A Fast Hough Transform for the Parametrisation of Straight Lines using Fourier Methods

The Hough transform is a useful technique in the detection of straight lines and curves in an image. Due to the mathematical similarity of the Hough transform and the forward Radon transform, the Hough transform can be computed using the Radon transform which, in turn, can be evaluated using the central slice theorem. This involves a two-dimensional Fourier transform, an $x$-$y$ to $r$-$\theta$ mapping and a 1D Fourier transform. This can be implemented in specialized hardware to take advantage of the computational savings of the fast Fourier transform. In this paper, we outline a fast and efficient method for the computation of the Hough transform using Fourier methods. The maxima points generated in the Radon space, corresponding to the parametrisation of straight lines, can be enhanced with a post transform convolutional filter. This can be applied as a 1D filtering operation on the resampled data whilst in the Fourier space, so further speeding the computation. Additionally, any edge enhancement or smoothing operations on the input function can be combined into the filter and applied as a net filter function.

Introduction

The Hough transform can be used to determine the parametrisation of straight lines and curves in an image [1]. It can be generalized for different classes of curves [2–4] but is most often used for the detection of straight-line segments in 2D image arrays.

For the parametrisation of straight lines, each point in the Cartesian $x$-$y$ image space is mapped to a sinusoidal curve in the Hough $r$-$\theta$ space using the parametric representation: $r = x \cos \theta + y \sin \theta$. If the points are collinear, the sinusoids intersect at a point $(r, \theta)$ corresponding to a parametrisation of the line. This produces a butterfly dispersion in the parameter space around each maximum point [5].

Many research groups have proposed fast robust algorithms for the detection of curves and lines in binary images using the Hough transform. Toft et al. [6–8] uses a fast curve estimation (FCE) algorithm which identifies curve parameters by first forming a pre-conditioning map, which takes into account pixels in the image with zero value (image point mapping), to determine regions in the parameter space which contain peaks. A generalized Radon transform is then applied to these regions,
thus reducing the computational cost. Illingworth and Kittler [9] and Li et al. [10] use a hierarchical Hough transform, which first quickly transforms the image using a coarse sampling interval to isolate areas of interest, and then transforms these areas with a slightly finer sampling interval. The method is repeated with increasing sampling resolution until the lines have been detected. Ballard [2] uses a gradient method to estimate the tangent of curves in the image. Each point in the image is then mapped to a point in parameter space, instead of a curve, given by the estimated gradient of the line. However accuracy is dependent upon the gradient operator used and the noise level in the image. Other groups such as Kultanen et al. [11] and Kälviäinen et al. [12] use an iterative algorithm called the random Radon transform, which maps two random non-zero pixels from the image to a common point in the parameter space. This is repeated until a suitable description of the parameter space has been obtained.

Due to the mathematical similarity of the Hough transform and the forward Radon transform [8,13–14] the Hough transform can be equivalently computed using the Radon transform [15], which can be very efficiently evaluated using the central slice theorem [16,17]. This involves a two dimensional Fourier transform (DFT), an \( x-y \) to \( r-\theta \) mapping and a 1D DFT which can be computed optically [18–20] or in hardware [21–23] using specialized digital signal processing (DSP) chips to take advantage of the computational savings of the fast Fourier transform (FFT).

Levers and Boyce [5] propose a convolution filter which, when applied to the sinogram, generates a much more consistent peak structure by detecting the butterfly distributions in the sinogram corresponding to continuous straight-line segments, and discriminating against those associated with discontinued colinearities. This can be applied as a filtering operation on the resampled 1D Fourier space prior to a 1D DFT being taken. Additionally, any edge enhancement or smoothing operations on the input function can be combined into the filter and applied as a net filter function to the 1D Fourier space.

In this paper we outline an algorithm for computing the Radon transform of a real grayscale image using Fourier methods. The interpolation method and the filtering processes used on the 1D resampled Fourier spectrum will also be discussed.

