Generalized Erlang and mortality levelling off distributions

C.D. Lai *

Institute of Fundamental Sciences – Statistics, Massey University, Palmerston North, New Zealand

ARTICLE INFO

Article history:
Received 5 October 2009
Received in revised form 15 January 2010
Accepted 25 January 2010

Keywords:
Bathtub shape
Burr distribution
Deceleration
Decreasing
Erlang
Hazard rate function
Increasing
Gompertz
Levelling-of
Log–logistics
Logistic model
Mortality rate
Mortality deceleration index
S shape
Unimodal

ABSTRACT

We propose some simple generalizations of the Erlang distribution and study their properties and their connections to other existing well known distributions. In particular, we examine the mortality rate functions and their shapes which informs us how the population ages under these models. With suitable selections of parameter values, we can achieve all the common monotonic or non-monotonic hazard rate shapes that occur in reliability and survival analysis. Some of these functions have an S shape which is suitable for the lifetime random variables that exhibit a late life deceleration phenomenon.

© 2010 Elsevier Ltd. All rights reserved.

1. Introduction

The Erlang distribution is well known, and often appears in a queueing waiting time context in telecommunication traffic engineering and others. Effectively, it is a gamma distribution with integer-valued shape parameter. From a reliability modelling perspective, we consider only a special case of this life distribution with hazard rate function (mortality rate) given by

$$h(t) = \frac{\lambda^2 t}{1 + \lambda t}.$$  \hfill (1)

The function increases but eventually it levels off to an asymptote and converges to $\lambda$. The first important observation of mortality levelling-off in humans was given by [1, p. 14] saying, “The increase of mortality rate with age advances at a slackening rate, that nearly all, perhaps all, methods of graduation of the type of Gompertz’s formula overstate senile mortality”. They also suggested “the possibility that with advancing age the rate of mortality asymptotes to a finite value”. Witten [2] and others observed that in human populations, the acceleration of mortality rate slows after 85 years. After 105 years, the mortality rate appears to cease increasing and may even decrease at this extremely advanced age. This mortality rate plateau phenomenon is quite puzzling as it goes against one’s intuition.

* Tel.: +64 6 3505799; fax: +64 6 3502261.
E-mail address: c.lai@massey.ac.nz.

0895-7177/$ - see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mcm.2010.01.011
Nomenclatures

IFR Increasing hazard rate
DFR Decreasing hazard rate
BT Bathtub shape, hazard rate decreases then increases
UBT (Unimodal) Upside-down bathtub shape, hazard rate increases then decreases
S shape curve Convex then concave
Reflected S shape Concave then convex

Economos [3] was the first researcher who described mortality levelling-off phenomenon in animals and manufactured products. He demonstrated mortality levelling-off at advanced ages for invertebrates (including fruit flies and house flies), rodents, and several manufactured products.

Vaupel et al. [4] also noted the late life deceleration phenomenon in many other organisms, and even in automobiles. Yashin et al. [5] also reported that mortality rates decelerate, level off, or even decline in at old age in some species.

Let $T$ be the lifetime random variable with $f(t)$, $F(t)$ being its probability density function (pdf) and cumulative distribution function (cdf), respectively.

The hazard rate (mortality rate, failure rate) function is defined as

$$ h(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)}, \quad (2) $$

where $R(t) = 1 - F(t)$ is the reliability or survival function of $T$. $h(t)$ $\Delta$ approximately denotes the probability of an immediate failure of an item given that it has already survived $t$ units of time. In insurance and demography studies, the hazard rate function is known as the force of mortality or simply the mortality rate. The cumulative hazard rate function is defined as

$$ H(t) = \int_0^t h(x)dx. \quad (3) $$

It is easy to show that the reliability function can be expressed as

$$ R(t) = e^{-H(t)}. \quad (4) $$

It is well known that the cumulative hazard rate function completely determines the lifetime distribution and it must satisfy the following three conditions in order to yield a proper lifetime distribution:

(i) $H(t)$ is nondecreasing for all $t \geq 0$,
(ii) $H(0) = 0$, and
(iii) $\lim_{t \to \infty} H(t) = \infty$.

