Hybrid Synthesis for Almost Asymptotic Regulation of Linear Impulsive Systems with Average Dwell Time

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Abstract—This paper deals with the hybrid output regulation problem for a class of linear impulsive systems with average dwell time. The hybrid regulator is constructed as a linear impulsive system that undergoes synchronous impulses with the controlled plant, and the hybrid synthesis conditions are formulated in terms of two matrix equations plus a set of linear matrix inequalities (LMIs). With the proposed hybrid synthesis scheme, both continuous-time and discrete-time dynamics of the hybrid regulator can be jointly synthesized by solving a convex optimization problem of minimizing the weighted $\mathcal{L}_2$ gain from the perturbation signal to the error output. A numerical example is used to demonstrate the proposed approach.

I. INTRODUCTION

The problem of output regulation, which concerns controlling a given plant such that its output tracks references or rejects disturbances generated by an exosystem, has been extensively studied over the past decades for various dynamical systems (see, [1], [2], [3] and the references therein). In particular, necessary and sufficient conditions for the solvability of the output regulation problem for linear time-invariant (LTI) systems were nicely established as a set of linear algebraic equations in [1], which paves the fundamentals for the subsequent researches on, for instance, optimal output regulation [4]; output regulation with saturations [5]; output regulator syntheses with multi-performance objectives [6], and also involves in many practical applications [7]. It is not until the nineties that the output regulation problem for general nonlinear systems was first pursued by the pioneering works [8] and [2]. Recent research on the output regulation problem has shifted to hybrid dynamical systems that combine continuous-time and discrete-time behaviors, such as [9], [10], [3], [11].

In this paper, we study the hybrid output regulation problem for a class of linear systems subject to impulse effects. This type of systems are classified as linear impulsive dynamical systems that are typically modeled by the combination of ordinary differential equations and instantaneous state jumps or resets (also referred to as impulse) [12], [13]. As a special case of hybrid systems, linear impulsive systems have been widely involved in the research in control community, due mainly to their presence in practical systems and potentials in overcoming limitations of traditional controllers (see, e.g., [12], [14]). Stability properties of such systems have been extensively investigated, see, for instance, [12], [15], [13]. However, few results on the output regulation problem of linear impulsive systems can be found in the literature. In particular, it is worth to mention the latest contribution [3] in which the classical output regulation concepts were extended to single-input single-output (SISO) linear systems with periodic state jumps and a series of fundamental theories for hybrid regulator design were proposed. This work was subsequently applied in [11] to design hybrid regulators with different structures. These results, however, rely heavily on the periodicity of the impulse sequences.

In this paper, the impulse instants of the linear impulsive system under consideration are allowed to occur non-periodically, but with a frequency constraint, which is formalized in terms of an average dwell time (ADT) condition [16], [13]. On the other hand, as opposed to the typical treatments in classical output regulation problem of considering an exosystem with autonomous dynamics and the reference/disturbance signals that are precisely known, we will also generalize the hybrid output regulation problem for linear impulsive systems with unknown perturbations on both controlled plant and the exosystem, which leads to the notion of almost asymptotic regulation. From this perspective, the hybrid regulator design problem is treated under a multiobjective framework, such that both objectives of output regulation and weighted $\mathcal{H}_\infty$ controlled performance (from [17]) can be achieved. The proposed hybrid output regulator is constructed as a linear impulsive system, and the resulting hybrid multiobjective synthesis conditions are formulated in terms of tractable conditions, consisting of linear algebraic equations and linear matrix inequalities (LMIs). Another novelty of the proposed hybrid synthesis scheme is that the hybrid regulator gain matrices with respect to both continuous-time and discrete-time dynamics can be jointly (simultaneously) synthesized in a systematic and unified framework, i.e., by solving a convex LMI-based optimization.