### Projection Generation and the Central Slice Theorem

#### Forward Radon transform

The forward Radon transform is used to transform a 2D function into its projections. For a continuous 2D function \( f(r) = f(x, y) \), a single 1D projection at an angle \( \phi \) relative to the x-axis can be derived by integration along lines normal to the angle of projection. For each set of integration lines at different angles relative to the x-axis, a different projection is derived. The complete set of 1D projections of the function is called a sinogram, since a point in Cartesian space maps to a sinusoid in

![Figure 1](image-url)
Radon space, and contains all the information in the original function.

We can obtain projections by integration over lines parallel to the \( y_0 \) axis in a system of coordinates \([x_0, y_0]\) rotated at angle \( \phi \) relative to the original \([x, y]\) axes (Figure 1).

\[
\begin{pmatrix}
  x' \\
  y'
\end{pmatrix} = \begin{pmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]  

(1)

Therefore for a point \((x, y)\) which lies a distance \( \rho \) along the \( x_0 \) axis:

\[
\rho = x \cos \phi + y \sin \phi
\]  

(2)

Thus a 1D projection can be formed by integration of the image intensity \( f(x, y) \) along a line, \( L \), that is perpendicular distance \( \rho \) from the origin and at angle \( \phi \) from the \( x \)-axis:

\[
\lambda(\rho, \phi) = \int_L f(x, y) dl
\]  

(3)

where \( \lambda(\rho, \phi) \) is a 1D projection of the function \( f(x, y) \) at angle \( \phi \).

The projection can also be defined as a 2D integral function using a 1D Dirac delta function:

\[
\lambda(\rho, \phi) = \int \int_D f(x, y) \delta(p - x \cos \phi - y \sin \phi) \, dx \, dy
\]  

(4)

where \( D \) is the space spanned by the variables of integration. For a continuous 2D function \( \phi \) can be limited to the region \((0 \leq \phi \leq \pi)\) since if \((-\infty \leq \rho \leq \infty)\), \( \lambda(\rho, \phi) = \lambda(-\rho, \phi + \pi) \) (Figure 2).

Equation (2) can also be expressed in vector notation:

\[
p = r \cos \theta \cos \phi + r \sin \theta \sin \phi = r \cos(\theta - \phi) = r \hat{n}
\]  

(5)

where \( r = (x, y) = |r|, \theta = [r \cos \theta, r \sin \theta] \) is the position vector of a point in Cartesian space. \( \hat{n} = 1/\phi = [\cos \phi, \sin \phi] \) is the unit vector orthogonal to the direction of projection.

Therefore Eqn (4) can be rewritten as:

\[
\lambda(p, \phi) = \int \int_D f(r) \delta(p - r \hat{n}) \, dx \, dy
\]  

(6)

The projection operator can also be expressed in operator notation:

\[
\mathbf{R}_2[f(r)] = \lambda(\rho, \phi)
\]  

(7)

where \( \mathbf{R}_2 \) is the Radon transform operator, which transforms the 2D function \( f(r) \) with axes \([x, y]\) to Radon space with axes \((\rho, \phi)\). The subscript 2 denotes the dimensionality of the function being transformed.

**Central slice theorem**

The central slice theorem [16,17] states that the Fourier transform of an \((n-1)\) dimensional projection is a
central section of the n-dimensional Fourier transform of the object, orthogonal to the projection direction.

The 2D DFT of a 2D function \( f(r) \) gives:

\[
F_2[f(r)] \equiv F(p) = \int \int \Delta f(r) e^{-j2\pi p \cdot r} \, dx \, dy
\]  
(8)

where \( F_2 \) is the 2D DFT operator, which transforms the 2D function \( f(r) \) with axes \([x, y]\) to 2D Fourier space with axes \([P_x, P_y]\).

