A characterization of $P_4$-comparability graphs

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This contribution is dedicated to the memory of Claude Berge, who inspired so many of us.

Abstract

A graph is a $P_4$-comparability graph if it admits an acyclic orientation of its edges which is transitive on every chordless path on four vertices. We give a characterization of $P_4$-comparability graphs in terms of an auxiliary graph being bipartite.

Keywords: Graph orientation; $P_4$-comparability; Recognition; Perfect graph

1. Introduction

An orientation of a graph is a $P_4$-transitive orientation if every $P_4$ (chordless path on four vertices) $abcd$ has either $a \rightarrow b$, $b \leftarrow c$, $c \rightarrow d$ or $a \leftarrow b$, $b \rightarrow c$, $c \leftarrow d$; in other words the orientation is transitive on every $P_4$. A graph is a $P_4$-comparability graph if it admits an acyclic $P_4$-transitive orientation. The $P_4$-comparability graphs were introduced in [10] as a subclass of perfectly orderable graphs. A graph is perfectly orderable [2] if it admits a linear ordering $\prec$ on its vertices such that, for every induced subgraph $H$ of $G$, the greedy coloring algorithm applied on $(H, \prec)$ produces an optimal coloring of the vertices of $H$. Chvátal [2] proved that $\prec$ satisfies this property if and only if no $P_4$ $abcd$ of $G$ has $a \prec b$ and $d \prec c$. See [8] for a survey. Since the recognition of perfectly orderable graphs is NP-complete [12], it is interesting to test the border between polynomially recognizable subclasses of perfectly orderable graphs and the whole class. It was established in [9] and later in [13–16] that $P_4$-comparability graphs form a polynomially recognizable class. The fastest algorithm is in [14,15] and takes $O(|V||E|)$ time to test whether a graph is $P_4$-comparability and, if it is, to find an acyclic $P_4$-transitive orientation in the same time bound.

Recall that a graph is a comparability graph if it admits an acyclic orientation $\rightarrow$ of its edges such that whenever we have arcs $a \rightarrow b$ and $b \rightarrow c$ we also have $a \rightarrow c$. Equivalently, a graph is a comparability graph if it admits an acyclic orientation $\rightarrow$ such that every $P_3$ $abc$ has either $a \rightarrow b$ and $b \leftarrow c$ or $a \leftarrow b$ and $b \rightarrow c$. It follows that comparability graphs are $P_4$-comparability graphs. Comparability graphs have been extensively studied in the literature, following...
the seminal works of Ghouila-Houri [5], Gilmore and Hoffman [6] and especially Gallai [4,11]. In particular, Gallai proved that a graph \( G = (V, E) \) is a comparability graph if and only if a certain auxiliary graph \( G' \) is bipartite. This graph is constructed as follows. Given a vertex \( x \), define a relation \( \wedge_x \) on the set \( E_x \) of those edges of \( E \) that are incident to \( x \) by putting \( xy \wedge_x xz \) whenever \( xy, xz \in E \), \( yz \notin E \); let then \( R_x \) be the transitive closure of \( \wedge_x \) on \( E_x \); so \( R_x \) is an equivalence relation on \( E_x \); let \( k(x) \) be the number of equivalence classes of \( R_x \). Now let \( G' \) have \( k(x) \) copies \( x_1, x_2, \ldots, x_{k(x)} \) of \( x \), and, for every edge \( e = uv (u, v \in V) \) of \( G \), let \( i, j \) be the integers such that \( e \) is in \( i \)th class of \( R_u \) and in the \( j \)th class of \( R_v \), and let \( G' \) have an edge \( u_i v_j \). Gallai proved that:

A graph \( G \) is a comparability graph if and only if \( G' \) is bipartite.

For an example, consider the graph \( G \) shown in Fig. 1. Its auxiliary graph \( G' \) is shown in Fig. 2.

Gallai’s theorem led to a deeper understanding of the structure of comparability graphs and put several researchers on the path to fast algorithms for the recognition of comparability graphs and for their orientation. It also allowed Gallai to find a characterization of comparability graphs by minimal forbidden induced subgraphs. Our aim here is to present a result that is similar in flavor to Gallai’s.

Let \( G = (V, E) \) be an undirected graph. Given a vertex \( x \), we define a relation \( \sim_x \) on the set \( E_x \) of those edges of \( E \) that are incident to \( x \), as follows: put \( xy \sim_x xz \) whenever \( xy, xz \in E \), \( yz \notin E \), and \( yxz \) extends to a \( P_4 \) in \( G \), i.e., there exists a vertex \( u \in V \) such that either \( uyxz \) or \( yxzu \) is a \( P_4 \) in \( G \). Let then \( R_x \) be the transitive closure of \( \sim_x \) on \( E_x \); so \( R_x \) is an equivalence relation on \( E_x \). We can define a graph \( G^* \) as follows. For every vertex \( x \), let \( k(x) \) be the number of equivalence classes of \( R_x \), and let \( G^* \) have \( k(x) \) copies \( x_1, x_2, \ldots, x_{k(x)} \) of \( x \). For every edge \( e = uv (u, v \in V) \), let \( i, j \) be the integers such that \( e \) is in \( i \)th class of \( R_u \) and in the \( j \)th class of \( R_v \), and let \( G^* \) have an edge \( u_i v_j \). Note that the edges of \( G^* \) are in a one-to-one correspondence with those of \( E \). Fig. 3 shows the auxiliary graph \( G^* \) of the graph \( G \) in Fig. 1.
The main results of this paper are the two theorems below.

**Theorem 1.** A graph $G$ is a $P_4$-comparability graph if and only if $G$ contains no antihole on at least seven vertices and $G^*$ is a bipartite graph.

It is easy to see that if $G$ is the antihole on six vertices then $G^*$ is isomorphic to $G$, and thus contains a triangle; and if $G$ is the antihole on $k$ vertices with $k \geq 7$ then $G^*$ consists of an even hole of length $2k$ plus isolated edges. Antiholes with at least seven vertices admit $P_4$-transitive orientations but do not admit acyclic $P_4$-transitive orientations; our next theorem shows that they are the only minimal graphs with this property.

**Theorem 2.** Let $G$ be a graph admitting a $P_3$-transitive orientation. Then $G$ is a $P_4$-comparability graph if and only if $G$ contains no antihole on at least seven vertices.

2. The proofs

We will generally follow the standard terminology from [1]. In addition and for simplicity, we say that a vertex $x$ of a graph sees another vertex $y$ if $x, y$ are adjacent, and we say that $x$ misses $y$ if they are not adjacent. We prove Theorem 2 first.

**Proof of Theorem 2.** The “only if” part is trivial. We will prove the “if” part. Let $G$ be a graph admitting a $P_3$-transitive orientation $T$ and suppose $G$ contains no antihole of length at least seven and is not a $P_4$-comparability graph. Thus, $G$ contains an induced subgraph $H$ that is minimally $P_4$-incomparable, that is, $H$ is not a $P_4$-comparability graph but each of its proper induced subgraphs is. Without loss of generality, we may assume $G$ is minimally $P_4$-incomparable.

From $T$, we define a partial orientation $P$ of $G$ in the following way: for each edge $uv$ that belongs to a $P_4$, let $uv$ have the direction it has in $T$; if $uv$ does not belong to any $P_4$ then it has no direction in $P$. From now on we will only deal with the partial orientation $P$.

