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In the proof of Theorem 4.21 we make use of an assumption not explicitly stated in the hypothesis of the statement. Here we make this hypothesis precise in order to make the argument used sound. Notations and definitions are the same.

1. The hypothesis

We first need to introduce some new definitions.

Definition 1.1. Given a multidotted word $\tilde{u}_1 \in \tilde{\Omega}_A$ and a letter $A \in \Delta$, we say that $A$ is simply dotted in $\tilde{u}$ if the letter $A$ occurs dotted only once in $\tilde{u}$; we say that $A$ is multidotted in $\tilde{u}$ if the letter $A$ occurs dotted strictly more than once in $\tilde{u}$.

We have that $A$ is simply dotted in $\dot{A}BABA$ but $A$ is multidotted in $\dot{A}B\dot{A}BA$.

Definition 1.2. Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let $T_{\gamma}$ be the associated transition system on $M^{k_2}$, and let $\mathcal{P}$ be a finite definable partition of $M^{k_2}$. Let us recall that $\Delta$ is the finite definable partition induced by the dynamical types on $\mathcal{P}$. We say that the set of multidotted words $\tilde{\Omega}_A$ is simple if the following conditions hold. For all multidotted word $\tilde{u}_1 \in \tilde{\Omega}_A$ and for all letter $A \in \Delta$ we have that

1. If $A$ is multidotted in $\tilde{u}_1$, then all the occurrences of $A$ in $\tilde{u}_1$ are dotted.
2. If $A$ is simply dotted in $\tilde{u}_1$, then for all word $\tilde{u}_2 \in \tilde{\Omega}_A$ containing the letter $A$, if $A$ is dotted in $\tilde{u}_2$ then $A$ is simply dotted in $\tilde{u}_2$.


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In fact, the previous conditions prevent the two following types of situation:
1. $\dot{A}B\dot{A}B\dot{A}A \notin \Omega_A$,
2. $A\dot{B}A \in \Omega_A$ and $\dot{A}B\dot{A} \in \Omega_A$.

We can now state Theorem 4.21 in a more precise way.

**Theorem 1.3.** Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let $T_\gamma$ be the associated transition system on $M^{k_2}$, and let $P$ be a finite definable partition of $M^{k_2}$ such that $\Omega_A$ is simple. If there exists a unique multidotted word associated with each $y \in M^{k_2}$, then there exists a finite bisimulation of $T_\gamma$ that respects $P$.

In this context we reformulate Corollary 4.22.

**Definition 1.4.** Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system. Given $y \in M^{k_2}$ we define $T(y) = \{t | \exists x \gamma(x, t) = y\}$.

The set $T(y)$ is clearly a definable subset of $M$. Hence it has a finite number $K$ of connected components. This number $K$ is related to the number of times that the trajectory $\Gamma_x$ crosses the point $y$.

**Definition 1.5.** Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system. We say that a point $y \in M^{k_2}$ has a looping behavior if there exists $x \in M^{k_1}$ such that $T(y)$ has at least two connected components.

**Definition 1.6.** Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system. We denote by $\text{Loop}$ the set of points of $M^{k_2}$ which have a looping behavior.

**Definition 1.7.** We say that $(\mathcal{M}, \gamma, P)$ is simple if given any trajectory $\Gamma_x$ any two points $y_1$ and $y_2 \in \Gamma_x \cap \text{Loop}$, we have that $y_1$ and $y_2$ belong to different pieces of $P$.

Let us remark that all the examples of system $(\mathcal{M}, \gamma, P)$ exhibited are simple.

**Corollary 1.8.** Let $(\mathcal{M}, \gamma)$ be an o-minimal dynamical system, let $T_\gamma$ be the associated transition system on $M^{k_2}$, and let $P$ be a finite definable partition of $M^{k_2}$ such that $(\mathcal{M}, \gamma, P)$ is simple. If there exists a unique trajectory (with possible self-intersections) associated with each $y \in M^{k_2}$, then there exists a finite bisimulation of $T_\gamma$ that respects $P$.

