable series can be recognized by a complete $\mathbb{R}_{\text{max}}$-weighted Muller-automaton.

In the same way as for Büchi-automata it can be shown that the behavior of a Muller-automaton does not depend on the values of the output-functions but only on their carrier:
Lemma 6. Let $A := (Q, T, q_0, \text{in}, \{\text{out}_i\}_{i \in I})$ be an $\mathbb{R}_{\max}$-weighted Muller automaton, $i \in I$ and $P := (p_1, a_{i_1+1}, x_{i+1}, p_{i+1})_{i \in \mathbb{N}}$ be a Muller-successful path with respect to $i$ in $A$. Then

$$
\text{cost}_A^M(P) := \text{in}(p_0) + \lim_{n \to \infty} \text{rcost}_A(P) = \text{in}(p_0) + \sum_{i=0}^{\infty} x_{i+1} \cdot d^i.
$$

Thus changing the output-functions to the characteristic functions of their carrier, does not change the behavior of the automaton. Muller-automata, whose input weight is 0 and whose output-functions take only values 0 or $-\infty$, are called normalized. The following lemma is a simple consequence of Lemma 6.

Lemma 7. Every $d$-Muller recognizable function can be computed by a complete, normalized $\mathbb{R}_{\max}$-weighted Muller automaton.

A normalized weighted Muller-automaton $A = (Q, T, q_0, \text{in}, \{\text{out}_i\}_{i \in I})$ is completely determined by the tuple $(Q, T, q_0, \mathcal{F})$ where $\mathcal{F} := \{\text{supp(out)}_i | i \in I\}$. This we call normalized representation. In the rest of the paper, we suppose that all $\mathbb{R}_{\max}$-weighted Muller automata are given in normalized representation.

Let us have a look now onto a necessary condition for a series to be $d$-Muller recognizable. An infinitary series $S$ is said to be Lipschitz-continuous if there exists a constant $\Delta \geq 0$ such that for all $\alpha, \beta \in \text{supp}(S)$ we have that

$$
|(S, \alpha) - (S, \beta)| \leq \Delta \cdot \text{dist}_d(\alpha, \beta).
$$

Proposition 8. All $d$-Muller-recognizable series are Lipschitz-continuous.

Proof. Let $A = (Q, T, q_0, \mathcal{F})$ be a weighted Muller-automaton and let $S$ be its behavior. Now let $M$ be the greatest weight that appears in the description of $A$. Without loss of generality we can assume that $M \geq 0$. Let $\alpha, \beta \in \text{supp}(S)$ and suppose that $\text{dist}_d(\alpha, \beta) = d^r$, in particular $|\alpha|_r = |\beta|_r$. Let $P_\alpha = (p_0^\alpha, a_1^\alpha, x_1^\alpha, p_2^\alpha)_{k \in \mathbb{N}}$ and $P_\beta = (p_0^\beta, a_1^\beta, x_1^\beta, p_2^\beta)_{k \in \mathbb{N}}$ be the two successful paths for $\alpha$ and $\beta$, respectively. Then

$$
|(S, \alpha) - (S, \beta)| = \sum_{k=0}^{\infty} (x_k^\alpha - x_k^\beta) d^k \leq \sum_{k=r}^{\infty} M d^k = d^r \cdot \frac{M}{1 - d}.
$$

Hence $S$ is $d$-Lipschitz-continuous with $\Delta = M/(1 - d)$.

The following lemmas summarize some closure properties of $\text{Rec}_d^M(\Sigma^\omega)$:

Lemma 9. An $\omega$-language is Muller recognizable if and only if its characteristic function is $d$-Muller recognizable.

Lemma 10. Let $S_1, S_2 \in \text{Rec}_d^M(\Sigma^\omega)$ and let $c \in \mathbb{R}_{\max}$. Then

1. $S_1 \oplus S_2 \in \text{Rec}_d^M(\Sigma^\omega)$
2. $c \cdot S_1 \in \text{Rec}_d^M(\Sigma^\omega)$.
Note that from the Lipschitz-continuity of $d$-Muller-recognizable series it follows that $\text{Rec}_d^\omega(S^\omega)$ cannot be closed with respect to pointwise maximum. Indeed, consider the series $S_1$ and $S_2$ with carriers $\{a^n b^n | n \in \mathbb{N}\}$ and $\{a^n\}$, respectively such that $(S_1, a^n b^n) = \frac{1}{n^2}$ and $(S_2, a^n) = \frac{1}{n^3}$. Then $S_1$ and $S_2$ are $d$-Muller-recognizable but their pointwise maximum is not even continuous.