Note that if an edge $uv$ of $G$ is oriented (say $u \rightarrow v$), it must extend to a $P_4$ and, considering the orientation of $uv$, this can happen in three different ways: either there exists a $P_4$ $uvw$ with $u \rightarrow v, x \rightarrow v, x \rightarrow y$, and we will say that the edge extends forward; or there exists a $P_4$ $stu$ with $s \rightarrow t, u \rightarrow t, u \rightarrow v$, and we will say that the edge extends backward; or there exists a $P_4$ $xuvw$ with $u \rightarrow x, u \rightarrow v, w \rightarrow v$, and we will say that the edge extends laterally.

To simplify, we will say that a $P_4$ of $G$ with this partial orientation is bad if it is not oriented transitively. Since $T$ is $P_4$-transitive, our partial orientation $P$ does not contain any bad $P_4$. We will prove that this partial orientation has no circuit. Clearly, this suffices to enable us to extend it to an orientation of all edges of $G$ without any circuit. So let us suppose on the contrary that our partial orientation of $P_4$ contains a circuit. Thus, $G$ contains no antihole of length at least seven and is not a $P_4$-comparability graph. Thus, $G$ contains an induced subgraph $H$ that is minimally $P_4$-incomparable, that is, $H$ is not a $P_4$-comparability graph but each of its proper induced subgraphs is.

From $T$, we define a partial orientation $P$ of $G$ in the following way: for each edge $uv$ that belongs to a $P_4$, let $uv$ have the direction it has in $T$; if $uv$ does not belong to any $P_4$ then it has no direction in $P$. From now on we will only deal with the partial orientation $P$.

**Lemma 1.** Any shortest circuit of $G$ induces a clique.

**Proof.** Let $k$ be the smallest integer that is the length of a circuit in $G$, and let $C = v_0v_1 \ldots v_{k-1}v_0$ be any circuit of length $k$, with arcs $v_0 \rightarrow v_1, \ldots, v_{k-2} \rightarrow v_{k-1}$, and $v_{k-1} \rightarrow v_0$ (subscripts are understood modulo $k$). If $k = 3$ the lemma holds trivially, so suppose $k \geq 4$. Observe that any chord of $C$ is a non-oriented edge, for otherwise there would be a shorter circuit; hence any chord of $C$ lies in no $P_4$ (it is an isolated edge of $G^*$).

**Claim 1.** If $uvw$ is a $P_4$, and $w$ sees $u$ and misses $v$, then the edge $uw$ forms a $P_4$ together with some edge of $uvw$. If $u \rightarrow v$ then $u \rightarrow w$; otherwise $w \rightarrow u$.

**Proof.** If $w$ misses $x$ then $uwvx$ is a $P_4$, and the lemma holds because there is no bad $P_4$. If $w$ sees $x$ and misses $y$, then $uwxy$ is a $P_4$ and the lemma holds similarly. If $w$ sees $x$ and $y$ then $uwxy$ is a $P_4$ and the lemma holds again. 

**Claim 2.** For each $i$ modulo $k$, $v_iv_{i+2}$ is an edge of $G$. 


Proof. Suppose without loss of generality that $v_0v_2$ is not an edge. As $v_1v_2$ is oriented, it must extend to a $P_4$; this can be done in three different ways.

(1) The edge $v_1v_2$ extends forward, along a $P_4$ $v_1v_2ab$, with $a \rightarrow v_2$ and $a \rightarrow b$. Here $av_0$ is an edge, or else $v_0v_1v_2$ is a bad $P_4$. Then $bv_0$ is an edge, or else $baav_0$ would be a bad $P_4$. But then $bv_0v_1v_2$ is a bad $P_4$.

(2) The edge $v_1v_2$ extends laterally, along a $P_4$ $av_1v_2b$, with $v_1 \rightarrow a$ and $b \rightarrow v_2$. Here $bv_0$ must be an edge, or else $v_0v_1v_2b$ is a bad $P_4$; then $av_0$ must be an edge, or else $bav_0v_1$ is a bad $P_4$. But now, $v_2bv_0a$ is a $P_4$, which implies $a \rightarrow v_0$, and so $v_0v_1$, $a$ form a circuit of length three, a contradiction to $k \geq 4$.

(3) The edge $v_1v_2$ extends backward, along a $P_4$ $abv_1v_2$ with $a \rightarrow b$ and $v_1 \rightarrow b$. By symmetry we may assume that $v_0v_1$ extends forward, along a $P_4$ $v_0v_1cd$, with $c \rightarrow v_1$ and $c \rightarrow d$. We may assume that $v_1v_k$ is an edge, for otherwise, shifting all subscripts by $-1$, we are as in case (1) above. Likewise we may assume that $v_1v_3$ is an edge. Here $av_0$ is not an edge, or else $av_0v_1v_2$ would be a bad $P_4$. Likewise $dv_2$ is not an edge. Then $bv_0$ is an edge, or else $v_0v_1ba$ would be a bad $P_4$. Likewise $cv_2$ is an edge. Suppose that $v_3c$ is not an edge. Then $vd_3$ is an edge, or else $v_3v_2cd$ is a bad $P_4$. Then $v_3v_0$ is an edge, or else the chord $v_3v_1$ of the circuit lies in the $P_4$ $d_3v_3v_1v_0$, a contradiction. Now the $P_4$ $cv_2v_3v_0$ implies $v_0 \rightarrow v_3$, and we find a shorter circuit, a contradiction. Thus $cv_3$ is an edge. Likewise $bv_{k-1}$ is an edge. Suppose that $v_{k-1}v_2$ is not an edge. Then $v_{k-1}d$ is not an edge, or else $dav_{k-1}v_2$ is a $P_4$, implying an orientation on the chord $v_{k-1}v_1$. Then $v_{k-1}e$ is an edge, or else $v_{k-1}v_1cd$ is a $P_4$, implying an orientation on the chord $v_{k-1}v_1$. But now $dcv_{k-1}v_0$ is a bad $P_4$. Therefore, $v_{k-1}v_2$ is an edge. Note that we do not have $v_{k-1} \rightarrow v_2$, for this would imply $k \geq 5$, and then $C - \{v_1, v_1\}$ would induce a shorter circuit. Now $av_{k-1}$ is an edge, or else $abv_{k-1}v_2$ would be a $P_4$, implying $v_{k-1} \rightarrow v_2$. Likewise $v_0v_3$ and $v_3d$ are edges. The edge $v_{k-1}v_0$ must extend to a $P_4$, and there are three ways to do this:

(a) $v_{k-1}v_0$ extends forward, along a $P_4$ $v_{k-1}v_0ef$ with $e \rightarrow v_0$ and $e \rightarrow f$. By Claim 1 applied to $v_{k-1}v_0ef$ and $v_2$ we have $v_{k-1} \rightarrow v_2$, a contradiction.

(b) $v_{k-1}v_0$ extends laterally, along a $P_4$ $fv_{k-1}v_0e$ with $e \rightarrow v_0$ and $v_{k-1} \rightarrow f$. Here $ev_2$ is an edge, for otherwise $v_{k-1}v_0ef$ is a $P_4$ implying $v_{k-1} \rightarrow v_2$, a contradiction. Then $fv_2$ is an edge, or else $fsv_{k-1}v_2$ is a $P_4$, implying $v_{k-1} \rightarrow v_2$ again. Then $v_0ev_2f$ is a $P_4$, implying $f \rightarrow v_2$. But we find a circuit $fv_2v_3 \cdots v_{k-1}f$ shorter than $C$, a contradiction.

