Hierarchical Fuzzy Sliding-Mode Control

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Abstract — A hierarchical fuzzy sliding-mode control is proposed to achieve asymptotic stability and favorable decoupling performance. In this approach, the nonlinear system is decoupled into several subsystems and the state response of each subsystem can be designed to be governed by a corresponding sliding surface. Then the whole system is controlled by a hierarchical sliding-mode controller. The proposed design method is applied to investigate the decoupling control of an inverted pendulum. Simulation is performed and a comparison between the proposed hierarchical fuzzy sliding-mode control and a conventional fuzzy sliding-mode decoupling control is made to demonstrate the effectiveness of the proposed design method.

Index terms — Fuzzy control, Hierarchical sliding-mode control, Adaptive law, Decoupling control

I. INTRODUCTION

Fuzzy logic control (FLC) using linguistic information can model the qualitative aspects of human knowledge, it is an alternative to conventional control techniques. It also possesses several advantages such as robustness, freedom from models, the universal approximation theorem and rule-based algorithms [1, 2]. However, the huge amount of the fuzzy rules required for a high-order system makes the analysis complex. Also, conventional FLCs still lack systematic or mathematical methodologies to guarantee system stability. Some useful criteria have been proposed to guarantee the stability of FLC systems based on certain special conditions such as the canonical state space model or the phase-plane dynamics model [3, 4].

Sliding-mode control (SMC) is a robust design methodology developed using a systematic scheme based on a sliding surface and Lyapunov’s stability theorem [5, 6]. The main advantage of SMC is that the system uncertainties and external disturbances can be handled under the invariance characteristics of system's sliding condition with guaranteed system stability. Recently, there has been much research on the design of fuzzy logic controllers based on the sliding-mode control scheme, referred to as fuzzy sliding-mode controls (FSMCs) [7]-[10]. FSMC has the advantages of both FLC and SMC. It can also reduce chattering of the control system compared to SMC. Moreover, a decoupled fuzzy sliding-mode control design method has been proposed to achieve decoupling performance of a class of nonlinear coupled systems [11]. Although an intermediate variable has been introduced to incorporate the state information of two subsystems, this intermediate variable must be pre-determined by time-consuming trial-and-error choice procedure to achieve the desired performance. From the authors’ simulations, it is determined that an inadequate choice of the intermediate variable will degrade the control performance, even causing the instability of the coupled system.

In this paper, a design method of hierarchical fuzzy sliding-mode (HFSM) control is proposed. In this approach, a class of nonlinear coupled systems can be decoupled into several subsystems, and a sliding surface, which will govern the states’ response, is defined for each subsystem. Then a hierarchical sliding-mode controller is proposed and an adaptive law is derived to tune the coupling factor of the hierarchical sliding-mode controller so as to achieve favorable decoupling performance with guaranteed stability. This proposed HFSM design method is applied for the decoupling control of an inverted pendulum. Simulation is performed and a comparison between the proposed hierarchical fuzzy sliding-mode decoupling control and a conventional fuzzy sliding-mode decoupling control is made. Simulation results demonstrate that hierarchical fuzzy sliding-mode control can cope with two coupling subsystems, not only guaranteeing the system stability, but also improving the system performances.

II. DECOUPLED SLIDING-MODE CONTROLLER DESIGN

The basic idea of sliding-mode control is to alter the system dynamics along some surfaces in the state system so that the states of the system are attracted to these surfaces. During the sliding motion of the state on the surface, the system remains insensitive to parameter variations and external disturbances. In order to derive the sliding-mode control law which forces the motion of the error to be along the sliding surface \( s = 0 \), a positive definite Lyapunov function is defined as

\[
V = \frac{1}{2} s^2 , \quad (1)
\]

if its derivative value \( \dot{V} \) is negative definite, then the system is stable and its system trajectory will approach to sliding surface till converging toward origin. This is a well-known sliding-mode condition:

\[
\dot{V} = s \cdot \dot{s} \leq 0 . \quad (2)
\]

