A Note on the Cross Gramian for Non-Symmetric Systems

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Abstract

The cross gramian matrix is a tool for model reduction and system identification, but it is only computable for square control systems. For symmetric control systems the cross gramian possesses a useful relation to the associated system’s Hankel singular values. Yet, many real-life models are neither square nor symmetric. In this work, concepts from decentralized control are used to generalize the cross gramian to be applicable for non-square and non-symmetric systems. To illustrate this new non-symmetric cross gramian, it is then applied in the context of model reduction.

Keywords: Cross Gramian, Controllability, Observability, Decentralized Control, Model Reduction

1 Introduction

The cross gramian was introduced in [5] for single-input-single-output (SISO) systems and extended to multiple-input-multiple-output (MIMO) systems in [12]. With many applications in model order reduction (MOR) and system identification applications such as decentralized control [13], parameter identification [10] or sensitivity analysis [18], a major hindrance in the use of the cross gramian is the constraint that it can only be computed for square systems and exhibits its core property only for symmetric systems.

This work can be seen as the a follow up to [12] and [4], which also expanded the scope of cross gramian.

The object of interest in this context is a linear time-invariant state space system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]

which consists of a vector field composed of a system matrix \( A \in \mathbb{R}^{N \times N} \) and an input matrix \( B \in \mathbb{R}^{M \times N} \), and a linear output functional composed of an output matrix \( C \in \mathbb{R}^{O \times N} \) and a feedforward matrix \( D \in \mathbb{R}^{O \times M} \). In the scope of this work, only a trivial feed-forward matrix \( D = 0 \) is considered. Furthermore, the system is assumed to be asymptotically stable, hence all eigenvalues of the system matrix \( A \) lie in the left half-plane \( \text{Re}(\lambda_i(A)) < 0 \).

In Section 2 the cross gramian is reviewed. Existing approaches and the new result for the non-symmetric cross gramian are presented in Section 3. Lastly, in Section 4 verification and validation for the new non-symmetric cross gramian and comparison with established methods is conducted.

2 The Cross Gramian

With the controllability operator \( \mathcal{C}(u) := \int_0^\infty e^{At}Bu(t)dt \) and the observability operator \( \mathcal{O}(x) := Ce^{Au}x \), the controllability and observability of a system can be evaluated through the associated controllability gramian \( W_C := \mathcal{C}^\ast \mathcal{C} \) and observability gramian \( W_O := \mathcal{O}\mathcal{O}^\ast \). A third system gramian, called cross gramian, combines controllability and observability into a single gramian matrix.

2.1 Square Systems

For a square system, a system with the same number of inputs and outputs, the cross gramian is defined as the product of controllability operator \( \mathcal{C} \)
and observability operator \( O \):\(^\text{[5]}\)
\[
W_X := CO = \int_0^\infty e^{At}BCe^{At}dt \in \mathbb{R}^{N \times N}.
\] (2)
Classically, the cross gramian is computed as solution to the Sylvester equation:
\[
AW_X + WXA = -BC,
\]
which relates to the definition in \( [2] \) through integration by parts:
\[
\int_0^\infty e^{At}BCe^{At}dt = A^{-1} |e^{At}BCe^{At}|_0^\infty
- A^{-1} \int_0^\infty e^{At}BCe^{At}dtA
\]
\[
\Rightarrow A \int_0^\infty e^{At}BCe^{At}dt = -BC - WXA.
\]
\[
\Rightarrow AW_X = -BC - WXA.
\]

2.2 Symmetric Systems
A system in the form of \( [2] \) is called symmetric if the system’s gain is symmetric\(^\text{[1]}\):
\[
CA^{-1}B = (CA^{-1}B)^T.
\]
In other words for a symmetric system exists a symmetric matrix \( P \) such that:
\[
AP = PA^T \text{ and } B = P^{-1}C^T
\] (3)
are fulfilled \( [2, 5] \). Since a SISO system has a scalar gain, all SISO systems are symmetric.