(c) $v_{k-1}v_0$ extends backward, along a $P_4$ $fsv_{k-1}v_0$ with $f \rightarrow e$ and $v_{k-1} \rightarrow e$. Here $fv_2$ is not an edge, or else $v_{k-1}v_0ve$ is a $P_4$, implying $v_{k-1} \rightarrow v_2$. Then $ev_2$ is an edge, or else $fsv_{k-1}v_2$ is a $P_4$, implying $v_{k-1} \rightarrow v_2$. If $ev_1$, $eb$ are not edges then $bav_2ve$ is a $P_4$, implying $e \rightarrow v_2$, and then $ev_2v_3 \cdots v_{k-1}e$ would be a circuit shorter than $C$. If $ev_1$ is not an edge and $eb$ is an edge then either (if $ae$ is not an edge) $abev_2$ is a $P_4$ implying again $e \rightarrow v_2$, or (if $ae$ is an edge) $aev_2v_1$ is a $P_4$ implying again $e \rightarrow v_2$. Hence $ev_1$ is an edge. Then $fv_1$ is an edge, or else $fev_1v_0$ is a bad $P_4$. Then $fa$ is an edge, or else $fsv_{k-1}v_0a$ is a $P_4$, implying an orientation on the chord $v_1v_{k-1}$. Now $fsv_{k-1}v_0$ is a $P_4$, implying $f \rightarrow a$, and $afv_1v_2$ is a bad $P_4$. This ends the proof of Claim 2. □

Claim 3. Any two vertices $v_i, v_j$ of the circuit are adjacent in $G$.

Proof. We prove this claim by induction on the value of $|j - i| \mod k$. If this value is 1, this is Claim 2. Now suppose without loss of generality that $v_0v_i$ is not an edge and that $v_iv_j$ is an edge whenever $|r - s| < i$, with $3 \leq i \leq k - 3$. The edge $v_0v_1$ must extend to a $P_4$, and there are three ways to do this:

(1) $v_0v_1$ extends backward, along a $P_4$ $abv_0v_1$ with $a \rightarrow b$ and $v_0 \rightarrow b$. By Claim 1 applied to $v_0v_0ba$ and $v_1$, we obtain that the chord $v_1v_1$ is in a $P_4$, a contradiction.

(2) $v_0v_1$ extends laterally, along a $P_4$ $av_0v_1b$ with $v_0 \rightarrow a$ and $b \rightarrow v_1$. If $v_1$ misses one of $a, b$, it is easy to check that $v_1v_1$ lies in a $P_4$ together with two vertices from $a, b, v_0$, a contradiction. Thus $av_1$ and $bv_1$ are edges. Then $v_0av_1v_2$ is a $P_4$, implying $v_1 \rightarrow b$. But now $bv_1v_2 \cdots v_2b$ is a shorter circuit.

(3) $v_0v_1$ extends forward, along a $P_4$ $v_0v_1ab$ with $a \rightarrow v_1$ and $a \rightarrow b$. Then $bv_1$ is not an edge, or else $v_0v_1v_2b$ is a $P_4$ implying an orientation on the chord $v_1v_1$. Then $bv_1$ is an edge, or else $baav_1v_2$ is a $P_4$ implying again an orientation on $v_1v_1$. By symmetry, we may assume that the edge $v_1v_1$ extends backward, along a $P_4$ $cdv_1v_1$ with $c \rightarrow d$ and $v_1v_1 \rightarrow d$. By symmetry $v_0$ misses $c$ and sees $d$. Here $av_1$ is an edge, or else $av_1v_1v_2$ would be a $P_4$, implying an orientation on the chord $v_1v_1$. Likewise $dv_1$ is an edge. Then $bv_1v_1$ is an edge, or else $v_0v_1v_1ab$ would be a $P_4$, implying an orientation on $v_0v_1v_1$. Likewise $cv_1$ is an edge. Then $bc$ is an edge, or else $v_1v_1v_1c$ would be a $P_4$, implying an orientation on the chord $v_1v_1$ (recall $i \geq 3$). Now $cbv_1v_0$ is a $P_4$, implying an orientation on the chord $v_0v_1$, a contradiction. This ends the proof of Claim 3. □
By Claim 3, Lemma 1 is proved. □

Recall that $k$ is the length of a shortest circuit $v_0, v_1 \ldots v_{k-1}$ in the graph.

**Lemma 2.** Suppose $k \geq 4$. Then every edge $v_i v_{i+1}$ of the circuit satisfies either:

- There exists a $P_4 v_i v_{i+1} b_i a_i$ such that $a_i$ sees every vertex of the circuit except $v_i$ and $v_{i+1}$.
- There exists a $P_4 c_i d_i v_i v_{i+1}$ such that $c_i$ sees every vertex of the circuit except $v_i$ and $v_i+1$.
- There exists a $P_4 e_i v_i v_{i+1} f_i$ such that $e_i$ sees every vertex of the circuit except $v_{i+1}$, and $f_i$ sees every vertex of the circuit except $v_i$.

**Proof.** If $v_i v_{i+1}$ extends forward, there exists a $P_4 v_i v_{i+1} b_i a_i$, with $b_i \rightarrow a_i$ and $b_i \rightarrow v_{i+1}$. Suppose that $a_i$ misses some vertex $v_j$ of the circuit, with $j \neq i, i + 1$. Then $v_j b_i$ is an edge, or else $v_j v_{i+1} b_i$ is a $P_4$ implying $v_j \rightarrow v_{i+1}$, a contradiction. Then $v_i v_j b_i a_i$ is a $P_4$ implying $v_i \rightarrow v_j$, a contradiction. So we obtain the desired property.

If $v_i v_{i+1}$ extends backward, the proof is similar.

Assume now that $v_i v_{i+1}$ extends laterally, along a $P_4 e_i v_i v_{i+1} f_i$ with $v_i \rightarrow e_i$ and $f_i \rightarrow v_{i+1}$. Remark that every vertex $v_j$ of the circuit, with $j \neq i, i + 1$, either sees both $e_i$, $f_i$ or misses both $e_i$, $f_i$; indeed, in the opposite case, either $e_i v_j v_{i+1} f_i$ is a $P_4$ implying $v_j \rightarrow v_{i+1}$, which is impossible since $j \neq i$, or $f_i v_j v_i e_i$ is a $P_4$ implying $v_i \rightarrow v_j$, which is impossible since $j \neq i + 1$. Suppose that some vertex $v_j$, with $j \neq i, i + 1$, is not adjacent to $e_i$ (and hence also not to $f_i$). To simplify notation we fix $i = 1$. First suppose $j = 3$. We distinguish between the three cases corresponding to how the edge $v_2 v_3$ extends.

(a) $v_2 v_3$ extends forward, along a $P_4 v_2 v_3 b_2 a_2$ with $b_2 \rightarrow a_2$ and $b_2 \rightarrow v_3$. Vertex $f_1$ misses $a_2$, or else $a_2 f_1 v_2 v_3$ is a bad $P_4$. Then $f_1$ sees $b_2$, or else $a_2 b_1 f_1 v_2$ is a bad $P_4$. But then $f_1 v_2 v_3 b_2$ is a bad $P_4$.

(b) $v_2 v_3$ extends backward, along a $P_4 c_2 d_2 v_2 v_3$, with $c_2 \rightarrow d_2$ and $v_2 \rightarrow d_2$. Vertex $f_1$ misses $c_2$, or else $c_2 f_1 v_2 v_3$ is a bad $P_4$. Then $f_1$ sees $d_2$, or else $f_1 v_2 d_2 c_2$ is a bad $P_4$. Then $v_1 d_2$ is not an edge, or else $f_1 d_2 v_1 v_3$ would be a $P_4$ implying an orientation on the chord $v_1 v_3$. Then $e_1 d_2$ is an edge, or else $e_1 v_1 v_2 d_2$ would be a bad $P_4$. But now one of $e_1 d_2 f_1$ and $e_1 d_2 v_3$ is a bad $P_4$.