The 1D DFT of the Radon projections \( \lambda(\rho, \phi) \) gives:

\[
F_1[\lambda(\rho, \phi)] \equiv \Lambda(v, \phi) = \int \left[ \int \Delta f(r) \delta(p - r \cdot \hat{n}) \, dx \, dy \right] e^{-j2\pi \rho \cdot v} \, dp
\]
\[
= \int \Delta f(r) \, dx \, dy \int \delta(p - r \cdot \hat{n}) e^{-j2\pi \rho \cdot v} \, dp
\]
\[
= \int \Delta f(r) e^{-j2\pi \rho \cdot v} \, dx \, dy
\]  
(9)

Therefore by comparing Eqns (8) and (9) we can see that:

\[
\Lambda(v, \phi) = F(p)|_{p \cdot \hat{n} \cdot \phi} = F(\hat{\nu})
\]  
(10)

Thus, the 1D DFT of a Radon projection at angle \( \phi \) relative to the x-axis gives a line through the origin of the 2D Fourier transform of the function \( f(r) \) relative to the \( P_x \)-axis (Figure 3). The central slice theorem can be represented in operator notation by

\[
F_2 = F_1 R_2
\]  
(11)

Thus if we take the 2D DFT of the image function \( f(r) \), perform a rectangular to polar coordinate transformation and then take the 1D DFT of the resampled 1D Fourier field, we obtain the desired projection, \( \lambda(\rho, \phi) \).

**Application of the Radon transform**

Figure 4 shows the processes used to generate the Radon transform of a 2D grayscale image \( f(r) \) using Fourier
methods. To take advantage of the computational savings of the fast Fourier transform, powers of 2 were used wherever possible.

To avoid aliasing effects, which arise from the $x$-$y$ to $v$-$\phi$ mapping, the image was padded with zeros, prior to being transformed, to increase its size by a factor of $4^b (b = 0, 1, 2, \ldots)$ from $u \times v$ to $(2^b u) \times (2^b v)$. This improves the resolution of its 2D Fourier spectrum, and reduces the sampling increments $P_x$ and $P_y$. However this results in an increase in the number of computations, and thus increases the computing time required.

Since the original image is real its 2D Fourier spectrum is symmetric about its origin (diametric Hermitian symmetry) and it is therefore only necessary to sample the top half of the Fourier spectrum. This is sampled to a $m \times n$ grid, where $m$ and $n$ are the number of radial and angular samples. The missing data can be obtained by conjugate reflection of the resampled data about the zeroth frequency (Figure 5).

The 2D Fourier spectrum can be filtered to edge enhance or smooth the image. However, since only the top half of the Fourier spectrum is sampled, it is not necessary to apply a 2D filter to the 2D Fourier spectrum. Instead the 1D Fourier spectrum can be filtered in one operation to both edge enhance and improve the peak structure of the sinogram. This can be done before the resampled data is conjugate mirrored.

To produce the sinogram, a 1D DFT of the filtered and resampled 1D Fourier spectrum is taken along $m$. Any zero padding is then removed by cropping the image from size $n \times 2m$ to produce a $n \times 2u$ sinogram.

**Figure 4.** Block diagram of the processing stages in the Fourier evaluation of the Radon transform to produce a peak enhanced filtered sinogram from a grayscale image $f(r)$.

**Figure 5.** Graphical representation of the main processing stages in the Fourier evaluation of the Radon transform to transform an image in Cartesian space (a) to a sinogram in Radon space (d). Due the diametric Hermitian symmetry of the 2D Fourier spectrum (b), the resampled 1D Fourier spectrum (c) can be obtained by sampling the top half of the 2D Fourier spectrum and mirroring the resampled data by conjugate reflection about the zeroth frequency.
This is then thresholded to locate the maxima in Radon space.