For a symmetric system, the absolute value of the eigenvalues of the cross gramian are equal to Hankel singular values \( [17] \):
\[
W_X^2 = W_CW_O
\Rightarrow |\lambda(W_X)| = \sqrt{|W_CW_O|}.
\] (4)
This core property of the cross gramian allows to evaluate controllability and observability information of a system by computing a single gramian matrix. This can be computationally more efficient than computing two, namely the controllability gramian and observability gramian, which additionally have to be balanced to be used in example for model order reduction through balanced truncation \( [2] \).

3 The Non-Symmetric Cross Gramian
In this section, existing approaches for cross gramians of non-symmetric systems and selected methods from decentralized control are briefly summarized; the latter is employed to expand the scope of the cross gramian from symmetric to non-symmetric systems.

3.1 Previous Work
To the authors’ best knowledge there exist two methods to broaden the scope of the cross gramian for non-square MIMO systems towards more general configurations.

The first approach \( [14] \) extends the applicability of the cross gramian from symmetric systems to the wider class of orthogonally symmetric systems. Given a symmetric matrix \( P = P^T \) for which \( AP = PA^T \) holds and an orthogonal matrix \( U \), with the property \( B = PCU^T \) if \( O \leq M \), or \( C = PBU^T \) if \( M \leq O \), then the system is orthogonally symmetric and the associated cross gramian:
\[
\hat{W}_X = \int_0^\infty e^{At}BUCe^{At}dt,
\]
satisfies the core property \( [4] \).

The second approach, presented in \( [16, 17] \), uses embedding of a non-square or non-symmetric system into a symmetric system, and relies on a symmetrizer matrix\(^2\) \( J = JT^T \):
\[
AJ = JA^T.
\]
Given a symmetrizer matrix \( J \) to \( A \), an embedding is given by:
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + (JC^T B) u(t) \\
y(t) &= \left( \begin{array}{c}
C \\
B^T J^{-1}
\end{array} \right) x(t);
\end{align*}
\]
the associated cross gramian has the form:
\[
\hat{W}_X = \int_0^\infty e^{At}(JC^T C + BB^T J^{-1})e^{At}dt.
\] (5)
While the first approach preserves the central property from \( [4] \) for \( \hat{W}_X \) of orthogonally symmetric systems, the second approach approximates it for \( \hat{W}_X \) of arbitrary systems by embedding.

\(^1\)Equivalently the symmetry of the impulse response, transfer function or Markov parameter can be used.

\(^2\)A symmetrizer can be computed for example by solving a Sylvester equation \( [3] \).
3.2 Decentralized Control

Decentralized control aims to associate inputs and outputs from MIMO systems to obtain a set of SISO systems for which each pairing associates an input with the output that indicates the strongest input-output coherence. To this end, an interaction measure for all combinations of inputs and outputs is computed, and assembled into a participation matrix or pairing matrix from which the dominant elements are selected for the decentralized SISO systems.

As a first step a MIMO system is decomposed into \( J \times O \) SISO systems, by partitioning the input matrix \( B \) and output matrix \( C \) column-wise and row-wise respectively:

\[
B = \begin{pmatrix} b_1 & \ldots & b_M \end{pmatrix}, b_i \in \mathbb{R}^{N \times 1},
\]

\[
C = \begin{pmatrix} c_1 \\ \vdots \\ c_O \end{pmatrix}, c_j \in \mathbb{R}^{1 \times N}.
\]

Each combination of \( b_i \) and \( c_j \) induces a SISO system with the following system gramians:

\[
W_{C,i} := \int_0^\infty e^{At}b_ite^{A^Td}dt,
\]

\[
W_{O,j} := \int_0^\infty e^{A^Td}c_jte^{At}dt,
\]

\[
W_{X,i,j} := \int_0^\infty e^{At}b_ie^{A^Td}dt.
\]

The gramians computed for the SISO subsystems relate to the full MIMO gramians as shown in [13]:

\[
W_C = \sum_{i=1}^M W_{C,i}, \quad W_O = \sum_{j=1}^O W_{O,j}
\]

\[
W_X = \sum_{i=1}^{M(O)} W_{X,i,j}.
\]