(c) $v_2 v_3$ extends laterally, along a $P_4 e_2 v_2 v_3 f_2$ with $v_2 \rightarrow e_2$ and $f_2 \rightarrow v_3$. Here $f_1 f_2$ is an edge, or else $f_1 v_2 v_3 f_2$ is a bad $P_4$. Then $e_2 f_1$ is an edge, or else $e_2 v_2 f_1 f_2$ is a bad $P_4$. Now $e_2 f_1 v_2 v_3$ is a $P_4$ implying $e_2 \rightarrow f_1$, and so $e_2, f_1, v_2$ induce a circuit of length three, a contradiction.

Now we may assume that $j \neq 3$, and, by symmetry, $j \neq 0$. Choose $j$ to be the smallest subscript such that $v_j$ misses $f_1$ ($4 \leq j \leq k - 1$). Suppose that $v_{j-1} v_j$ extends forward into a $P_4 v_{j-1} v_j b_{j-1} a_{j-1}$. Note that $a_{j-1}$ sees all vertices of $C$ except $v_{j-1}$, $v_j$ as shown in the first paragraph of this proof. Then vertex $f_1$ misses $a_{j-1}$ for otherwise the chord $v_1 v_j$ of the circuit belongs to the $P_4 f_1 a_{j-1} v_j v_{j-1}$, a contradiction. But now the chord $v_1 v_{j-1}$ belongs to the $P_4 f_1 v_{j-1} v_j a_{j-1}$, a contradiction. Similarly, $v_{j-1} v_j$ cannot extend backward. So we know that $v_{j-1} v_j$ extends laterally into a $P_4 e_{j-1} v_{j-1} v_j f_j$. Then $f_1 f_j$ must be an edge, for otherwise the $P_4 f_j v_j v_{j-1} f_1$ implies $v_{j-1} \rightarrow f_1$, and thus there is a circuit $v_2 v_3 \ldots v_{j-1} f_1 v_2$, of length $j - 1 < k$, a contradiction. Then $f_1 e_{j-1}$ must be an edge, for otherwise the $P_4 e_{j-1} v_{j-1} f_1 f_j$ implies $v_{j-1} \rightarrow f_1$ again. But then the $P_4 v_j f_j f_1 e_{j-1}$ implies $e_{j-1} \rightarrow f_1$, and so there is a circuit $v_2 v_3 \ldots v_{j-1} e_{j-1} f_1 f_2$, of length $j < k$, a contradiction. □

Among all circuits of length $k$ in $G$, we choose a circuit $C$ that has as many edges extending forward or backward as possible. (If an edge of the circuit can extend laterally and also forward or backward, it is understood that we count it as extending forward or backward.)

**Lemma 3.** Suppose $k \geq 4$. Then $G$ contains an antihole.

**Proof of Lemma 3.** For each edge $v_i v_{i+1}$ of the circuit $C$, select either one vertex $x_i$ ($x_i = a_i$ if the edge extends forward; $x_i = e_i$ if the edge extends backward) or the two vertices $e_i$, $f_i$ (if the edge extends laterally) given by Lemma 2, whichever applies. For simplicity we call these the “selected vertices”. Recall that $x_i$ sees all vertices of the circuit except $v_i$ and $v_{i+1}$, while $e_i$ sees all vertices of the circuit except $v_{j+1}$, and $f_i$ sees all vertices of the circuit except $v_j$. Let $H$ be the subgraph of $G$ induced by all the vertices of the circuit and all the selected vertices. Consider the vertices $v_0, \ldots, v_{k-1}, v_0$ of the circuit listed in this order and, between $v_i$ and $v_{i+1}$, insert the corresponding selected vertices: either $x_i$ or the two vertices $f_i, e_i$ (whichever applies). Doing this for every $i$, we obtain a (cyclic) ordering of
the vertices of $H$ such that any two consecutive vertices in that ordering are non-adjacent in $G$, i.e., we obtain a cycle in the complement $\overline{G}$ of $G$. This cycle of $\overline{G}$ has length at least $2k$. Our aim now is to show that this is actually an antihole of $G$. To prove this, since the vertices of the circuit induce a clique, and because of the properties stated in Lemma 2, we need only show that distinct selected vertices are pairwise adjacent (except, of course, for the non-adjacent pairs of type $e_i f_j$). We have not assumed that all the selected vertices are pairwise different. However, because of the adjacencies between the vertices of $C$ and the selected vertices, as stated in Lemma 2, we know that the only equalities that could possibly occur between the selected vertices are $e_i = f_{i+1}$ if these vertices are defined (i.e., if both $v_i v_{i+1}$ and $v_{i+1} v_{i+2}$ extend only laterally). But that is also not possible:

Claim 4. We never have $e_i = f_{i+1}$ (whenever these vertices are selected).

Proof. Suppose $e_i = f_{i+1}$. Then replacing $v_{i+1}$ by $e_i$ in $C$ we get a circuit $C'$ of length $k$. In $C'$, the edge $v_i e_i$ extends backward (because of $f_{i+1} v_{i+1} v_i e_i$). So $C'$ has more edges extending forward or backward than $C$, a contradiction to the choice of $C$.

Claim 5. If $x_i$ and $x_j$ are selected vertices ($i \neq j$), then $x_i x_j$ is an edge.

Proof. If $x_i x_j$ is not an edge, then at least one of the three sets $\{x_i, v_i, v_j, x_j\}$, $\{x_i, v_{i+1}, v_j, x_j\}$, $\{x_i, v_{i+1}, v_j, x_j\}$ induces a $P_4$ that contains an orientation on the chord of the circuit. This proves the claim.

Claim 6. If $x_i$ and $e_j$, $f_j$ are selected vertices, with $i \neq j$, then $x_i$ sees $e_j$ and $f_j$.

Proof. Vertex $x_i$ sees $e_j$ when $j \neq i - 1$ and $j \neq i - 2$, for otherwise $x_i v_{j+1} v_i e_j$ is a $P_4$ implying an orientation on the chord $v_{j+1} v_i$. Vertex $x_i$ also sees $e_{i-2}$, for otherwise the $P_4 x_i v_{i-1} v_{i+1} e_{i-2}$ implies an orientation on the chord $v_{i-1} v_{i+1}$, a contradiction. By symmetry $x_i$ sees $f_j$ when $j \neq i + 1$. Now we show that $x_i$ also sees $f_{i+1}$. If not, then $x_i e_{i+1} v_i f_{i+1}$ is a $P_4$. We cannot have $e_{i+1} \rightarrow v_i$, for then $e_{i+1}, v_i, v_{i+1}$ would induce a circuit of length three. Thus on the $P_4 x_i e_{i+1} v_i f_{i+1} v_i$ we must have $v_i \rightarrow f_{i+1}$. Now replacing in $C$ the vertex $v_{i+1}$ by $f_{i+1}$ we obtain a new circuit $C'$ of length $k$. Along $C'$ the edge $v_i f_{i+1}$ extends backward (because of $x_i e_{i+1} v_i f_{i+1}$), and the edge $f_{i+1} v_{i+2}$ extends forward (because of $f_{i+1} v_{i+2} v_{i+1} e_{i+1}$); this contradicts the choice of $C$. So $x_i$ sees $f_{i+1}$. Likewise, $x_i$ sees $e_{i-1}$. This proves the claim.