Many parametric families of life distributions have been proposed in the literature; they have been nicely summarized in Part IV of [6]. Pham and Lai [7] also gave a brief review on life distributions based on generalizations of the Weibull distribution. Two recent generalizations of Weibull were proposed: (i) the generalized power Weibull family by [8], and (ii) the generalized modified Weibull by [9]. Although the hazard rate functions of the two families can yield bathtub and upside-down bathtub (unimodal) shapes, they do not have the mortality leveling-off property. The present paper aims to provide a family of lifetime distributions that gives flexible hazard rate shapes, and also possesses the mortality leveling-off/late life deceleration phenomenon which is prevalent in some biological species. The generalized Erlang family meets our criteria. In addition, $S$-shaped hazard rates are obtainable from this family.

In order to provide a more informative background to our readers, we give some further reliability properties of this basic Erlang distribution. The cumulative hazard function that corresponds to (1) is given by

$$ H(t) = \int_0^t h(x)dx = \lambda t - \log(1 + \lambda t), \quad (5) $$

and the survival function is

$$ \bar{F}(t) = e^{-H(t)} = (1 + \lambda t)e^{-\lambda t}. $$

This lifetime distribution is in fact the sum of two exponentials so it is a gamma distribution with shape parameter 2. Thus the mean time to failure is simply $2/\lambda$. The quantiles of the distribution can be solved numerically using the fixed-point iteration method.

The derivative of $h(t)$ is

$$ h'(t) = \lambda^2 \frac{(1 + \lambda t) - \lambda t}{(1 + \lambda t)^2} = \frac{\lambda^2}{(1 + \lambda t)^2}.$$
thelatelifemortalitydecelerationtobediscussedinSection
for
Here
possibleapplicationtothestudyofhumandemography.
functionswhichareafundamentalmeasureofriskoffailures.AnothernewgeneralizationoftheErlangdistributionis
andsurvivalanalysis.Thesalientfeatureoftheproposedlifedistributionsistheirsimplicityintermsoftheirhazardrate
generalizationsenableustoconstructseveralfamilynon-monotonichazardrateshapesthatoccurofteninreliability
whichisalwayspositivebut
whichisnegativefor

Itcanbeshownthatforthislifedistribution,

\[
F(t) = \begin{cases} 
1 - e^{-\lambda t} & \text{for } 0 < \lambda t < 1 \\
1 - \sum_{k=1}^{\infty} \frac{(-\lambda t)^k}{k!} & \text{for } \lambda t \geq 1 
\end{cases}
\]

Inthispaper,westudysomesimplegeneralizationsofthebasicErlangdistributiongivenin(5).Wewillseethatthese
generalizationsenableusetoconstructseveralfamiliarnon-monotonichazardrateshapeswhichoccurofteninreliability
andsurvivalanalysis.Thesalientfeatureoftheproposedlifedistributionsistheirsimplicityintermsoftheirhazardrate
functions which are a fundamental measure of risk of failures. Another new generalization of the Erlang distribution is
obtained by exponentiating the distribution function of a simple Erlang distribution. We study its properties and propose a
possible application to the study of human demography.

2. Generalized Erlang family

A life distribution may be characterized by a survival function \(\bar{F}(t)\) or a hazard rate function \(h(t)\). Generalized Weibull
distributions were generally constructed through the first approach [7] but many others were derived via the second.
We now consider a simple generalization of (1) which has the following form:

\[
h(t) = A + a\lambda^{b+1}t^b/(1 + \lambda t)^c, \quad A \geq 0, \ a > 0, \ \lambda > 0, \ b, \ c \ \text{real}.
\]  

(6)

Here \(A\) is a ‘lifting’ factor which may be set to 0 in most applications so unless explicitly stated otherwise we consider only
the distribution that has hazard rate given by

\[
h(t) = a\lambda^{b+1}t^b/(1 + \lambda t)^c, \quad a > 0, \ b, \ c \ \text{real}.
\]  

(7)