3.3 Main Result

Next, the previous approach from decentralized control is utilized for the computation of a cross gramian of non-square or non-symmetric systems. The central idea is to exploit, that for any SISO system a cross gramian with the property (4) can be computed. With (4), the product of controllability and observability can be expressed as sum of squared cross gramians:

\[
W_CW_O = \sum_{i=1}^M \sum_{j=1}^O W_{C,i}W_{O,j} = \sum_{i=1}^M \sum_{j=1}^O W_{X,i,j}W_{X,i,j}.
\]

Due to the squaring, this ansatz is numerically not efficient. Therefore, an alternative non-symmetric cross gramian, related to (7), can be defined as the sum of the cross gramians of all combinations of SISO subsystems:

\[
\tilde{W}_X := \sum_{i=1}^M \sum_{j=1}^O W_{X,i,j}.
\]

Obviously, this gramian does not preserve the cross gramian’s property (4). Yet, for a linear system, \( \tilde{W}_X \) yields the following representation:

\[
\tilde{W}_X = \sum_{i=1}^M \sum_{j=1}^O \int_0^\infty e^{At}b_ie^{A^Td}dt
\]

\[
= \int_0^\infty e^{At} \sum_{i=1}^M b_i e^{A^Td}dt
\]

\[
= \int_0^\infty e^{At} (\sum_{i=1}^M b_i)(\sum_{j=1}^O c_j)e^{A^Td}dt.
\]

Hence, this non-symmetric cross gramian is equal to the cross gramian of the SISO system given by \( A \), the row sum of the input matrix \( B \) and the column sum of the output matrix \( C \).

Both approaches in section 3.1 share the common drawback, that they are limited to linear systems [4], and additionally may require a, potentially computationally expensive, symmetrizer. The new approach, introduced in [8], does not require the linear structure of the underlying system or derive system properties using linear algebra. Only a cross gramian of a SISO system needs to be computable. Thus, the non-symmetric cross gramian can even be computed for nonlinear systems if a nonlinear cross gramian is available, this could be for example an empirical cross gramian [10]. Empirical gramians [11] are computed from (simulated) trajectories of
the underlying system with perturbations in input and initial state. The empirical cross gramian has been introduced for SISO systems in [18] and extended to MIMO systems in [10]; and as shown in [10], the empirical cross gramian is equal to the cross gramian [2] for linear systems [1], hence the empirical cross gramian can be used for the following experiments.

4 Numerical Results

The presented method is implemented as part of the empirical gramian framework (emgr) [3] introduced in [9]; and the following numerical experiments are conducted using emgr, which is compatible with OCTAVE and MATLAB®. The source code for reproducing the experiments can be found under an open license in the supplemental materials and at http://www.runmycode.org/companion/view/913.

From a computational point of view, the proposed method has the advantage, that for all SISO subsystem cross gramians the trajectories for perturbed initial states (observability), which consume the dominant fraction of overall computational time, have to be computed only once.

Next, the presented non-symmetric cross gramian is tested in the context of projection-based model reduction [2]. For a linear system (1) a reduced order model is obtained through projection-based model reduction using the projection \( U_1, V_1 \) by:

\[
\begin{align*}
\dot{x}_r &= V_1 A U_1 x_r(t) + V_1 B u(t), \\
y_r &= C U_1 x_r(t).
\end{align*}
\]

Among others, such projections can be computed by balancing approaches, like balanced truncation [13] utilizing the controllability and observability gramian, or approximate balancing utilizing the cross gramian. By truncating the left singular vectors from the singular value decomposition of the cross gramian a (one-sided) Galerkin projection is generated, for further details see [10].

\[
W_X^{SV} \equiv UDV \rightarrow U = (U_1 \quad U_2) \rightarrow V_1 = U_1^T.
\]

First, the approximate non-symmetric cross gramian is compared to balanced truncation [14] [2].