Claim 7. If $e_i$, $f_i$ and $e_j$, $f_j$ are two pairs of selected vertices, with $i \neq j$, then each of $e_i$, $f_i$ sees each of $e_j$, $f_j$.

Proof. First suppose $j = i + 1$. If $e_i e_{i+1}$ is not an edge, then $e_i v_{i+2} v_{i+1} e_{i+1}$ is a $P_4$ implying $e_i \rightarrow v_{i+2}$. But then, replacing $v_{i+1}$ by $e_i$ in $C$, we obtain a circuit $C'$ of length $k$, in which the edge $v_i e_i$ extends backward (because of $f_{i+1} v_{i+1} v_i e_i$), contradicting the choice of $C$ (as the two edges $v_i v_{i+1}, v_{i+1} v_{i+2}$ of $C$, which extend laterally only, have been replaced by $v_i e_i, e_i v_{i+2}$, of which at least one extends forward or backward). Thus $e_i e_{i+1}$ is an edge. Likewise $f_i f_{i+1}$ is an edge. If $e_i f_{i+1}$ is not an edge, then $e_{i+1} e_i v_{i+2} f_{i+1}$ is a $P_4$ implying $e_{i+1} \rightarrow v_{i+2}$, which is a contradiction exactly as above. So $e_i f_{i+1}$ is an edge. Finally, $f_i e_{i+1}$ is an edge, or else $f_{i+1} v_{i+2} v_{i+1} e_i$ would be a $P_4$ implying an orientation on the chord $v_i v_{i+2}$. The proof is similar if $j = i - 1$. Now suppose $j \neq i - 1, i + 1$. Assume $e_i f_j$ is not an edge. Then $e_i v_{j+1} v_i f_j$ is a $P_4$ implying an orientation of $v_{j+1} v_i$. So it must be that $j = i + 2$, and the orientation is such that $e_i \rightarrow v_j$. Now, replacing $v_{i+1}$ by $e_i$ in $C$, we obtain a circuit $C'$ of length $k$. In $C'$, the two edges $v_i e_i$ and $e_i v_j$ extend, respectively, backward (along $f_{i+1} v_{i+1} e_i$) and forward (along $e_i v_{j+1} v_{i+1} f_j$), contradicting the choice of $C$. So $e_i f_j$ is an edge. Then $e_i e_j$ too must be an edge, for otherwise the $P_4 e_i v_{j+1} v_{i+1} e_j$ implies a direction on the chord $v_{i+1} v_{i+1}$ (recall that $j \neq i + 1, j \neq i - 1$.) We have shown $e_i$ sees $f_j, e_j$. By symmetry, $f_j$ sees $e_i, f_i$. So the claim is proved. Now the four preceding claims imply that $H$ induces an antihole.

We call directed pyramid the graph featured in Fig. 4.

Lemma 4. Suppose that $G$ has a circuit of length three. Then $G$ contains an antihole or a directed pyramid as in Fig. 4.
Proof of Lemma 4. We assume that none of the conclusions of the lemma hold. We choose a directed triangle $T$ that has the most edges extending forward or backward. Let $T$ be on vertices $v_0, v_1, v_2$ with $v_i \rightarrow v_{i+1}$ ($i = 0, 1, 2$, all subscripts being modulo 3). We will always use the notation given in Lemma 2. We first establish some technical facts.

Claim 8. If $v_i v_{i+1}$ extends forward, then $a_i v_{i+2}$ is an edge. If $v_i v_{i+1}$ extends backward, then $c_i v_{i+2}$ is an edge.

Proof. To prove the first part of the claim, suppose $a_i v_{i+2}$ is not an edge. Then either $a_i b_i v_{i+1} v_{i+2}$ is a bad $P_4$ (if $b_i$ misses $v_{i+2}^+$), or $a_i b_i v_{i+2} v_i$ is a bad $P_4$ (if $b_i$ sees $v_{i+2}$), a contradiction. The second part of the claim also holds true, by symmetry.

Claim 9. If $v_i v_{i+1}$ extends laterally, then $v_{i+2}$ either sees both or misses both $e_i, f_i$.

Proof. In the opposite case, one of $e_i v_{i+2} v_{i+1} f_i$ or $e_i v_i v_{i+2} f_i$ is a bad $P_4$.

Claim 10. If $v_i v_{i+1}$ extends laterally, and $v_{i+1} v_{i+2}$ extends forward, then $e_i \neq a_{i+1}$, and $v_{i+2}$ sees both $e_i, f_i$.

Proof. First suppose that $e_i = a_{i+1}$. By Claim 9, $v_{i+2} f_i$ is not an edge. Then $f_i b_{i+1}$ is an edge, or else $f_i v_{i+1} v_{i+2} b_{i+1}$ would be a bad $P_4$. But then $v_{i+1} f_i b_{i+1} a_{i+1}$ is a bad $P_4$, a contradiction. So $e_i \neq a_{i+1}$. Now suppose that $v_{i+2}$ misses $f_i$ (for otherwise we are done by Claim 9). Then $f_i b_{i+1}$ is an edge, or else $f_i v_{i+1} v_{i+2} b_{i+1}$ would be a bad $P_4$. Then $f_i a_{i+1}$ is an edge, or else $a_{i+1} b_{i+1} f_i v_{i+1}$ would be a bad $P_4$. But then $a_{i+1} f_i v_{i+1} v_{i+2}$ is a bad $P_4$.

Claim 11. If $v_i v_{i+1}$ extends laterally, and $v_{i+1} v_{i+2}$ extends backward, then $e_i \neq c_{i+1}$.

Proof. Suppose on the contrary that $e_i = c_{i+1}$. By Claim 9, $v_{i+2} f_i$ is not an edge. Then $f_i d_{i+1}$ is an edge, or else $f_i v_{i+1} d_{i+1} e_i$ would be a bad $P_4$. Then $v_{i+1} d_{i+1}$ is an edge, or else $f_i d_{i+1} e_i v_i$ would be a bad $P_4$. But now the six vertices induce a directed pyramid.

Claim 12. If $v_0 v_1$ and $v_1 v_2$ extend laterally, then $v_2$ sees $e_0$ and $f_0$, and $v_0$ sees $e_1$ and $f_1$.

Suppose that this claim is false: by symmetry and by Claim 9 we may assume that $v_2$ misses $e_0$ and $f_0$ (thus $e_0 \neq f_1$). Then $f_0 f_1$ is an edge, or else $f_0 v_1 v_2 f_1$ is a bad $P_4$. Then $f_0 e_1$ is an edge, or else $f_1 v_0 v_1 e_1$ is a bad $P_4$. But now the $P_4 e_1 f_0 f_1 v_2$ implies $e_1 \rightarrow f_0$, and $f_0, e_1, v_1$ form a circuit $T'$. However, all edges of $T'$ extend forward or backward (along $f_0 v_1 v_0 e_0, e_1 v_1 v_2 f_1$ and $e_1 f_0 f_1 v_2$), a contradiction to the choice of $T$.

Claim 13. If $v_0 v_1$ and $v_1 v_2$ extend laterally, then $e_0 \neq f_1$, and each of $e_0, f_0$ sees each of $e_1, f_1$.