**x-y to \(\nu-\phi\) mapping**

The 2D Fourier spectrum is reordered from a Cartesian x-y grid to a polar \(\nu-\phi\) grid of size \(m \times n\). This is achieved by computing the location of the each \(\nu-\phi\) pixel on the Cartesian grid at \(p = \hat{n}\nu\) and interpolating the surrounding pixels. We found that a combination of bilinear interpolation of the four nearest neighbors and zero supplementation of the data produced very good results. Figure 6 shows the sensor geometry used.

The \(n\) angular samples span a range from 0 to \(\pi\). Therefore,

\[
\Delta \phi = \frac{\pi}{n - 1} \tag{12}
\]

\(n\) has to be sufficiently high to ensure optimum pixel coverage. It is not necessary for \(n\) to be a power of 2 since \(n\) 1D DFTs of length \(2m\) are taken along \(m\) to produce the sinogram. The radial sampling increment \(\Delta \nu\) is defined as:

\[
\Delta \nu = \sqrt{\Delta P_x^2 + \Delta P_y^2} = \frac{\sqrt{2}}{2\hat{n}} \tag{13}
\]

where \(\Delta P_x = \Delta P_y = 1/2^h\) are the sampling increments in the 2D Fourier space. Enlarging the input image by a factor, \(h\), with zero data causes the Fourier field to be enlarged and \(\Delta \nu\) to decrease, leading to a more accurate sinogram. This also removes any interperiod artefacts present. \(\Delta \nu\) can be further decreased by a factor of \(2^k (k = 0, 1, 2, \ldots)\) to \(\Delta \nu/2^k\) to ensure optimum pixel coverage for different interpolation methods. Thus the \(m\) radial samples span a range from \(\Delta \nu/2^k\) to \(R\). Therefore,

\[
m = \frac{R}{\Delta \nu/2^k} = (u/2) \times 2^{(h+k)} \tag{14}
\]

where \(R = \nu_{\text{max}} = (u/2)\sqrt{2}\) is defined as the field of view that the mapping covers. A circular visual field which totally encloses the 2D Fourier spectrum was chosen so that the data sampled would not be truncated. Any pixels outside the Cartesian grid were set to zero.

Since \(R = (u/2)\sqrt{2}\), the sinogram will be expanded by \(\sqrt{2}\) to \(2R\sqrt{2} \times n\) i.e. \(2u \times n\). Due to the Fourier similarity theorem an expansion in the spatial domain results in a contraction in the corresponding Fourier

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**Figure 6.** x-y to \(\nu-\phi\) mapping sensor geometry. The top half of the 2D Fourier spectrum (a) of the function is resampled to an \(m \times n\) grid (b) using the sensor geometry shown. \(m\) and \(n\) are the number of radial and angular samples and are related to \(\Delta \phi\) by Eqns (12) and (13). The sampling density of the x-y to \(\nu-\phi\) mapping gets denser as the mapping approaches the centre of the Fourier spectrum. Therefore, at high frequencies, the data samples are further apart and so \(n\) has to be sufficiently high to ensure adequate coverage in these regions and so prevent high frequency distortion of the image.
The length of the resampled 1D Fourier spectrum, \( m \), along which the \( n \) 1D DFTs are taken, is dependent on the values of \( b \) and \( k \) and the size of the original image. Figure 8 shows the effect of decreasing the radial sampling increment by \( 2^k \) from \( \Delta \nu \) to \( \Delta \nu/2^k \). With no Fourier enlargement (\( b = 0 \)) aliasing effects occur from the high contrast pixels near the side of the image, which cause shadows to appear near the opposite side of the sinogram. The sinogram produced when \( b \) and \( k \) are both zero, [Figure 8(b)], is truncated since \( m = u/2 \). Increasing \( k \) results in a decrease in \( \Delta \nu/2^k \) and produces a more accurate sinogram with slightly less aliasing effects. However, this increases \( m \) by \( 2^k \) and increases the computation time required. If \( m > u \) the sinogram will have to be cropped to \( 2u \times n \) to remove any zero padding. Increasing \( k \) beyond 2 does not yield any significant improvement and results in oversampling of the data present in the Fourier domain. This can be increased by increasing the amount of zero padding of the input image in order to reduce the Fourier domain sampling increments, \( \Delta P_x \) and \( \Delta P_y \).