If the lifting factor \(A\) is used, we may set the multiplicative constant \(a\) to 1 so the model (6) becomes

\[
h(t) = A + \lambda^{b+1}t^b/(1 + \lambda t)^c, \quad A > 0, \ \lambda > 0, \ b, \ c \ \text{real}.
\]  

(8)

Here, both \(b\) and \(c\) are shape parameters. If \(b\) and \(c\) are integers, then (7) is simply a rational polynomial as discussed in

The cumulative hazard function of (7) is

\[
H(t) = a\lambda^{b+1} \int_0^t \frac{x^b}{(1 + \lambda x)^c} dx = a \int_0^{\lambda t} \frac{y^b}{(1 + y)^c} dy = \frac{a(\lambda t)^{b+1}}{b + 1} \sum_{k=1}^{\infty} \frac{(-\lambda t)^k}{k!} F_1(c, b + 1; b + 2; -\lambda t),
\]  

(9)

for \(b > -1\) (see page 313 of [12]). Here \(F_1(a, b; c; t)\) is the Gauss hypergeometric function defined by

\[
\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} t^k, \quad (a)_k = \frac{a(a + 1) \cdots (a + k - 1)}{k!},
\]

or

\[
F_1(a, b; c; t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a + k)\Gamma(b + k)}{\Gamma(c + k) k!} t^k.
\]  

(10)

The survival function can be obtained from (9) and (4).

We now study several of its special cases in the forthcoming subsections.
2.1. The case \( b = 0 \)

For \( b = 0, -1 \leq c \leq 1, h(t) = a\lambda(1 + \lambda t)^{-c}, a > 0, t > 0, \) where \( a > 0, \) and \( c \) is real. Various shapes for the hazard rate function can be obtained:

- horizontal if \( c = 0, \)
- straight line if \( c = -1, \)
- convex if \( c < -1 \)
- concave, if \(-1 < c < 0\)
- increasing, if \( c < 0, \)
- decreasing if \( c = 1. \) For \( a = 1, \) we have the Lomax distribution with

\[
\tilde{F}(t) = (1 + \lambda t)^{-1}, \quad a > 0.
\]  

(11)

2.2. The case \( b = c = 1 \)

It follows from (9) that

\[
H(t) = \frac{a(\lambda t)^2}{2} F_2(1, 2; 3; -\lambda t) = a\lambda t - a \log(1 + \lambda t),
\]  

(12)

as would have been obtained by integrating (7) directly with \( b = c = 1. \)

**Logistic model**

If we further set \( \lambda = 1 \) and replace \( t \) by \( ae^{\beta t} \) in (7), we then have a hazard rate function given as

\[
h(t) = \frac{ae^{\beta t}}{1 + ae^{\beta t}}.
\]  

(13)

(13) is known as the logistic model in mortality literature; see for example, [13] who pointed out the logistic model gives a better fit to the demography data sets than other traditional models such as the Makeham–Gompertz model.

Integrating (13) and exponentiating, we obtain the survival function

\[
\tilde{F}(t) = \left[1 + ae^{\beta t}\right]^{1/\beta}.
\]  

(14)

The case \( b = c \) will be considered in Section 5 below.

2.3. The \( c > b + 1 \)

For this case, we will consider Eq. (8) instead of (7) as the latter will have \( \lim_{t \to \infty} H(t) = k < \infty. \) This would result in an improper distribution. Rewrite (8) as:

\[
h(t) = A + \lambda(\lambda t)^b/(1 + \lambda t)^c, \quad A > 0, \ b \geq 0, \ c > b + 1.
\]

Now,

\[
\lambda \int_0^t \frac{(\lambda x)^b}{(1 + \lambda x)^c} \, dx = \int_0^\lambda \frac{x^b}{(1 + x)^c} \, dx = \int_0^1 y^{c-b-2}(1-y)^b \, dy
\]

\[
= B(c - b - 1, b + 1) \left[1 - B(c - b - 1, b + 1; (1 + \lambda t)^{-1}) \right]
\]  

(15)

where \( B(u, v; t) \) is the incomplete beta function defined by

\[
B(u, v; t) = \int_0^t x^{u-1}(1-x)^{v-1} \, ds, \quad u, v > 0.
\]