Proof. Note that we have the conclusion of the preceding claim. If $e_0 = f_1$, then $e_0, v_2, v_0$ induce a circuit of length three, of which at least two edges extend forward or backward (along $e_0 v_0 v_1 f_0$ and $e_0 v_2 v_1 e_1$), a contradiction to the choice of $T$. If $f_0 e_1$ is not an edge, then $f_0 v_2 v_0 e_1$ is a $P_4$, implying $e_1 \rightarrow v_0$, and then $e_1, v_0, v_1$ form a circuit with two edges extending forward or backward (along $f_0 v_2 v_1 e_1$ and $e_1 v_1 v_2 f_1$), a contradiction. So $f_0 e_1$ is an edge. If $e_0 e_1$ is not an edge, then $e_0 v_0 e_1 f_0$ is a $P_4$, which implies $f_0 \rightarrow v_1$; then $e_1 f_0 v_2 e_0$ is a $P_4$, which implies $e_0 \rightarrow v_2$. Now $e_0, v_2, v_0$ form a circuit, with at least two edges extending forward or backward (along $e_0 v_0 v_1 f_0$ and $e_1 f_0 v_2 e_0$), a contradiction.
to the choice of $T$. So $e_0e_1$ is an edge. Likewise $f_0f_1$ is an edge. If $e_0f_1$ is not an edge, then $e_1e_0v_2f_1$ is a $P_4$, implying again $e_0 \rightarrow v_2$, a contradiction as above. This completes the proof of the claim. Finally, suppose $e_0, f_1$ is not an edge. The $P_4 f_1v_2e_0e_1$ implies $e_0 \rightarrow v_2, e_0 \rightarrow e_1$. The circuit $e_0v_2v_0$ has one edge $(e_0v_0)$ extending backward. Therefore, the circuit $v_0v_1v_2$ must have one edge extending forward or backward, and this edge can only be $v_0v_2$. But then the circuit $e_0v_2v_0$ has two edges extending forward or backward, a contradiction. □

Now we go on with the proof of Lemma 4, distinguishing between cases.

Case 1. All three edges of $T$ extend forward or backward. For $i = 0, 1, 2$, if $v_i v_{i+1}$ extends forward, set $x_i = a_i$; if it extends backward, set $x_i = c_i$; in either case we know that $x_i$ sees $v_{i+2}$. Call $H$ the subgraph of $G$ induced by the six vertices $v_0, v_1, v_2, x_0, x_1, x_2$, and call $X$ the subgraph of $G$ induced by $x_0, x_1, x_2$. Now it is easy to check that: if $X$ has zero edge then $G^*$ contains a triangle; if $X$ has exactly one edge then $G^*$ contains a 5-cycle; if $X$ has two or three edges then $G$ contains an antihole of length five or six. In either case we have a contradiction.

Case 2. Exactly one edge of $T$ extends laterally. Assume that $v_2v_0$ extends laterally, with the usual notation. Each of the edges $v_0v_1$ and $v_1v_2$ must extend forward or backward, which, by symmetry, leads to three subcases.

Subcase 2.1. $v_0v_1$ extends forward and $v_1v_2$ backward. As usual we have $P_4$’s $v_0v_1b_0a_0$ and $c_1d_1v_1v_2$. By Claim 8, $a_0v_2$ and $c_1v_0$ are edges. By Claim 10, $v_1e_2$ and $v_1f_2$ are edges. Suppose $a_0f_2$ is not an edge. Then $f_2v_1v_2a_0$ is a $P_4$, implying $v_1 \rightarrow f_2$. Then $f_2, v_0, v_1$ induce a circuit whose three edges extend forward or backward (along $f_2v_0v_2e_2, v_0v_1b_0$ and $f_2v_1v_2a_0$), a contradiction. Thus $a_0f_2$ is an edge. Likewise $c_1v_2$ is an edge.

Suppose $a_0e_2$ is not an edge. Then $a_0f_2v_1e_2$ is a $P_4$. If this $P_4$ is oriented in such a way that $e_2 \rightarrow v_1$, then $e_2, v_1, v_2$ would be a circuit whose three edges extend forward or backward (along $c_1d_1v_1v_2, e_2v_0v_2f_2$ and $a_0f_2v_1e_2$), a contradiction. Thus the $P_4 a_0f_2v_1e_2$ is oriented in such a way that $a_0 \rightarrow f_2$. But then $f_2a_0v_2e_2$ is a bad $P_4$. It follows that $a_0e_2$ is an edge. Likewise, $c_1f_2$ is an edge. Now $a_0f_1$ is an edge, or else the six vertices $v_0, v_1, e_2, v_2, c_1$ would induce an antihole, a contradiction. But then, the seven vertices $a_0, v_0, e_2, v_2, c_1, v_1$ induce an antihole, a contradiction.

Subcase 2.2. $v_0v_1$ extends backward and $v_1v_2$ forward. We have $P_4$’s $c_0d_0v_0v_1$ and $v_1v_2b_1a_1$. By Claim 11 we have $e_2 \neq c_0$, and similarly $f_2 \neq a_1$. By Claim 8, $c_0v_2$ and $a_1v_0$ are edges. Suppose that $v_1$ sees both $e_2, f_2$. Then $f_2b_2$ is an edge, for otherwise $f_2v_1v_2b_1$ would be a $P_4$ implying $v_1 \rightarrow f_2$, and thus $v_0, v_1, f_2$ would induce a circuit whose three edges extend forward or backward (along $c_0d_0v_0v_1, f_2v_0v_2e_2$ and $f_2v_1v_2b_1$), a contradiction. Likewise $e_2d_0$ is an edge. Then $f_2a_1$ is an edge, or else $v_1f_2b_1a_1$ is a $P_4$ implying $v_1 \rightarrow f_2$, and thus $v_0, v_1, f_2$ would be a circuit whose three edges extend forward or backward (along $c_0d_0v_0v_1, f_2v_0v_2e_2$ and $v_1f_2b_1a_1$), a contradiction. Likewise $e_2d_0$ is an edge. Then $f_2e_0$ is an edge, or else $f_2v_1v_2c_0$ would be a $P_4$ implying again that $v_0, v_1, f_2$ form a circuit contradicting the choice of $T$. Likewise $e_2a_1$ is an edge. But now the vertices $c_0, v_0, e_2, v_2, a_1, v_1$ induce a subgraph that contains an antihole of length six (if $a_1c_0$ is not an edge) or seven (if $a_1c_0$ is an edge), a contradiction. The conclusion of this paragraph (with Claim 9) is that $v_1$ misses both $e_2, f_2$. Now $b_1v_0$ must be an edge, or else either $b_1v_2v_0f_2$ is a bad $P_4$ (if $b_1$ misses $e_2$), or $b_1f_2v_0v_1$ is a bad $P_4$ (if $b_1$ sees $f_2$). Likewise $d_0v_2$ is an edge. Then $c_0f_2$ is not an edge, or else $c_0f_2v_0v_1$ would be a bad $P_4$. Likewise $a_1e_2$ is not an edge. Then $d_0f_2$ is an edge, or else $d_0v_2d_0c_0$ is a bad $P_4$. Likewise $b_1e_2$ is an edge. Since $v_1v_2d_0f_2$ is a $P_4$, we have $d_0 \rightarrow v_2$. Then $d_0e_2$ is an edge, or else $d_2v_0d_0v_2e_2$ would be a bad $P_4$. But now the vertices $v_0, v_1, d_0, e_2, f_2$ induce a directed pyramid.