In the following we will show that all $d$-Muller-recognizable series are also $d$-Büchi-recognizable.

**Lemma 11.** Every $d$-Büchi subsequential series is $d$-Muller recognizable.

**Proof.** Let $S : \Sigma^\omega \to \mathbb{R}^\text{max}$ be $d$-Büchi subsequential and let $A := (Q, T, q_0, F)$ be a deterministic $\mathbb{R}^\text{max}$-weighted Büchi automaton which computes $S$. Then consider the weighted Muller-automaton $A' := (Q, T, q_0, F)$ where $F := \{ A \subseteq Q | A \cap F \neq \emptyset \}$ and observe that $A$ and $A'$ recognize the same series.

**Theorem 12.** [Characterization Theorem] Let $S : \Sigma^\omega \to \mathbb{R}^\text{max}$ be an FPS. Then $S$ is $d$-Muller recognizable if and only if there is a $d$-Büchi subsequential function $R : \Sigma^\omega \to \mathbb{R}^\text{max}$ and a Muller recognizable $\omega$-language $L \subseteq \Sigma^\omega$ such that $S = R \oplus 1_L$.

**Proof.** (“$\Rightarrow$“): This follows directly from Lemmas 9, 10, and 11.

(“$\Leftarrow$“): Let $S : \Sigma^\omega \to \mathbb{R}^\text{max}$ be a $d$-Muller recognizable function recognized by $A := (Q, T, q_0, F)$ where $F = \{G_1, \ldots, G_k\}$. Let $G := \bigcup_{i \in \mathbb{Z}} G_i$ and consider the $\mathbb{R}^\text{max}$-weighted Büchi automaton $A_R := (Q, T, q_0, G)$. Let $R := \|A_R\|_1^\omega$ and let $L := \text{supp}(S)$. Then $S = R \oplus 1_L$.

**Corollary 13.** Every $d$-Muller recognizable function is $d$-Büchi recognizable.

## 6 Lipschitz-continuous $d$-Büchi-recognizable series

We noted in the previous section that $d$-Muller-recognizable series are Lipschitz-continuous. In this section we will analyze weighted Büchi-automata whose behavior is Lipschitz-continuous.

For an $\mathbb{R}^\text{max}$-weighted Büchi-automaton $A = (Q, T, q_0, F)$, a word $\alpha \in \Sigma^\omega$, and a natural number $r > 0$ we define

$$m_r(\alpha) := \max\{\text{cost}_A(P) \mid P \text{ is } \alpha\text{-labeled, Büchi-successful path}\},$$

$$M_r(\alpha) := \max\{\text{cost}_A(P) \mid P \text{ is } \alpha\text{-labeled, initial path}\}.$$

As it was shown in Theorem 5, if $S_A$ is continuous, then $m_r(\alpha)$ and $M_r(\alpha)$ are both convergent to the same number. The next proposition shows that in the case that $S_A$ is Lipschitz-continuous, then this convergence is even rather fast:

**Proposition 14.** Let $A = (Q, T, q_0, F)$ be a weighted Büchi-automaton with Lipschitz-continuous behavior $S$ with Lipschitz-constant $\Delta$ and let $M$ be the maximal weight in $A$. Then for all $\alpha \in \text{supp}(S)$ and for all $r > 0$ we have that

$$|m_r(\alpha) - M_r(\alpha)| \leq \left(\Delta + 2 \cdot \left[\frac{M}{1 - d}\right]\right) \cdot d^r$$

□
The previous proposition is a strong hint, that perhaps all Lipschitz-continuous \( d \)-Büchi-recognizable series \( S \) are in fact \( d \)-Muller-recognizable. The idea for the construction of a Muller-automaton is, to follow for a word \( \alpha \in \Sigma^* \) all paths \( P \) such that 
\[
| \text{rcost}_A(P) - M_r(\alpha)| \leq \left( \Delta + 3 \cdot \left\lceil \frac{M}{1/d} \right\rceil \right) \cdot d',
\]
then in this set of paths there are enough Büchi-successful paths such that the least upper bound of their costs is equal to \((S, \alpha)\). Indeed, if we consider an \( \alpha \)-labeled Büchi-successful path \( P \) such that \( \text{rcost}_A(P) = m_r(\alpha) \), then for all Büchi-successful paths \( P' \) for \( \alpha \) with \( \text{rcost}_A(P') \geq \text{rcost}_A(P) \) we have that 
\[
\text{rcost}_A(P'|_r) \geq \text{rcost}_A(P|_r) - \left\lceil \frac{M}{1/d} \right\rceil.
\]

7 A weighted version of McNaughton’s theorem

In this section we will give a construction to determinize weighted Büchi-automata with Lipschitz-continuous behavior. In order to obtain that this Algorithm certainly terminates we have to make some assumptions. From now on we assume that \( d = 1/b \) for some natural number \( b > 1 \). Moreover we assume that all weights that appear in our automata are in fact non-negative rational numbers.