![Figure 1: L2 output error of reduced order models from balanced truncation, cross gramian and non-symmetric cross gramian for a symmetric system.](image)

Figure 1: \( L_2 \) output error of reduced order models from balanced truncation, cross gramian and non-symmetric cross gramian for a symmetric system.

A state-space symmetric system \( A = A^T, B = C^T \) of state-space dimension \( N = \dim(x(t)) = 256 \) and input dimension \( J = \dim(u(t)) = \dim(y(t)) = O = 16 \) is selected, with a negative Lehmer matrix as system matrix \( A \) and uniformly random generated input matrix \( B = C^T \). For this test zero initial state \( x_0 = 0 \) and impulse input \( u(t) = \delta(t) \) is applied and the relative \( L_2 \) output error is evaluated for varying reduced order state-space dimensions.

To confirm [9], the error between the proposed non-symmetric cross gramian \( \tilde{W}_X \) and the cross gramian \( W_X \) of the SISO system \( A, b, c \), with \( b_i = \sum_{j=1}^{J} B_{ij} \) and \( c_j = \sum_{i=1}^{O} C_{ij} \), is compared in the Frobenius norm:

\[
\|\tilde{W}_X - W_X\|_F \approx 2.2e-12.
\]

Fig. 4 shows, that the reduced order models obtained by balanced truncation and the cross gramian exhibit the same behavior due to state-space symmetric nature of the system. The newly proposed non-symmetric cross gramian does not achieve the same accuracy, but provides a lower output error for lesser order reduced models. A lower accuracy is to be expected due to the use of a one-sided projection; while balanced truncation uses a two-sided Petrov-Galerkin projection, which in the state-space symmetric case is equal to the projection obtained from the (symmetric) cross gramian. Notably, the output error of the reduced order model from the non-symmetric cross gramian drops already at a very low order \( n \geq 5 \).
to the steady error level while balanced truncation and the cross gramian reach this error not until a reduced order of $n \geq 30$.

Second, the non-symmetric cross gramian is compared to balanced truncation and the cross gramian of the symmetric system derived by embedding from \cite{15} for a non-square (and thus non-symmetric) system.

To prevent the computation of a symmetrizer matrix $J$, the symmetric system matrix $A \in \mathbb{R}^{256 \times 256}$ from the first example is reused, yet now, a uniformly random generated input matrix $B \in \mathbb{R}^{256 \times 8}$ and a uniformly random output matrix $C \in \mathbb{R}^{24 \times 256}$ is selected, thus the system is non-square and non-symmetric, since $J = 8$ and $O = 24$. Also for this example, zero initial state $x_0 = 0$ and impulse input $u(t) = \delta(t)$ is applied and the relative $L_2$ output error is evaluated for varying reduced order state-space dimensions in Fig. 2.

With the reference of balanced truncation, the cross gramian of the embedded system performs worse as predicted in \cite{15} \cite{17}. Again, the non-symmetric cross gramian yields reduced models with less relative output error for small (reduced) orders.

For both experiments, the model reduction error of the presented non-symmetric cross gramian is worse than for balanced truncation, but the descent of the error is steeper. Hence, if the lower accuracy is acceptable, a smaller reduced order model can be constructed with this (non-symmetric) variant of the cross gramian method.

5 Conclusion

In this work a non-symmetric cross gramian, based on concepts from decentralized control, is proposed, and demonstrated to provide viable results for linear non-symmetric and non-square systems, which are outside the scope of the regular cross gramian. Future work will evaluate the effectiveness of the non-symmetric cross gramian for nonlinear systems. Furthermore, since the procedure suggested in \cite{15} did not yield stable reduced order models in this setting, an investigation of alternative two-sided projections \cite{7} for the (non-symmetric) cross gramian, may yield errors comparable to balanced truncation.

Acknowledgement

This work was supported by the Deutsche Forschungsgemeinschaft, the Open Access Publication Fund of the University of Münster, DFG EXC 1003 Cells in Motion - Cluster of Excellence, Münster, Germany as well as by the Center for Developing Mathematics in Interaction, DEMAIN, Münster, Germany.

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