Notation. $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}_+$ for the positive real numbers. $\mathbb{R}^{m \times n}$ is the set of real $m \times n$ matrices. The transpose of a real matrix $M$ is denoted by $M^T$. The hermitian operator $He\{\cdot\}$ is defined as $He\{M\} = M + M^T$ for real matrices. The identity matrix of any dimension is denoted by $I$. $\mathbb{S}^n$ and $\mathbb{S}_+^n$ are used to denote the set of real symmetric $n \times n$ matrices and positive definite matrices, respectively. A block diagonal matrix with matrices $X_1, X_2, \ldots, X_p$ on its main diagonal is denoted by $diag\{X_1, X_2, \ldots, X_p\}$. Furthermore, we use the symbol $*$ in LMIs to denote entries that follow from symmetry. For $x \in \mathbb{R}^n$, its norm is defined as $\|x\| := (x^T x)^{1/2}$. The space of square integrable functions is denoted by $\mathcal{L}_2$, that is, for
any \( u \in L^2 \), \( \|u\|^2 := \left[ \int_0^1 u^T(t)u(t)dt \right]^{1/2} \) is finite. For two integers \( k_1 < k_2 \), we denote \( I_{[k_1, k_2]} = \{k_1, k_1+1, \ldots, k_2\} \).

II. PROBLEM STATEMENT AND BASIC DEFINITIONS

We consider a linear impulsive plant

\[
\begin{aligned}
&\dot{x}_p(t) = A_p x_p(t) + A_w w(t) + B_{1p} d(t) + B_{2p} u(t), \\
&x^+_p(t) = J_p x_p(t), \\
&w(t) = A_c w(t) + Q d(t), \\
&e(t) = C_p x_p(t) + C_{sw} w(t) + D_{p1d} d(t) + D_{p2u} u(t), \\
y(t) = C_{px} x_p(t) + C_{sw} w(t) + D_{p2d} d(t)
\end{aligned}
\]

where \( \{t_1, t_2, \ldots\} \) is a strictly increasing sequence of impulse times in \((t_0, \infty)\) for some initial time \( t_0 \), it is assumed to contain a finite number of elements on any finite time interval. The plant state \( x_p(t) \in \mathbb{R}^n \) is absolutely continuous between impulses and assumed to be left-continuous with \( x^+(t) := \lim_{s \to t^-} x(s), \ w(t) \in \mathbb{R}^{n_w} \) is the exogenous variable governed by the “exosystem” \( \dot{w}(t) = A_c w(t) + Q d(t), \ u(t) \in \mathbb{R}^{n_u} \) is the control input, \( d(t) \in \mathbb{R}^{n_d} \) is the unknown perturbation input that injects to both plant and exosystem dynamics, and \( e(t) \in \mathbb{R}^{n_e} \) is the error output to be regulated, \( y(t) \in \mathbb{R}^{n_y} \) is the measurement output available for feedback control use. We further assume that \( (A1) A_c \) is anti-Hurwitz (i.e., all eigenvalues of \( A_c \) are in the closed right half-plane); \( (A2) \) the pair \( (A_p, B_{2p}) \) is stabilizable; \( (A3) \) the pair \( \left( \begin{bmatrix} A_p & A_w \\ 0 & A_{c2} \end{bmatrix}, \begin{bmatrix} C_{px} \\ C_{sw} \end{bmatrix} \right) \) is detectable. These three assumptions do not cause a loss of generality, we follow the same discussions of these assumptions as presented in [5] for LTI case. In this paper, we focus our study of system (1) subject to a class of impulse time sequences with average dwell time (ADT).

**Definition 1:** Given an impulse time sequence \( \{t_k\} \), we said that the linear impulsive system (1) possesses the property of ADT impulse, if there exist two positive numbers \( N_0 \) and \( \tau_a \) such that \( N(t,s) \leq N_0 + \frac{s}{\tau_a}, \ \forall t \geq s \geq t_0, \) where \( N(t,s) \) denotes the number of impulses in the semi-open interval \((s,t]\), \( N_0 \) is the chatter bound, and \( \tau_a \) is called the “average dwell time”.

We will consider a hybrid dynamic output-feedback law of the form

\[
\begin{aligned}
&\dot{x}_c(t) = A_c x_c(t) + B_c y(t), \quad \text{if} \quad t \neq t_k, \quad k \in \{1, 2, \ldots\} \\
&x^+_c(t) = J_c x_c(t), \quad \text{if} \quad t = t_k, \quad k \in \{1, 2, \ldots\} \\
&u(t) = C_c x_c(t) + D_c y(t)
\end{aligned}
\]

where \( x_c(t) \in \mathbb{R}^{n_c} \) is the continuous-time state of the controller that undergoes instantaneous jumps at each impulse time instant \( t_k \). The impulse time sequence is assumed to be synchronous between the impulsive plant and the controller.