Consider a single-input multi-output nonlinear coupled system expressed in the following form:
\[ \dot{x}_1(t) = x_2(t) \]
\[ \dot{x}_2(t) = g_1(x_1, x_2) + b_1(x_1, x_2)u(t) + d_1(t) \]
\[ \dot{x}_3(t) = x_4(t) \]
\[ \dot{x}_4(t) = g_2(x_1, x_2) + b_2(x_1, x_2)u(t) + d_2(t) \]
\[ y(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T \]

where \( x_1(t), x_2(t), x_3(t) \) and \( x_4(t) \) are the state variables; \( g_1(x_1, x_2), g_2(x_1, x_2), b_1(x_1, x_2) \) and \( b_2(x_1, x_2) \) are the nominal nonlinear functions; \( d_1(t) \) and \( d_2(t) \) are the bounded lumped disturbances which include the parameter variations and external disturbances; and \( u(t) \) is the control input. They are all abbreviated as \( x, x_2, x_3, x_4, g, b, d_1, d_2 \) and \( u \) in the following description. This system can be treated as two subsystems with second order canonical form including the states \( (x_1, x_2) \) and \( (x_3, x_4) \), respectively. The decoupling control tries to design a single input \( u \) to simultaneously control the states \( (x_1, x_2) \) and \( (x_3, x_4) \) to achieve desired performance.

The hierarchical sliding-mode control (HSMC) is characterized by first defining a suitable pair of sliding surfaces as

\[ s_1 = \dot{s}_1 + \lambda_1 x_1 = x_2 + \lambda_1 x_1, \]
\[ s_2 = \dot{s}_2 + \lambda_2 x_3 = x_4 + \lambda_2 x_3, \]

where \( \lambda_1 \) and \( \lambda_2 \) are real positive constants. Their time derivatives are obtained as

\[ \dot{s}_1 = \ddot{x}_2 + \lambda_1 \dot{x}_1 = \ddot{x}_2 + \lambda_1 x_2, \]
\[ \dot{s}_2 = \ddot{x}_4 + \lambda_2 \dot{x}_3 = \ddot{x}_4 + \lambda_2 x_4. \]

Then define a hierarchical coupled sliding surface as

\[ s_h = s_1 - n^* s_2, \]

where \( n^* \) is set as a real positive constant and is referred to as the optimal coupling factor. This coupling factor will play an important role for the interactive control between the sliding surface \( s_1 \) and \( s_2 \). The derivative of this hierarchical sliding surface is obtained as

\[ \dot{s}_h = \dot{s}_1 - n^* \dot{s}_2. \]

From (3) and (6), it is obtained that

\[ \dot{s}_1 = \ddot{x}_2 + \lambda_1 x_2 = g_1 + b_1 u + d_1 + \lambda_1 x_2. \]

In (10), if the lumped disturbance \( d_1 \) is not existent, i.e. \( d_1 = 0 \), by the condition of sliding mode \( \dot{s}_1 = 0 \), an equivalent control law can be obtained as

\[ u_{eq} = \frac{-g_1 - \lambda_1 s_2}{b_1}. \]

However, this control law does not consider the lumped disturbance \( d_1 \); and it only considers the control of the \( (x_1, x_2) \) subsystem while disregards the control of the \( (x_3, x_4) \) subsystem. Similarly, one can define the equivalent control as

\[ u_{eq}^* = \frac{-g_2 - \lambda_2 s_4}{b_2}; \]

however, this only considers the control of the \( (x_3, x_4) \) subsystem. For achieving favorable decoupling performance, a hierarchical fuzzy sliding-mode control is proposed in the next section.

### III. Hierarchical Fuzzy Sliding-Mode Control

In (9), for different initial conditions, different \( n^* \) should be appropriately chosen case by case to achieve satisfactory decoupling performance. However, this is a time-consuming trial-and-error process. Thus, to improve the decoupling performance of the coupled system, an adaptive fuzzy tuning method of the coupling factor is proposed below. In order to estimate the optimal \( n^* \), a fuzzy inference system is employed to tune a coupling factor \( \hat{n} \) to estimate the \( n^* \). The fuzzy inference system is expressed as

\[ R_i: \text{If } s_h = A_{i1} \text{ and } \dot{s}_h = A_{i2} \text{ then } \hat{n} = B_i. \]