Subcase 2.3. Both $v_0v_1$ and $v_1v_2$ extend forward. Thus we have $P_4$’s $v_0v_1b_0a_0$ and $v_1v_2b_1a_1$. By Claim 8, $a_0v_2$ and $a_1v_0$ are edges. By Claim 10, $v_1$ sees $e_2$ and $f_2$. Suppose $a_1e_2$ is not an edge. Then $a_1v_0v_1e_2$ is a $P_4$, implying $e_2 \rightarrow v_1$; therefore $e_2, v_1, v_2$ form a circuit, whose three edges extend forward or backward (along $a_1v_0v_1e_2, e_2v_2v_0f_2$ and $v_1v_2b_1a_1$), a contradiction. Thus $a_1e_2$ is an edge. Suppose $a_0a_1$ is not an edge. Then $a_0v_2v_0d_0$ is a $P_4$ implying $v_2 \rightarrow a_0$. Then $a_0f_2$ is an edge, or else $a_0v_2v_1f_2$ is a bad $P_4$. If $a_0e_2$ is not an edge, then $e_2v_0v_2f_2$ and $a_0f_2v_1e_2$ are $P_4$’s implying $f_2 \rightarrow e_0$ and $e_0 \rightarrow v_1$; but then $e_2, v_1, v_2$ form a circuit whose three edges extend forward or backward (along $a_0f_2v_1e_2, e_2v_0v_2f_2$ and $v_1b_1a_1$), a contradiction. Thus $a_0e_2$ is an edge. But now the subgraph induced by $a_0, a_1, v_1, f_2$ contains an antihole of length five or six, a contradiction. Therefore $a_0d_1$ is an edge. It follows that $a_1d_0v_2v_1$ is a $P_4$, implying $a_0 \rightarrow v_2$. Then $a_0f_2$ is an edge, or else $a_0v_2v_0f_2$ would be a bad $P_4$. Consider the subgraph $H$ induced by the seven vertices $a_0, v_1, a_1, v_2, e_2, v_0$; its complement $H$ is a cycle, whose only possible chords (in $G$) are $a_0e_2$ and $a_1f_2$; but, whichever are chords or not, $H$ contains an antihole of length five, six or seven, a contradiction.

Subcase 2.4. Both $v_0v_1$ and $v_1v_2$ extend backward. This case is similar to Subcase 2.3, by symmetry, and we omit its proof.
Case 3. Exactly two edges of $T$ extend laterally. Let us assume that $v_0v_1$ and $v_1v_2$ extend laterally, with the usual notation. By Claims 12 and 13, vertex $v_0$ sees $e_1$ and $f_1$, vertex $v_2$ sees $e_0$ and $f_0$, we have $e_0 \neq f_1$, and $f_0e_1$, $e_0e_1$, $f_0f_1$, and $e_0f_1$ are edges. By symmetry we may assume that $v_2v_0$ extends forward, along a $P_4$-$v_2v_0b_2a_2$. Here $v_1a_2$ is an edge, for otherwise either $a_2b_2v_0v_1$ or $a_2b_2v_1v_2$ is a bad $P_4$. Then $a_2f_1$ is an edge, or else either $a_2v_1v_2f_1$ or $a_2v_1v_0f_1$ is a bad $P_4$. Likewise $a_2e_0$ is an edge, or else either $a_2v_1v_2e_0$ or $a_2v_1v_0e_0$ is a bad $P_4$. Now, the eight vertices $v_0$, $v_1$, $v_2$, $e_0$, $f_0$, $e_1$, $f_1$, $a_2$ form a subgraph that contains (depending on the existence of the edges $a_2f_0$ and $a_2e_1$) an induced antihole of length six, seven or eight, a contradiction.

Case 4: All edges of $T$ extend laterally. Thus, each edge $v_iv_{i+1}$ lies in a $P_4$ $e_iv_iv_{i+1}f_i$, with $v_i \rightarrow e_i$ and $f_i \rightarrow v_{i+1}$. By Claims 12 and 13, each $v_i$ sees $e_{i+1}$ and $f_{i+1}$, we have $e_0 \neq f_1$, $e_1 \neq f_2$, $e_2 \neq f_0$, and the six vertices $e_0$, $f_0$, $e_1$, $f_1$, $e_2$, $f_2$ are pairwise adjacent except, of course, for the three non-adjacent pairs $e_i f_i$ ($i = 0, 1, 2$). Now, the nine vertices $v_i$, $e_i$, $f_i$ ($i = 0, 1, 2$) form an antihole, a contradiction. This completes the proof of Lemma 4.

Now, we need to introduce two definitions. A homogeneous set of $G$ is a set $S$ of vertices that contains at least two vertices but not all vertices of $G$ such that each vertex outside $S$ sees either all or none of the vertices in $S$. A good partition is a partition of the vertex set of $G$ into sets $C$, $S$, $P$, $Q$, $R$ such that

- $C$ is a clique with at least two vertices, and $S$ is a stable set,
- every vertex in $P$ sees every vertex in $C \cup S$,
- every vertex in $R$ sees all of $C$ and none of $S$,
- every vertex in $Q$ sees none of $C \cup S \cup R$,
- $P \cup Q \cup R$ is non-empty.

The following results were proved by Hoàng and Reed [9].

Lemma 5 (Hoàng and Reed [9]). Let $G$ be a graph admitting an orientation that contains no bad $P_4$, and contains a directed pyramid. Then $G$ contains a homogeneous set or a good partition. \[\square\]

Lemma 6 (Hoàng and Reed [9]). No minimally $P_4$-incomparable graph contains a homogeneous set. \[\square\]

Lemma 7 (Hoàng and Reed [9]). No minimally $P_4$-incomparable graph contains a good partition. \[\square\]

We continue the proof of the theorem. Lemmas 3 and 4 show that $G$ must contain a directed pyramid (with the $P_4$-transitive orientation $T$.) By Lemma 5, $G$ has a homogeneous set or a good partition. But then Lemmas 6 and 7 show that $G$ cannot be minimally $P_4$-incomparable, a contradiction. This completes the proof of Theorem 2. \[\square\]

Remark 1. Two edges are $P_4$-adjacent if they belong to the same $P_4$. The equivalence classes of the transitive closure of this $P_4$-adjacency relation are called $P_4$-components. A $P_4$-component $C$ of $G$ corresponds to a component $C^*$ of the auxiliary graph $G^*$ (there is a one-to-one correspondence between the edges of $C$ and those of $C^*$). Hoàng and Reed did not state their result as in Lemma 5. Lemma 3.6 in [9] states that if a graph $G$ admits an orientation such that (i) there is no bad $P_4$, (ii) no $P_4$-components contain a circuit, and (iii) there is a directed pyramid, then $G$ contains a homogeneous set or a good partition. But a close examination of the proof in [9] reveals that condition (ii) is not used to prove the lemma. Thus, it can be restated as Lemma 5.

Remark 2. In [13], Nikolopoulos and Palios give a construction of an infinite family of graphs that admit $P_4$-transitive orientations but no acyclic $P_4$-transitive orientations, and are minimal with respect to this property. Theorem 2 shows that these graphs must be antiholes. Indeed, an examination of Nikolopoulos and Palios’s definition shows that their examples are antiholes.

Now, we prove Theorem 1.