Let \( A = (Q, T, q_0, F) \) be a weighted Büchi-automaton with Lipschitz-continuous behavior with respect to \( \Delta \). Without loss of generality we can assume that all weights of \( A \) are natural numbers because otherwise we simply multiply all weights with a suitable constant \( c \). The behavior of the resulting automaton is then \( c \cdot S_A \). After this the determinization this step can be taken back by multiplying everything with \( 1/c \).

Let \( M \) be the maximal weight of \( A \) and define \( \Delta_{\text{max}} := \Delta + 3 \cdot \left\lceil \frac{M}{1/d} \right\rceil \). In the following we will define a deterministic weighted Büchi-automaton \( A_{\text{max}} = (Q', T', q_0', F') \) whose behavior coincides with the one of \( A \) on the support of \( S_A \).

The states of \( A_{\text{max}} \) are sets of pairs \( (q, c) \) where \( q \in Q \) and where \( c \in \{0, 1, \ldots, \Delta_{\text{max}}\} \).

We require that for every \( q \in Q \) there is at most one pair \( (q, c) \) in each state of \( A_{\text{max}} \).

When \( A_{\text{max}} \) reads an \( \omega \)-word \( \alpha \), then it follows simultaneously all paths of \( A \) that are sufficiently close to the maximal path for each prefix of \( \alpha \). Such paths we will call good paths. Since \( A_{\text{max}} \) can not take into account the weights of all paths at once, it is performing in each step always the transition of the worst good path. For all other good paths it is just memorizing their advantage over the worst good path. So each pair of a state of \( A_{\text{max}} \) corresponds to the actual state of a good path and its current advantage.

In order to get a finite automaton we restrict the maximal advantage to be \( \Delta_{\text{max}} \).

Now we must define for a state \( U \) of \( A_{\text{max}} \) and a letter \( a \in \Sigma \), the next state and the weight of the transition. First we look, which good paths can be most successfully extended. For this we consider each good path \( (q, c) \in U \) and all transitions \((q, a, x, q')\) that extend this path and consider the actual cost of this step, that is \( x + c \). Over all these costs we take the maximum and denote it by \( K_a(U) \):
\[
K_a(U) := \max \{ x + c \mid (q, c) \in U, (q, a, x, q') \in T \}.
\]

As next step we have to consider all extensions of paths that are good in comparison with \( K_a(U) \) and to choose the worst one:
\[
k_a(U) := \min \{ x + c \mid (q, c) \in U, (q, a, x, q') \in T, (K_a(U) - x - c) \cdot b \leq \Delta_{\text{max}} \}.
\]
It remains to collect the new good branches and their advantages:

\[ Q_\alpha(U) := \{ q' \mid \exists (q, c) \in U, (q, a, x, q') \in T, x + c - k_\alpha(U) \geq 0 \} \]

\[ \hat{Q}_\alpha(U) := \{ (q', c_\alpha(U, q')) \mid q' \in Q_\alpha(U) \} \quad \text{where} \]

\[ c_\alpha(U, q') = \max \{ (x + c - k_\alpha(U)) \cdot b \mid (q, c) \in U, (q, a, x, q') \in T, x + c - k_\alpha(U) \geq 0 \}. \]

We are now ready to define the transition set \( T' \) of \( A_{\Delta_{\max}} \):

\[ T' := \bigcup_{a \in \Sigma} \{ (U, a, k_\alpha(U), \hat{Q}_\alpha(U)) \mid U \in Q' \}. \]

The initial state \( q'_0 \) of \( A_{\Delta_{\max}} \) is going to be \( \{(q_0, 0)\} \). As set \( F' \) of final states we simply take \( Q' \).

**Proposition 15.** Let \( S \) and \( S_{\Delta_{\max}} \) be the behaviors of \( A \) and \( A_{\Delta_{\max}} \), respectively and let \( \alpha \in \text{supp}(S) \). Then \( (S, \alpha) = (S_{\Delta_{\max}}, \alpha) \).

**Proof.** Let us first show that there is an \( \alpha \)-labeled path in \( A_{\Delta_{\max}} \). We will show by induction on the length \( r \) of the prefix that was already read that in every state \( U_r \) there is a good path with weight \( m_r(\alpha) \) and enough good paths that can be completed in \( A \) to an \( \alpha \)-successful path.