where \( R_i \) denotes the \( i \)th control rule, \( i = 1, 2, \ldots, r \); \( A_{ij} \) is the fuzzy set in the antecedent part associated with the \( j \)th input variable at the \( i \)th control rule characterized by the fuzzy membership functions \( \mu_{A_{ij}}(\cdot) \); and \( B_i \) is the fuzzy set in the consequent part characterized by an adjustable singleton fuzzy membership functions \( \hat{\theta}_i \). The Gaussian-type membership function of \( A_{ij}(\cdot) \) is given as

\[ \mu_{A_{ij}}(x_j) = e^{-\frac{1}{2} \sum_{i=1}^{r} (x_j - w_i)^2 w_i^2} \]

where \( x_j \) is the fuzzy input, and \( w_i, w_c \) and \( w_d \) are the function parameters [12]. These fuzzy rules indicate that the slope of the estimated hierarchical sliding surface is time varying and is on-line tuned by the values of \( s_h \) and \( \dot{s}_h \). Then the fuzzy output can be inferred as

\[ \hat{n} = \frac{\sum_{i=1}^{r} \zeta_i \hat{\theta}_i}{\sum_{i=1}^{r} \zeta_i}, \]

where

\[ \zeta_i = u_{A_{i1}}(s_h) \cdot u_{A_{i2}}(\dot{s}_h) \]

is the inferred grade of the \( i \)th fuzzy rule. Equation (14) can be rewritten as

\[ \hat{n} = w^T \hat{\theta} \]

where \( \hat{\theta} = [\hat{\theta}_1 \ \hat{\theta}_2 \ \cdots \ \hat{\theta}_r]^T \) is a parameter vector, and \( w = [w_1 \ w_2 \ \cdots \ w_r]^T \) is a regressive vector with \( w_i \) defined as
\[ wi = \frac{\zeta_i}{r} \sum_{i=1}^{r} \zeta_i \]  

(17)

Since \( n^* \) is a positive value, the estimated value \( \hat{n} \) is also restrained to be positive.

Assume the optimal \( n^* \) can be also formulated as:

\[ n^* = \frac{\sum_{i=1}^{r} \zeta_i \theta_i^*}{\sum_{i=1}^{r} \zeta_i} = w^T \theta^* \]  

(18)

where \( \theta^* \) stands for the constant optimal parameter vector of the fuzzy consequent part.

Define the estimation error of the optimal parameter vector

\[ \phi = \hat{\theta} - \theta^* . \]  

(19)

Considering the disturbed system in (10), i.e. \( d_1 \neq 0 \), the control input is defined as

\[ u = u_{eq} + u_{vs} , \]  

(20)

where \( u_{eq} \) is given in (11a), and \( u_{vs} \) is the variable structural term to cope with the lumped disturbance and interactive coupling influence.

From (1) and (19), a Lyapunov function is defined as

\[ V = \frac{1}{2} s_h^2 + \frac{1}{2\alpha} \phi^T \phi , \]  

(21)

where \( \alpha \) is a positive constant presenting as the learning rate of the adaptive law, as shown in the following.

From (8), (10), (19) and (20), the Lyapunov stability condition can be derived as follows:

\[ \dot{V} = s_h \dot{s}_h + \frac{1}{\alpha} \phi^T \dot{\phi} \]

\[ = s_h [\dot{s}_h - n^* \dot{s}_h] + \frac{1}{\alpha} \phi^T \dot{\theta} \]

\[ = s_h [g_1 + b_1 u + d_1 + \lambda_1 x_2 - n^* \dot{s}_h] + \frac{1}{\alpha} \phi^T \dot{\theta} \]

\[ = s_h [g_1 + b_1 (u_{eq} + u_{vs}) + d_1 + \lambda_1 x_2 - \dot{n} \dot{s}_2] \]

\[ + (\dot{n} - n^*) s_h \dot{s}_2 + \frac{1}{\alpha} \phi^T \dot{\theta} \]

\[ = s_h [b_1 u_{vs} + d_1 - \dot{n} \dot{s}_2] + w^T \phi s_h \dot{s}_2 + \frac{1}{\alpha} \phi^T \dot{\theta} \]

\[ = s_h [b_1 u_{vs} + d_1 - \dot{n} \dot{s}_2] + \frac{1}{\alpha} \phi^T (\dot{\theta} + \alpha s_h \dot{s}_2 w) . \]  