Figure 9 shows sinograms of the image in Figure 8(a), with different spatial enlargement factors, \( b \). The aliasing effects and interperiod artefacts in the sinogram are reduced with \( b = 1 \). Higher values of \( b \) result in a greater improvement in the sinogram, but result in an increase in the size of the Fourier field and in the computing time required. We found that values of \( b = 1 \) and \( k = 1 \) gave the best results with minimal increase in computation time. The number of radial samples, \( n \), has to be sufficiently high so as to ensure that enough data is stored in the sinogram for proper reconstruction. Also, since the sampling density of the \( x \)-\( y \) to \( \nu \)-\( \phi \) mapping gets less dense as the mapping approaches the edges of the Fourier spectrum, \( n \) has to be sufficiently high to ensure sufficient coverage in these regions, so that high frequency components of the input image are not distorted where the data samples are further apart.

Accuracy can be further improved using higher order interpolation functions, such as cubic splines, which involve more than the immediate neighbors of the Fourier spectrum to be interpolated. However, we found that a bilinear interpolation of the four nearest neighbors produced accurate results with minimal computational overhead. Accuracy can further be improved by bandlimiting the projections prior to, or after, interpolation. This can be incorporated into the
Figure 7. Radon transform of a straight line segment (a) to a sinogram in Radon space (d) with parameters \((n, b, k) = (256, 0, 1)\). This is obtained by resampling the 2D Fourier spectrum (b) of the image, in \(n\) angular slices, into its 1D Fourier spectrum (c), and then taking \(n\) 1D DFTs across (c). With a spatial enlargement factor of \(b = 0\), the sinogram exhibits some aliasing, which can be reduced by zero supplementation of the data.

Figure 8. Radon transform of an image (a) with parameters \((n, b, k) = (256, 0, k)\). \(b\) is set to zero resulting in aliasing and interperiod artefacts. Increasing \(k\) decreases the radial sampling increment by \(2^k\) from \(\Delta \nu\) to \(\Delta \nu/2^k\). This reduces the aliasing effects but increases the size of the reordered grid by \(2^k\).
algorithm as a 1D filtering operation on the 1D Fourier spectrum. In the next sections we will discuss the filtering processes used to enhance the peak structures in the sinogram.

1D Filtering

Peaks in the Radon transform of $f(r)$ are correlation peaks indicating rays, parametrised by $\rho$ and $\phi$, along which straight line segments of $f(r)$ lie. The image is usually edge enhanced with a 2D convolutional filter prior to application of the Radon transform. This can be equivalently applied as a filtering operation in the 2D Fourier space. Further, by invoking the filter theorem [17], it can be reduced to a 1D filtering operation, by taking projections of the 2D filter function. If the filter is circular symmetric in the frequency domain then a cross-section can be taken and applied to each angular slice in the 1D Fourier spectrum independent of the angle. Any other post transform processes can be combined into the filter to produce a net filter function.

Figure 9. Radon transform of Figure 8(a) with parameters $(n, b, k) = (256, b, 1)$. $k$ is set at 1. Increasing $b$ increases the size of the Fourier field by $4^b$, thus decreasing the sampling increments in the 2D Fourier space $\Delta P_x$, $\Delta P_y$ by $1/2^b$. This reduces the aliasing effects and interperiod artefacts present in the sinogram. Higher values of $b$ result in a greater improvement in the sinogram, but also result in a huge increase in the computing time required.
Leavers and Boyce [5] propose a convolutional filter, which enhances the peak structure of the butterfly dispersions in the sinogram, based on the angle between the boundaries of the butterfly distribution. This convolution filter has a high positive response to distributions which have their largest value in the center pixel, which falls off to approximately 50% to either side and vanishes rapidly above and below the center pixel. However, the butterfly distribution is determined by the length of the line in the image and the number of angular slices taken, and so each filter has to be tailored to the parameters of the mapping. A low number of samples results in a very broad butterfly distribution, whereas a very large number of samples results in a much thinner distribution. A more efficient approach is to apply a 1D filter to the resampled 1D Fourier space. This can be incorporated into our algorithm to filter the image whilst in the Fourier space.