It follows that

\[
\tilde{F}(t) = \exp \left\{-At - B(c - b - 1, b + 1) \left[1 - B(c - b - 1, b + 1; (1 + \lambda t)^{-1}) \right] \right\}.
\]  

(16)

In this case, the median \( t_M \) can be found by solving

\[
At_M + aB(c - b - 1, b + 1) \left[1 - B(c - b - 1, b + 1; (1 + \lambda t_M)^{-1}) \right] = -\log(1/2) = 0.693.
\]  

(17)

Note that in this case we must have the lifting constant \( A > 0; \) otherwise \( h(t) \) decreases rapidly to zero so that \( H(t) \) converges to a finite value and thus \( F(t) \) will become an improper distribution function.
2.4. UBT shape with \( c > b > 0 \)

In reliability and survival analysis, empirical hazard rate function often exhibits an upside-down (inverted) bathtub shape. Marshall and Olkin [6, p. 131], commented that “In the past, inverted bathtub hazard rates have not attracted much interest, at least in reliability theory, perhaps because the bathtub hazard rates have been a focus of attention. But there are good reasons for giving consideration to inverted bathtub hazard rates”.

On differentiating either (6) or (7), we have

\[
h'(t) = a \lambda t^{b-1} e^{-\lambda t} - b (c - b) \lambda t / (1 + \lambda t)^{c+1}\]

(18)

which is positive for \( t < \frac{b}{(c-b)\lambda} \) and negative for \( t > \frac{b}{(c-b)\lambda} \); thus \( h(t) \) has an UBT shape. That is, \( h(t) \) is unimodal. 

Again, we note that if \( c > b + 1 \), Eq. (8) should be used to avoid an improper distribution.

2.5. Bathtub shape with \( c < b < 0 \)

Bathtub shaped hazard rates have been a focus in reliability theory for several decades. Lai and Xie [14] devote a whole chapter (Chapter 4) of their book to discussing this class of life distribution as well as their relationships with inverted bathtub mean residual life functions.

Let \( c = -2 \), \( b = -1 \), then

\[
h(t) = a(1 + \lambda t)^2 / t.
\]

(19)

So

\[
h'(t) = \frac{2a(1 + \lambda t)\lambda t - (1 + \lambda t)^2}{t^2}
\]

\[
= \frac{a(1 + \lambda t)(\lambda t - 1)}{t^2},
\]

(20)

which is negative for \( t < \frac{1}{\lambda} \) and positive when \( t > \frac{1}{\lambda} \).

A positive constant \( \phi \) needs to be added to the denominator of (19) to prevent the hazard rate function going to infinity at \( t = 0 \). As a result,

\[
h(t) = \frac{a(1 + \lambda t)^2}{t + \phi}, \quad \lambda, \phi > 0.
\]

(21)

It can be shown that for \( \lambda \phi < 1/2 \), \( h(t) \) has a bathtub shape with the turning point occurring at \( (1 - 2\lambda \phi)/\lambda \).

Further

\[
\tilde{h}(t) = \left( \frac{\phi}{t + \phi} \right)^{a(1 - \phi \lambda)^2} \exp \left\{ -a(2\lambda - \phi \lambda^2)t - a\lambda^2 t^2/2 \right\}.
\]

(22)

3. Relationship with log–logistic and Burr XII life distributions

Consider the case of (7) with \( a = 1 \), \( b = 0 \). This becomes the Lomax distribution (12) with \( \tilde{h}(t) = 1/(1 + \lambda t) \). Let the random variable that corresponds to it be given by \( X \). Define \( T = X^{1/a} \), then the hazard rate function of \( T \)

\[
h(t) = a \lambda t^{a-1} / (1 + \lambda t^a), \quad c > 0
\]

(23)

which is the hazard rate function of the log–logistic model. Note that \( 0 < a \leq 1 \), \( h(t) \) is decreasing, UBT (increases initially and decreases later) if \( a > 1 \). Kiefer [15] fitted this distribution to economic duration (strike length) data. The log–logistic distribution provides one parametric model for survival analysis when mortality from cancer following diagnosis or treatment can have a non-monotonic (unimodal) hazard rate function [16].