Proof of Theorem 1. First assume that $G$ is a $P_4$-comparability graph, and consider an acyclic $P_4$-transitive orientation of the edges of $G$. Consider the corresponding orientation of the edges of $G^*$. It follows from the construction of $G^*$,
and from the fact that every $P_4$ of $G$ is transitive, that in $G^*$ every vertex $v$ is either a source (i.e., all edges incident to $v$ are directed away from $v$) or a sink (i.e., all edges incident to $v$ are directed toward $v$). Thus $G^*$ is a bipartite graph. Now we prove the converse. Assume that $G$ contains no antihole on at least seven vertices and that $G^*$ is bipartite. Note that $G$ also contains no antihole on five or six vertices, for this would imply that $G^*$ contains a cycle on, respectively, five or three vertices. Observe that any edge of $G$ that is not in a $P_4$ of $G$ is isolated in $G^*$. Since $G^*$ is bipartite, we can label its vertices either “left” or “right” in such a way that every edge is between a left vertex and a right vertex. Let us orient from left to right every non-isolated edge of $G^*$. If we keep the same orientation on the corresponding edges of $G$, we obtain a (partial) orientation of $G$; it is partial in the sense that only the edges that do not lie in a $P_4$ (if any) are not oriented. It is clear that this is an orientation in which every $P_4$ is oriented as desired, because every $P_4$ of $G$ is also a $P_4$ of $G^*$. Thus, $G$ admits a $P_4$-transitive orientation. Since $G$ contains no antihole, by Theorem 2, $G$ must be a $P_4$-comparability graph.

3. Some consequences

Hoàng and Reed established the following result.

**Lemma 8** (Hoàng and Reed [9, Theorem 3.1]). A graph $G$ is a $P_4$-comparability graph if and only if each of its $P_4$-components admits an acyclic $P_4$-transitive orientation.

Lemma 8 suggests a natural procedure to recognize a $P_4$-comparability graph. Given a graph $G$, the procedure

(i) computes its $P_4$-components (or, the components of $G^*$), and
(ii) verifies that each $P_4$-component admits an acyclic $P_4$-transitive orientation.

The $P_4$-comparability graph recognition algorithms in [9,13,16] are implementations of this procedure. It is customary to write $n = |V|$ and $m = |E|$. Two edges $ab, cd$ uniquely determine the $P_4$ on vertices $a, b, c, d$. Thus, a graph has at most $O(m^2)$ $P_4$s. This implies steps (i) and (ii) can be executed in $O(m^2)$ time. The contribution of [13] is an $O(nm)$ algorithm to compute all $P_4$-components of a graph. This algorithm does not enumerate explicitly all $P_4$s of the graph. We have to do more work to find an acyclic $P_4$-transitive orientation of a $P_4$-comparability graph. Again, Lemma 8 suggests a natural way to do this: given a graph $G$,

(i) compute its $P_4$-components,
(ii) to each $P_4$-component assign an acyclic $P_4$-transitive orientation, and
(iii) put the orientations of the $P_4$-components together to obtain an acyclic $P_4$-transitive orientation of $G$.

The directed pyramid shows that in step (iii) we may have to reverse the directions of the edges in some $P_4$-components. We show here how the results from the preceding section can be used to devise an algorithm to find an acyclic $P_4$-transitive orientation in a graph $G = (V, E)$ (or decide that it admits none). However, the complexity of our algorithm is $O(m^2)$, which is not as good as the current best from [14,15], so we only give a brief description of the algorithm; details can be found in the research report version [3]. We note that the algorithm of [14,15] constructs the orientation of $G$ in a way different from ours. Our algorithm goes as follows. Let $G = (V, E)$ be the input graph. If $G$ is a $P_4$-comparability graph then the algorithm returns the answer “yes” and an acyclic $P_4$-transitive orientation, otherwise it returns “no”.

1. Construct the auxiliary graph $G^*$ and check if it is bipartite; if it is not then $G$ is not a $P_4$-comparability graph; return “no” and stop.
2. Compute the connected components $G_1^*, \ldots, G_p^*$ of $G^*$, and let $E_1^*, \ldots, E_p^*$ be their edge-sets; orient the edges of each $E_i^*$ from left to right (with respect to the bipartition of $G^*$); call $E_i$ the set of edges of $G$ that correspond naturally with the edges of $G^*$ in $E_i^*$.
3. For $i = 1, \ldots, p$, compute the set $V_i$ of those vertices of $V$ that are incident to an edge in $E_i$; the graph $G_i = (V_i, E_i)$ is connected; we may assume $|V_1| \geq |V_2| \geq \cdots \geq |V_p|$; call $D_i$ the orientation of $G_i$ that results from the orienting part of step 2.
4. Check each $D_t$ for acyclicity; if some $D_t$ has a circuit, then $G$ is not a $P_4$-comparability graph, and return “no” and stop.

5. Set $i = 2$. Say that an edge $xy$ of $E_i \cup \cdots \cup E_p$ is forced, with $x \rightarrow y$, if there exists a directed path from $x$ to $y$ whose arcs all belong to $E_1 \cup \cdots \cup E_{i-1}$.

There are two cases:

- **Some edge $e$ of $E_i$ is forced.** Assume $e = xy$ and $e$ is forced with $x \rightarrow y$. If this orientation of $e$ is in $D_i$ then keep $D_i$ as the orientation of $E_i$; else reverse every arc of $D_i$.

- **No edge $e$ of $E_i$ is forced.** Then keep $D_i$ as the orientation of $E_i$.

**In either case:** If $i = p$ we stop, else we repeat Step 5 with $i = i + 1$.

At any step of the algorithm, many different edges may be forced, but we can show that the forcings are “coherent”. The correctness of this algorithm is stated in some details in Lemma 9 below, whose proof can be found in the research report version [3]. A result of Raschle and Simon [16], which we reformulate as follows, is used in the proof of Lemma 9.

**Theorem 3.** For any $h < j$, no edge of $E_h$ has its two endpoints in $V_j$.

**Lemma 9 (de Figueiredo et al. [3]).** Consider the situation at step $i \geq 2$, where the edges of $E_1 \cup \cdots \cup E_{i-1}$ are oriented without creating a circuit, and the other edges are not directed.

- Let $abcd$ be a $P_4$ whose edges are in a class $E_j$ with $j \geq i$. If the edge $ab$ is forced, with $a \rightarrow b$, then the edge $bc$ is forced, with $c \rightarrow b$.

- Let $abcd$ be a $P_4$ whose edges are in a class $E_j$ with $j \geq i$. If the edge $bc$ is forced with $b \rightarrow c$, then the edge $ab$ is forced, with $a \rightarrow b$ and the edge $cd$ is forced, with $d \rightarrow c$.

- For $j \geq i$, suppose that any edge of $E_j$ is forced from “left” to “right” with respect to the bipartition of $G^*$. Then every edge of $E_j$ is forced from left to right.

- The algorithm correctly returns an acyclic $P_4$-transitive orientation of $G$.

Finally, let us analyze the complexity of our algorithm. In step 1, the auxiliary graph $G^*$ can be built in time $O(m^2)$ by listing all $P_4$s. Furthermore, [14,15] describes an algorithm to construct all $P_4$-components (hence the graph $G^*$) in $O(nm)$ time. It is easy to see that each graph $G_i = (V_i, E_i)$ can be checked for acyclicity in time $O(|V_i| + |E_i|)$. Since $\sum |E_i| = O(m)$, steps 2–4 can be implemented in time $O(n + m)$. Step 5 is the bottleneck of our algorithm. To check whether an edge $ab$ of $E_j$ is forced, we have to find a directed path, in $E_1 \cup \cdots \cup E_{i-1}$, from $a$ to $b$, or from $b$ to $a$. This can be done in time $O(n + m)$. However, a graph may have $O(m)$ $P_4$-components, i.e. $p = O(m)$. Thus, step 5 may go through $O(m)$ iterations. This gives an $O(m^2)$ time bound for step 5. If we could implement step 5 in time $O(nm)$ then our algorithm would run in $O(nm)$, matching the current fastest algorithm of [14,15]. We pose this as an open problem.

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**References**


Further reading