Filter theorem

The filter theorem [17] states that the projection of the 2D convolution of two functions is the 1D convolution of the projections of each of the functions. Since convolution in the spatial domain is equivalent to multiplication in the Fourier domain, the filtering process can be incorporated into our algorithm as a 1D filter function, whilst in the Fourier domain.

If we wish to convolve the 2D continuous function \( f(\mathbf{r}) \) with a filter function \( h(\mathbf{r}) \):

\[
g(\mathbf{r}) = f(\mathbf{r})^{**} h(\mathbf{r})
\]

where the number of asterisks denotes the dimensionally of the convolution.

Therefore,

\[
F_2[g(\mathbf{r})] = F_2[f(\mathbf{r})] \ast F_2[h(\mathbf{r})]
\]

This can be rewritten using the operator notation for the central slice theorem. From Eqn (11):

\[
F_1 R_2 [g(\mathbf{r})] = F_1 R_2 [f(\mathbf{r})] \ast F_1 R_2 [h(\mathbf{r})]
\]

Applying the Radon transform operator using Eqn (7):

\[
F_1 [\hat{\lambda}_g(\rho, \phi)] = F_1 [\hat{\lambda}_f(\rho, \phi)] \ast F_1 [\hat{\lambda}_h(\rho, \phi)]
\]

Therefore,

\[
\hat{\lambda}_g(\rho, \phi) = \hat{\lambda}_f(\rho, \phi)^{*} \hat{\lambda}_h(\rho, \phi)
\]

Thus, the projection of a 2D convolution of two functions is the 1D convolution of the projections of each of the functions. The same relationship holds for 2D correlation. Therefore the projection of a 2D filtering or correlation operation can be derived by simply performing 1D filtering or correlation of the projections of the original functions.

It is computationally faster to perform this convolution as a filtering process whilst in the frequency domain. Therefore, from Eqn (20) we have

\[
\Lambda_g(\nu, \phi) = \Lambda_f(\nu, \phi) \ast \Lambda_h(\nu, \phi)
\]

1D difference of Gaussian filter

The peak structure of the sinogram can be enhanced by bandpass filtering the 1D Fourier spectrum to enhance
the high frequency components of the butterfly distribution. This was implemented using a 1D difference of Gaussian (DOG) filter, which can be formed by taking slices through the 2D frequency domain DOG filter. However, since the 2D DOG filter is circular symmetric in the 2D frequency space, the set of projections obtained about its center will be identical for all angular slices. Thus only one slice through the 2D frequency domain filter is required to form the 1D DOG filter. This is applied to each angular slice in the 1D Fourier spectrum prior to it being conjugate mirrored.

The DOG filter is a function which approximates the $\nabla^2$ Gaussian operator and acts as a second-differential operator on the image intensities to produce an edge map of the reduced resolution image [24]. In the frequency domain this is equivalent to a bandpass filtering operation. The DOG filter, $h(x, y)$, can be written in the space domain as:

$$h(x, y) = d_e^2 \exp(-d_e^2(x^2 + y^2)) - d_i^2 \exp(-d_i^2(x^2 + y^2))$$

where

$$d_{e,i} = \frac{1}{\sqrt{2\pi}\sigma_{e,i}}$$

where $\sigma_e$ and $\sigma_i$ represent the excitatory and inhibitory standard deviations of the two Gaussian functions. The DOG filter can be expressed in the frequency domain as:

$$F_2[h(x, y)] = H(p) = \exp\left(-\frac{p_x^2 + p_y^2}{d_e^2}\right) - \exp\left(-\frac{p_x^2 + p_y^2}{d_i^2}\right)$$