If \( r = 0 \), then \( A_{\Delta_{\max}} \) is in the state \( \{(q_0, 0)\} \). All \( \alpha \)-labeled Büchi-successful paths in \( A \) start with \( q_0 \).

Suppose that after reading \( r \) letters of \( \alpha \) the automaton \( A_{\Delta_{\max}} \) is in the state \( U_r \) and that we have that in \( U_r \) there is a good path with weight \( m_r(\alpha) \) and that all good paths with costs at least \( m_r(\alpha) - d^r \left\lceil \frac{M}{t - \varpi} \right\rceil \) are in \( U_r \). By construction \( U_{r+1} \) contains the best paths for \( \alpha|_{r+1} \) and all extensions of paths that are at most \( d^r \cdot \Delta_{\max} \) behind the best paths. All the good extensions of the good \( r \)-paths are in this range, by Proposition 14. Hence \( U_{r+1} \) contains the good paths with weight \( m_{r+1}(\alpha) \) and all those good paths that have costs at least \( m_r(\alpha) - d^r \left\lceil \frac{M}{t - \varpi} \right\rceil \).

This shows that there is an initial \( \alpha \)-labeled path in \( A_{\Delta_{\max}} \). Since \( F' = Q' \), this path is also Büchi-successful.

Let us go on showing that \( (S, \alpha) = (S_{\Delta_{\max}}, \alpha) \). Let us consider the unique \( \alpha \)-labeled path \( P = (U_t, a_{t+1}, x_{t+1}, U_{t+1}) \in \mathbb{N} \) of \( A_{\Delta_{\max}} \). Then by construction we have that \( M_r(\alpha) - d^r \cdot \Delta_{\max} \leq \text{rcost}_{\alpha}(A_{\Delta_{\max}})(P) \leq M_r(\alpha) \). Hence

\[ (S_{\Delta_{\max}}, \alpha) = \lim_{r \to \infty} \text{rcost}_{\alpha}(A_{\Delta_{\max}})(P) = \lim_{r \to \infty} M_r = (S, \alpha). \]

\[ \square \]

**Theorem 16.** Let \( d = 1/b \) for \( b \in \mathbb{N}, b > 1 \). Let \( A \) be an \( \mathbb{R}_{\max} \)-weighted Büchi-automaton with weights from the non-negative rational numbers. Let \( S = \|A\|_d^B \) then \( S \) is Lipschitz-continuous if and only if it is d-Müller-recognizable.

**Proof.** If \( S \) is Lipschitz-continuous then we take \( S_{\Delta_{\max}} \) as constructed above and observe with Proposition 15, Lemma 11, Lemma 10, 9, and McNaughton’s theorem that \( S = S_{\Delta_{\max}} \oplus 1_{\text{supp}(S)} \) is \( d \)-Müller-recognizable.

If \( S \) is Muller-recognizable, then by Proposition 8, \( S \) is Lipschitz-continuous. \( \square \)
From this theorem we can derive a useful decidability-result:

**Theorem 17.** Let $A_1$ and $A_2$ be $\mathbb{R}_{\text{max}}$-weighted Muller-automata with weights from $\mathbb{Q}_{\geq 0}$, let $d = 1/b$ for some natural number $b > 1$ and let $S_1 = \|A_1\|^M_d$ and $S_2 = \|A_2\|^M_d$, respectively. Then it is decidable whether $S_1 = S_2$.

**Proof.** We just sketch the decision procedure. First we decide if the supports of $S_1$ and $S_2$ are equal. This is not a problem because these languages are Büchi-recognizable.

If the supports are equal then we observe that the maximum of $S_1$ and $S_2$ is Lipschitz-continuous. The Lipschitz-constant is the maximum of the Lipschitz-constants for $S_1$ and $S_2$, respectively. Now we take deterministic weighted Büchi-automata $A'_1$ and $A'_2$ that recognize $S_1$ and $S_2$, respectively. Then the maximum of $S_1$ and $S_2$ is computed by the disjoint union of $A'_1$ and $A'_2$. For normalization the behavior of the two initial states is simulated by an additional new initial state that is a source. Let us call this automaton $A'$. Its states are all the states of $A'_1$ and of $A'_2$ plus one more initial state. As the next step we construct the automaton $A'_{\Delta_{\text{max}}}$ as above and make it trim (i.e. we remove all those states that are not on some Büchi-successful path. It is not hard to see that $S_1 = S_2$ if and only if every of the remaining states is either the initial state or it contains at least one state from $A'_1$ and one state from $A'_2$. $\Box$

**References**