Consider a series structure of \( n \) independent components, each having a hazard rate given by (22). Then the system hazard rate is given by

\[
h_s(t) = \frac{n a \lambda t^{a-1}}{(1 + \lambda t^a)}
\]

(24)

which is the Burr XII distribution introduced by [17] although \( n \) is a real positive number in the original derivation. For further properties and its applications in reliability see, for example, Chapter 2 of [14]. As expected, the shape of this hazard rate is the same as the log–logistic.
4. Mortality levelling-off and S shape hazard rate distribution

The late life mortality levelling-off phenomenon is well known in biological organisms; see for example, [18]. Our literature search for a definition for an S-shape function gives one the following:

(i) a function with small beginnings that accelerates and approaches a climax over time,
(ii) a curve that is convex followed by concave, or
(iii) a function that is first increasing, then decreasing.

Within the context of a hazard rate function $h(t)$, definition (i) indicates $h(t)$ is increasing throughout the whole life but levelling-off to an asymptote. Definition (iii) seems to correspond to $h(t)$ having an UBT shape or an inverted bathtub. (Essentially it means $h(t)$ is unimodal). Only definition (ii) appears to give rise to a proper S-shape. All three definitions may be considered as a form of late life mortality deceleration.

In this section, we consider a special case of the generalized Erlang family that will have an S-shaped hazard rate function. Without loss of generality, letting $a = 1$ and setting $b = c$ in Eq. (7), we have

$$h(t) = \frac{\lambda^{b+1} t^{b}}{(1 + \lambda t)^{b}}, \quad b > 0. \quad (25)$$

Clearly the hazard rate function is levelling-off as $t$ increases (see Fig. 1 below).

Since the case $b = 1$ has already been dealt with, we now restrict ourselves to $b > 1$.

It is easy to verify that

$$h'(t) = \frac{\lambda^{b+1} t^{b-1}}{(1 + \lambda t)^{b+1}}, \quad b > 1 \quad (26)$$

and

$$h''(t) = \frac{\lambda^{b+1} t^{b-2} (b - 1 - 2\lambda t)}{(1 + \lambda t)^{b+2}}, \quad b > 1$$

showing that $h(t)$ is convex for $t < (b - 1)/(2\lambda)$ and concave for $t > (b - 1)/(2\lambda)$ so it truly has an S-shape distribution.

The point $t = (b - 1)/(2\lambda)$ may be considered as the onset of deceleration.

It follows from (25) and (9) that

$$H(t) = \frac{(\lambda t)^{b+1}}{(b + 1)^2} F_1(b, b + 1; b + 2, -\lambda t) = \sum_{k=0}^{\infty} \frac{(k + 1)(-1)^k(b\lambda t)^{b+k+1}}{(b + k + 1)}. \quad (27)$$

Consider the case when $b = 2$. It is proved in the Appendix that

$$H(t) = \sum_{k=0}^{\infty} \left[ \frac{1}{k + 3} - \frac{2}{k + 3} \right] (-1)^k (\lambda t)^{k+3}$$

$$= \sum_{k=0}^{\infty} (-1)^k (\lambda t)^{k+3} - 2 \sum_{k=0}^{\infty} \frac{1}{k + 3} (-1)^k (\lambda t)^{k+3}$$

$$= (1 + \lambda t) - \frac{1}{1 + \lambda t} \log(1 + \lambda t)^2.$$

$$H(t) = (1 + \lambda t) - \frac{1}{1 + \lambda t} \log(1 + \lambda t)^2.$$

$$\tilde{F}(t) = e^{-H(t)} = (1 + \lambda t)^{-2} \exp \left[ -(1 + \lambda t) + (1 + \lambda t)^{-1} \right].$$

If the multiplicative constant $a$ is required, then the last equation becomes

$$\tilde{F}(t) = e^{-H(t)} = (1 + \lambda t)^{-2a} \exp \left[ -a(1 + \lambda t) + a(1 + \lambda t)^{-1} \right].$$