Figure 11. 1D Difference of Gaussian (DOG) function $\Lambda_{\phi}(v, \phi)$ at angle $\phi$ with parameters $(d_e, d_i) = (120, 75)$ for $m = 256$. This is equivalent to an angular slice from the center to the edge of the corresponding 2D DOG function. The filter is applied to each slice of the resampled 1D Fourier spectrum prior to it being conjugate mirrored.

Figure 12. Radon transforms and 1D Fourier spectrums of the image in Figure 8(a) with Gaussian white noise of zero mean and variance 0.05 added: (b),(e) unfiltered (c),(f) ideal (d),(g) filtered with parameters $(d_e, d_i) = (120, 75)$. (n, b, k) = (256, 1, 1). This attenuates the low and high frequency components in the Fourier spectrum (d) to produce a peak enhanced sinogram (g). This can be thresholded to produce the maxima in Radon space (f).
Taking angular slices by substituting for $p = \hat{n}\nu$:

$$\Lambda_\theta(\nu, \phi) = \left. H(p) \right|_{p = \hat{n}\nu}$$

$$= \exp\left(\frac{-(\nu \cos \phi)^2 + (\nu \sin \phi)^2}{d_c^2}\right) - \exp\left(\frac{-(\nu \cos \phi)^2 + (\nu \sin \phi)^2}{d_l^2}\right)$$

$$= \exp\left(\frac{-\nu^2}{d_c^2}\right) - \exp\left(\frac{-\nu^2}{d_l^2}\right) \quad (25)$$

This is independent of $\phi$, since the DOG filter is a circular symmetric 2D filter. Therefore by specifying the values of $d_c$ and $d_l$ we can define our desired filter. Marr and Hildreth [24] show that the best approximation to the $\nabla^2$ of a Gaussian operator occurs if the ratio between the standard deviations of the inhibitory and excitatory Gaussians, $\sigma_i$ and $\sigma_c$ is 1.6. Figure 10 shows 2D DOG functions and their corresponding 1D DOG functions for various values of $(d_c, d_l)$. Larger values of $d_c$ and $d_l$ produce less pronounced positive and negative swings either side of the zero crossing of the filter and lead to a greater reduction in the resolution of the image.

**Results of 1D filtering**

Figure 11 shows the 1D DOG filter function $\Lambda_\theta(\nu, \phi)$ at angle $\phi$ with parameters $(d_c, d_l) = (120, 75)$ for $m = 256$. This is independent of $\phi$ and so it is applied to each angular slice along $m$. The resampled Fourier spectrum is conjugate mirrored after being sampled and so it is only necessary to filter the sampled half of the Fourier spectrum. The filter of size $m \times n$ is dependent on the parameters $(n, b, k)$ of the mapping and the size of the input function. Thus the filter function will change with the length of the sampled half Fourier spectrum, $m$ and will have to be scaled with different values of $m$ to produce the desired 1D DOG function.

This filter enhances the peak structures in the sinogram by attenuating the low frequency components.
of the 1D Fourier spectrum. Figure 12 shows the Radon transform and unfiltered 1D Fourier spectrum of the image in Figure 8(a) with Gaussian white noise of zero mean and variance 0.05. Parameters are \((n, b, k) = (256, 1, 1)\), so \(m = 256\) and \((d_e, d_i) = (120, 75)\). The Fourier spectrum is bandpass filtered to attenuate the both the low and high frequency components in the Fourier spectrum. This produces a peak enhanced sinogram which can be thresholded to locate the maxima points in the sinogram.