2. Why is this hypothesis needed?

We exhibit an o-minimal dynamical system $(\mathcal{M}, \gamma)$, with a unique trajectory associated with each point $y$ of $M^{k_2}$, together with a finite partition $P$ such that $(\mathcal{M}, \gamma, P)$ is not simple; and we see that $T_{\Omega_A}$ (the multidotted symbolic transition system) is not bisimilar to $T_\gamma$ (the transition system associated with $(\mathcal{M}, \gamma)$). This shows the necessity of the previously introduced hypothesis.

**Example 2.1.** Let us consider the dynamical system $(\mathcal{M}, \gamma)$ of Fig. 2. In order to understand correctly how the trajectory evolves, it is decomposed according to time evolution in Fig. 1. One can easily be convinced that $(\mathcal{M}, \gamma)$ is an o-minimal dynamical system.
Without loss of generality we assume that $(\mathcal{M}, \gamma)$ consists only in the trajectory drawn in Fig. 1. We denote this trajectory by $\Gamma_x$.

Let us consider the partition of the plane $\mathcal{P} = \{A, B\}$ where $B$ is the shaded region (see Fig. 2). We show that the multidotted words encoding introduced in the original paper are not sufficient to recover a bisimulation w.r.t $\mathcal{P}$. First one can easily see that there is a unique word $(\omega_x)$ associated with $(\mathcal{M}, \gamma)$ w.r.t $\mathcal{P}$. In particular we have that

$$\Omega = \{ABABA\}.$$ 

The set of dotted words associated with $(\mathcal{M}, \gamma)$ w.r.t $\mathcal{P}$ is given by

$$\hat{\Omega} = \{\hat{A}BABA, \ldots, ABA\hat{B}A\}.$$ 

We now consider the six dynamical types:

$$W_1 = \{\hat{A}BABA\}, \ldots, W_5 = \{ABAB\hat{A}\}, W_6 = \{A\hat{B}ABA, ABA\hat{B}A\}.$$ 

Let us notice that $y_2$ is the only point whose dynamical type is $W_6$. Although the trajectory crosses $y_1$ several times, it has a “simple” dynamical type (i.e., a dynamical type that contains a single dotted word).

The dynamical types induce a new partition $\Delta$. Again there is a unique word $(\omega_x)$ associated with $(\mathcal{M}, \gamma)$ w.r.t $\Delta$. In particular we have that

$$\Omega_\Delta = \{W_1 W_2 W_6 W_2 W_3 W_4 W_6 W_4 W_5\}.$$
A unique multidotted word is associated with each point $y$ of the trajectory $\Gamma_x$. Let us denote $\bar{u}_{y_1}$ (resp. $\bar{u}_{y_2}$ and $\bar{u}_{y_3}$) the multidotted word associated with $y_1$ (resp. $y_2$ and $y_3$) on $A$. We have that
\[
\bar{u}_{y_1} = W_1 \dot{W}_2 W_6 \dot{W}_2 W_3 W_4 W_6 W_4 W_5,
\]
\[
\bar{u}_{y_2} = W_1 W_2 \dot{W}_6 W_2 W_3 W_4 \dot{W}_6 W_4 W_5,
\]
\[
\bar{u}_{y_3} = W_1 W_2 W_6 \dot{W}_2 W_3 W_4 W_6 W_4 W_5.
\]

Following the proof of Theorem 4.21 the binary relation $\sim$ should be a bisimulation between $T_{\bar{\Omega}_A}$ and $T_{\bar{\gamma}}$. We show that $T_{\bar{\gamma}}$ does not simulate $T_{\bar{\Omega}_A}$.

Let us take the two multidotted words $\bar{u}_{y_1}$ and $\bar{u}_{y_3}$. We have that $\bar{u}_{y_3} \rightarrow_{\bar{\Omega}_A} \bar{u}_{y_1}$ using the definition of the transition in $T_{\bar{\Omega}_A}$. We also clearly have that $y_3 \sim \bar{u}_{y_3}$ (where $\sim$ is the pretended bisimulation relation defined in the original paper). However, it is impossible to find a point $y_3'$ on $\Gamma_x$ such that $y_3 \rightarrow_{\bar{\gamma}} y_3'$ and $y_3' \sim \bar{u}_{y_1}$ due to the definition of the transition system $T_{\bar{\gamma}}$ which is not transitive.