(22)

Choose

\[ u_{vs} = \frac{\dot{n} \dot{s}_2 - \epsilon \text{sgn}(s_h)}{b_1} , \]  

(23)

and

\[ \dot{\theta} = -\alpha s_h \dot{s}_2 w , \]  

(24)

where \( \epsilon \) is a positive constant which represents the bound of the lumped disturbance, i.e. \( |d_1| \leq \epsilon \), and \( \text{sgn}() \) is a sign function. Then (22) becomes

\[ \dot{V} = -\epsilon s_h \text{sgn}(s_h) s_h + s_h d_1 \]

\[ \leq -|s_h| (\epsilon - |d_1|) \]

\[ \leq 0 . \]  

(25)

This guarantees the stability of the system.

In summary, the control law is given as

\[ u = u_{eq} + u_{vs} = \frac{\dot{n} \dot{s}_2 - \epsilon \text{sgn}(s_h) - g_1 - \lambda_1 x_2}{b_1} \]  

(26)

where the coupling factor \( \hat{n} \) is adjusted by (16) with \( \hat{\theta} \) adapted by (24). This control law reveals the desired decoupling control through the coupling factor \( \hat{n} \) and the information from two subsystems; also the disturbance rejection is achieved by the switching control term. In general, the switching sign function can be replaced by a saturation function in order to reduce chattering of the control signal.

Similarly, this idea can be extended to even higher order systems by introducing the hierarchical sliding surface. The concept diagram of this hierarchical fuzzy sliding-mode control system is depicted in Fig. 1.

**IV. SIMULATION RESULTS**

A conventional fuzzy sliding-mode (FSM) decoupling control method has been proposed to deal with a class of nonlinear coupled systems [11]. However, for their design, an intermediate variable should be pre-determined by trial-and-error. In this paper, the proposed HFSM control method is applied to control the same systems so as to verify the favorable decoupling performance. The system is an inverted pendulum system which is single input multi-output nonlinear coupled system. A comparison between the proposed HFSM and the FSM decoupling control is demonstrated to illustrate the effectiveness of the proposed design method.

The structure and controlled states of inverted pendulum are shown in Fig. 2. Its dynamics are described as:

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1 u + d \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2 u + d
\end{align*} \]  

(27)

where \( x_1 \) is the angle of the pole with respect to the vertical axis;

\( x_2 \) is the angular velocity of the pole with respect to the vertical axis;

\( x_3 \) is the position of the cart;

\( x_4 \) is the velocity of the cart;

\( u \) is the applied force to move the cart;

\( d \) is the disturbance.
The system equations are given in APPENDIX. In the simulation, the parameters are chosen as \( m_p = 0.05 \) Kg; \( m_c = 1 \) Kg; \( L = 0.5 \) m; \( g = 9.8 \) m/s\(^2\) and \( d = 0.05 \). Five Gaussian-type membership functions are constructed from (13) with the centers at \( w_c = -1.5, -0.5, 0, 0.5 \) and 1.5, respectively. For each membership function, \( w_s \) and \( w_d \) are set at \( w_s = 1 \) and \( w_d = 0.15 \). From (12), 25 singletons rules will be generated. And the consequent part characterized by an adjustable singleton fuzzy membership functions \( \theta_i \) are initialized as a vector which is defined within a reasonable range and got from human knowledge. In order to express this initial vector, it is reshaped as a 5x5 matrix as

\[
\hat{\theta}_0 = \begin{bmatrix}
\hat{\theta}_1 \\
\hat{\theta}_2 \\
\vdots \\
\hat{\theta}_{21} \\
\hat{\theta}_{22} \\
\hat{\theta}_{23} \\
\hat{\theta}_{24} \\
\hat{\theta}_{25}
\end{bmatrix} = 
\begin{bmatrix}
-0.5 & -0.5 & -0.5 & -0.5 & 0 \\
-0.5 & -0.5 & -0.5 & 0 & 0.5 \\
-0.5 & 0 & 0.5 & 0.5 \\
0 & 0.5 & 0.5 & 0.5 \\
\end{bmatrix}
\]