When applying the Radon transform to real-world images, the image is usually first edge enhanced prior to transformation (Figure 13). However, when evaluating the Radon transform using Fourier methods the edges may be equivalently detected with a 2D bandpass filter in the 2D frequency space. This can be combined into our filter to produce a net filter function. As with the 2D DOG filter the values of \((d_e, d_i)\) need to be tailored to the image being transformed. Increasing the values of \(d_e\) and \(d_i\) highpasses the 1D Fourier spectrum to preserve high frequencies and attenuate low frequencies. Figure 14 shows the effects of the filtering the 8-bit grayscale image of Figure 13(a), with parameters \((n, b, k) = (256, 1, 1)\). A filter with \((d_e, d_i) = (200, 125)\) has the effect of edge detecting the image and enhancing the structures in the resulting sinogram. This gives a sharper sinogram than the one corresponding to \((d_e, d_i) = (120, 75)\). However the edges of the image have also been detected causing four spurious maxima points. These have to be taken into account when detecting the peaks in the sinogram.

Figure 15 shows the unfiltered (g) and filtered (h) sinograms of an 8-bit grayscale image (a), and the unfiltered sinogram (i) obtained from its edge enhanced image (c). Parameters are \((n, b, k) = (512, 1, 1)\) and \((d_e, d_i) = (400, 250)\) for \(m = 512\). Bandpass filtering the 1D Fourier spectrum of the image (e) has the effect of detecting its edges and enhancing the peak structures in the filtered sinogram. As with the previous example, the
edges of the image have also been detected in the filtered sinogram. These effects can be seen by comparing the edge detected image (c) and its unfiltered sinogram, with the reconstructed image (b), which was obtained by reconstructing the filtered sinogram (h) using the inverse Radon transform.

Conclusion

In this paper, we have outlined a fast and efficient method for the computation of the Hough transform. This was achieved by computing the Hough transform via the central slice theorem, so that the algorithm becomes a 2D DFT, an x-y to r-ϕ mapping and a 1D DFT. This can be efficiently realized using the fast Fourier transform implemented in DSP hardware.

By zero supplementing the input function prior to transformation, the 2D Fourier space can be enlarged, reducing the aliasing errors in the sinogram. Due to the diametric Hermitian symmetry of the 2D Fourier spectrum it is only necessary to sample the top half of the 2D Fourier spectrum. The missing data can be obtained by conjugate reflection to produce the 1D Fourier spectrum. The sampling process used is critical in producing accurate results. A bilinear interpolation of the nearest four neighbors was found to produce accurate results with minimal computational overhead. The number of angular slices taken have to be high.

Figure 15. Comparison of unfiltered (g) and filtered (h) sinograms of a real 8-bit grayscale image (a), with the unfiltered sinogram (i) obtained from the edge enhanced image (c). Parameters are \((n, b, k) = (512, 1, 1)\) and \((d, d') = (400, 250)\) for \(m = 512\). The reconstructed image (b) was obtained by reconstructing the filtered sinogram (h) using the inverse Radon transform.
enough to ensure sufficient coverage at high frequencies, where the sample points are relatively far apart. In sampling the 2D Fourier spectrum up to its corners, the sinogram produced was scaled by a factor of \( \sqrt{2} \). However sampling the 2D Fourier spectrum up to its sides results in high frequencies being truncated, causing distortion of edges and other high frequency spatial components.

The Hough transform of straight line segments produces butterfly dispersions around each maximum point in the sinogram. To detect these maxima, the 1D Fourier spectrum can be filtered using a 1D difference of Gaussian filter to enhance the peak structure of the Fourier spectrum. By increasing the values of the standard deviations of the inhibitory components of the butterfly distribution. By increasing the values of the standard deviations of the inhibitory and excitatory Gaussians, \( \sigma_i \) and \( \sigma_e \), the filter can be altered to emphasise the higher frequencies of the input function. The algorithm can be efficiently implemented in DSP hardware and utilised in machine vision applications to detect straight lines in 2D image arrays.

References