4.1. Mortality deceleration index

Horiuchi and Coale [19] use the age-specific rate of mortality change with age, defined by

$$\varphi(t) = \frac{d\log h(t)}{dt} = \frac{h'(t)}{h(t)}. \quad (28)$$

It follows from (25) and (26) that the age-specific rate of mortality change in our model is $[(1 + \lambda t) t^{-1}, a$ a decreasing function of $t$. Obviously, the rate of deceleration is only meaningful after the onset of deceleration which is $(b - 1)/(2\lambda)$ in this case (see Fig. 1).
4.2. Time required for mortality rate to double

Species comparisons in mortality rates are added by calculations of MRD (time required for mortality rate to double); see for example, [20]. For the model specified in (25) with a lifting factor $A > 0$,

$$\text{MRD} = \frac{\lambda}{A^{1/b}} \left[ (\lambda)^{1/b} - A^{1/b} \right]$$

provided $\lambda > A$.

5. Exponentiated Erlang distribution

Recall from (5), the distribution function of the Erlang distribution is $F(t) = 1 - \tilde{F}(t) = 1 - (1 + \lambda t)e^{-\lambda t}$. We obtain a simple generalization by exponentiating this function:

$$F_\theta(t) = \left[ 1 - (1 + \lambda t)e^{-\lambda t} \right]^{\theta}$$

so the survival function is

$$S_\theta(t) = 1 - \left[ 1 - (1 + \lambda t)e^{-\lambda t} \right]^{\theta}.$$

The probability density function that corresponds to (29) is

$$f_\theta(t) = \theta \lambda^2 t e^{-\lambda t} \left[ 1 - (1 + \lambda t)e^{-\lambda t} \right]^{\theta-1}.$$  

with hazard rate function

$$h_\theta(t) = \frac{\theta \lambda^2 t e^{-\lambda t} \left[ 1 - (1 + \lambda t)e^{-\lambda t} \right]^{\theta-1}}{1 - \left[ 1 - (1 + \lambda t)e^{-\lambda t} \right]^{\theta}}.$$  

In order to study the shape of $h(t)$, we first let $k(t) = 1 - (1 + \lambda t)e^{-\lambda t}$ so

$$k'(t) = \lambda^2 t e^{-\lambda t} \quad \text{and} \quad k''(t) = \lambda^2 e^{-\lambda t} (1 - \lambda t).$$

Now

$$h_\theta(t) = \frac{\theta k'(t)[k(t)]^{\theta-1}}{1 - [k(t)]^{\theta}},$$

$$h_\theta(t) = \frac{[k(t)]^{\theta-2} \left\{ \left[ k''(t)k(t) + [k'(t)]^2(\theta - 1) \right] \{ 1 - [k(t)]^{\theta} \} + [k'(t)]^2[k(t)]^{\theta} \right\}}{(1 - [k(t)]^{\theta})^2/\theta}.$$  

It is shown in the Appendix that $h_\theta(t) = 0$ iff

$$\left\{ k''(t)k(t) - [k'(t)]^2 \right\} (1 - [k(t)]^{\theta}) + [k'(t)]^2 \theta = 0.$$  

The solutions to the above functional equation seem non-trivial. We note however, the sign of $h_\theta(t)$ depends only on the sign of the function on the left of (31). It is easy to verify by L'Hospital's rule that

$$\lim_{t \to \infty} h_\theta(t) = \lim_{t \to \infty} \frac{-k''(t)/k'(t) + (\theta - 1)k'(t)}{\lambda}.$$  

It is easy to verify by L'Hospital's rule that

$$\lim_{t \to \infty} h_\theta(t) = \lim_{t \to \infty} \frac{-k''(t)/k'(t) + (\theta - 1)k'(t)}{\lambda} = \lambda.$$  

![Fig. 1. Hazard rate function $h(t)$ with $\lambda = 1.25$, $b = 6$.](image-url)
It now follows that if \( h(t) \) have turning points, the last one has to be a local minimum because \( h'(t) \) is always positive as \( t \) becomes large.