For the HFSM decoupling control, the hierarchical sliding surface is defined as in Eq. (8), where \( s_1 \) and \( s_2 \) are built with \( \lambda_1 = 5 \) and \( \lambda_2 = 0.5 \), respectively. The control law is designed as in Eq. (26) with the adaptive law given in (24) where \( \alpha = 0.1 \) and \( \epsilon = 10 \). The simulation results for the proposed HFSM decoupling control and the FSM decoupling control [11] from initial conditions \( x_1 = -60^\circ, \ x_2 = 0^\circ / s, \ x_3 = 0 \) m and \( x_4 = 0 \) m/s are shown in Fig. 3. It is shown that the optimal coupling factor has been tuned to be 0.55. By moving the cart back and forth along the horizontal direction, the angle of the pole can be balanced to zero and the position of cart can be returned to the origin. Meanwhile, it is demonstrated that the overshoot and the settling time of \( x_1 \) and \( x_3 \) with HFSM decoupling control is better than that with FSM decoupling control. For a large initial condition \( x_1 = -85^\circ, \ x_2 = 0^\circ / s, \ x_3 = 0 \) m and \( x_4 = 0 \) m/s, the simulation results for the HFSM decoupling control and FSM decoupling control are shown in Fig. 4. It is shown that the optimal coupling factor has been tuned to be 0.27, and the system performance can be clearly improved by using the proposed HFSM decoupling control.

## V. CONCLUSIONS

A hierarchical fuzzy sliding-mode controller has been proposed to achieve decoupling performance for a single-input multi-output nonlinear coupled system. The sliding-mode control method can be combined with the adaptive fuzzy control to treat coupled system problems such as instabilities, coupling effects, and disturbances, etc. The advantages of this approach are that the adaptive law of coupling factor is tuned based on a Lyapunov function, so this control method can guarantee that system stability and decoupling performance can be improved for the coupled nonlinear system. In the simulation example, it is shown that the system performance is considerably improved and that the system also exhibits the desired stability and robustness.

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## APPENDIX

**System Equations for the Inverted Pendulum System**

\[
f_1 = \frac{m_p \sin x_1 - m_p \sin x_1 \cos x_1 x_2^2}{L \cdot \left( \frac{4}{3} m_t - m_p \cos^2 x_1 \right)}
\]

\[
b_1 = \frac{\cos x_1}{L \cdot \left( \frac{4}{3} m_t - m_p \cos^2 x_1 \right)}
\]

\[
f_2 = \frac{4}{3} m_p L x_2^2 \sin x_1 + m_p g \sin x_1 \cos x_1
\]

\[
b_2 = \frac{\frac{4}{3} m_t - m_p \cos^2 x_1}{3 \cdot \left( \frac{4}{3} m_t - m_p \cos^2 x_1 \right)}
\]

\( m_c \) mass of the cart; \( m_p \) mass of the pole; \( m_t = m_c + m_p \); \( L \) length of pole; \( g \) acceleration of gravity.

## REFERENCES


Fig. 1. The concept diagram of HFSM control system

Fig. 2. The structure of inverted pendulum

Fig. 3(a) The $x_1$ (angle) response diagram of inverted pendulum ($x_1(0) = -60^\circ$)

Fig. 3(b) The $x_3$ (position) response diagram of inverted pendulum ($x_1(0) = -60^\circ$)

Fig. 3(c) The control input diagram of inverted pendulum ($x_1(0) = -60^\circ$)
Fig. 3(d) The coupling factor $\hat{n}$ ($x_1(0) = -60^\circ$)

Fig. 4(a) The $x_1$ (angle) response diagram of inverted pendulum ($x_1(0) = -85^\circ$)

Fig. 4(b) The $x_3$ (position) response diagram of inverted pendulum ($x_1(0) = -85^\circ$)

Fig. 4(c) The control input diagram of inverted pendulum ($x_1(0) = -85^\circ$)

Fig. 4(d) The coupling factor $\hat{n}$ ($x_1(0) = -85^\circ$)