For example, for \( \lambda = 2, \ \theta = 0.3 \), we have a bathtub shape as shown in Fig. 2.

The model is suitable to describe the combined phenomenon of an early life ‘infant mortality’ together with a late life deceleration at advanced age. Most of the known bathtub shaped hazard rate functions go to \( \infty \) as \( t \to \infty \); see a recent summary of available bathtub hazard rates functions given by [21].

6. Discussion and conclusion

In lifetime analysis, there are two approaches to hazard rate modelling: (i) fit a distribution function \( F(t) \) to a given data set and consider its goodness of fit and then derive its hazard rate function, or (ii) select an appropriate \( h(t) \) based on the empirically estimated hazard rates. In this paper, we have adopted the second approach, that is, we will select and fit a member of the proposed generalized Erlang class to the lifetime data concerned, based on the shape of the empirical hazard plots.

The proposed generalized Erlang family is a class of parametric life distributions whose hazard rate has a simple form and yet is able to give various shapes that are common in reliability and survival analysis. In particular, the family contains members that have mortality plateau and S-shape phenomenon which are important in mortality study. We believe these particular features give the family a competitive edge over many other life models proposed in the literature.

Estimates of parameters have not yet been considered here. In principle, the maximum likelihood estimates based the density function \( f(t) = h(t)f(t) \) can be derived. In its general form, this family of distributions is expressed in terms of some special functions (Gauss hypergeometric functions; see Eq. (9)), and thus numerical maximum likelihood estimates are required. Statistical packages such as MATLAB and R should be able to compute such estimates.

Acknowledgement

The author wishes to thank the anonymous referee for several helpful comments that led to a substantial improvement in the paper.

Appendix

\[
H(t) = \sum_{k=0}^{\infty} \left[ 1 - \frac{2}{k+3} \right] (-1)^k (\lambda t)^{k+3} = \sum_{k=0}^{\infty} (-1)^k (\lambda t)^{k+3} - 2 \sum_{k=0}^{\infty} \frac{1}{k+3} (-1)^k (\lambda t)^{k+3} \\
= \{ (\lambda t)^3 - (\lambda t)^4 + (\lambda t)^5 + \cdots \} - 2 \left\{ \frac{(\lambda t)^3}{3} - \frac{(\lambda t)^4}{4} + \frac{(\lambda t)^5}{5} + \cdots \right\} \\
= \{ -1 + \lambda t - (\lambda t)^2 + (\lambda t)^3 - (\lambda t)^4 + (\lambda t)^5 + \cdots \} + (1 - \lambda t + (\lambda t)^2) \\
- 2 \left\{ \lambda t - \frac{(\lambda t)^2}{2} + \frac{(\lambda t)^3}{3} - \frac{(\lambda t)^4}{4} + \frac{(\lambda t)^5}{5} + \cdots \right\} \\
= \left\{ (1 - t + t^2) - \frac{1}{1 + t} \right\} - 2 \left\{ -\lambda t + \frac{(\lambda t)^2}{2} - \log(1 + \lambda t) \right\} \\
= (1 + \lambda t) - \frac{1}{(1 + \lambda t)} + \log(1 + \lambda t)^2.
\]
The derivative of \( h_\theta(t) \) is

\[
h'_\theta(t) = \frac{[k'(t)[k(t)]^{\theta-1} + [k'(t)]^2(\theta - 1)[k(t)]^{\theta-2}}{(1 - [k(t)]^\theta)^2/\theta}
\]

The numerator of \( h'_\theta(t) \) is

\[
[k'(t)[k(t)]^\theta + [k'(t)]^2(\theta - 1)] (1 - [k(t)]^\theta) + [k'(t)]^2[k(t)]^\theta
\]

\[
= k'(t)[k(t)]^\theta + [k'(t)]^2(\theta - 1) - k'(t)[k(t)]^\theta + [k'(t)]^2[k(t)]^\theta
\]

\[
= [k'(t)[k(t)] - [k'(t)]^2] (1 - [k(t)]^\theta) + [k'(t)]^\theta
\]

References