The closed-loop system formed by the interconnection of the impulsive plant (1) and the hybrid controller (2) can be expressed in a general linear hybrid impulsive system as below

\[
\begin{aligned}
&\dot{x}_{cl}(t) = A_{cl} x_{cl}(t) + A_{clw} w(t) + B_{cl} d(t), \quad \text{if} \quad t \neq t_k, \quad k \in \{1, 2, \ldots\} \\
&x^+_{cl}(t) = A_{cl} x_{cl}(t), \quad \text{if} \quad t = t_k, \quad k \in \{1, 2, \ldots\} \\
&\dot{w}(t) = A_c w(t) + Q d(t), \\
e(t) = C_{cl} x_{cl}(t) + C_{clw} w(t) + D_{cl} d(t)
\end{aligned}
\]

where \( x_{cl} := \begin{bmatrix} x^T_p & x^T_c \end{bmatrix}^T \) and

\[
\begin{bmatrix}
A_{cl} & A_{clw} \\
C_{cl} & C_{clw}
\end{bmatrix} =
\begin{bmatrix}
A_p + B_{1p} D_{c2} C_{px} + B_{2p} C_{c2} \\
B_p C_{px} + D_{c21} D_{c22} C_{px}
\end{bmatrix}
\]

with \( A_f := \begin{bmatrix} J_p & 0 \\ 0 & J_c \end{bmatrix} \).

Before proceeding further, we have the following basic definitions.

**Definition 2:** Given an impulse time sequence \( \{t_k\} \), the linear hybrid impulsive system (3) with \( w(t) \equiv 0 \) and \( d(t) \equiv 0 \) is globally uniformly asymptotically stable (GUAS) if there exists a class \( \mathcal{K} \) function \( \beta \) such that for all \( t \geq t_0 \) and \( x_{cl}(t_0) \), the corresponding solution to (3) exists globally and satisfies \( \|x_{cl}(t)\| \leq \beta(\|x_{cl}(t_0)\|, t-t_0) \).

**Definition 3:** Given an impulse time sequence \( \{t_k\} \), the linear hybrid impulsive system (3) is said to achieve almost asymptotic output regulation, if there exists a function \( \beta \in \mathcal{K} \) such that the error output signal \( e(t) \) satisfies \( \|e(t)\| \leq \beta(\|x_{cl}(t_0)\| + \|w(t_0)\|-t_0) \) for all \( t \geq t_0 \), some \( \epsilon > 0 \) and any \( x_{cl}(t_0), w(t_0) \), \( e(t) \) is said to be exactly asymptotically regulated if the above condition holds and \( d(t) \equiv 0 \).

**Definition 4:** Given an impulse time sequence \( \{t_k\} \), the linear hybrid impulsive system (3) is said to be GUAS with a weighted \( L_2 \) gain \( \gamma > 0 \) from \( d(t) \) to \( e(t) \), if under zero initial conditions, the system (3) is GUAS and satisfies

\[
\int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \|e(t)\|^2 dt \leq \gamma^2 \int_{t_0}^{\infty} \|d(t)\|^2 dt, \quad \forall t \geq t_0
\]

for some \( \lambda > 0 \) and any \( d(t) \) with finite energy \( \int_{t_0}^{\infty} \|d(t)\|^2 dt < \infty \).

Our objective in this paper is to (i) construct a hybrid output-feedback law in the form of (2) that renders the hybrid closed-loop system (3) GUAS with guaranteed weighted \( L_2 \) performance, and achieves almost asymptotic output regulation; (ii) formulate the synthesis problem with computationally tractable conditions such that the hybrid controller gain matrices with respect to both continuous-time and discrete-time dynamics can be jointly designed in a systematic way.

III. LYAPUNOV-BASED ANALYSIS CONDITIONS

In this section, with the ADT impulse constraint, sufficient conditions for almost asymptotic regulation with weighted \( L_2 \) performance of the hybrid impulsive system (3) will be established by using a quadratic Lyapunov-like function. Note that different from classical Lyapunov functions, the Lyapunov-like function adopted in this paper for hybrid impulsive system analysis is allowed to have discontinuities at each impulse time instant with bounded jumps, but required to be continuous and to decrease between impulses with a lower bounded rate.

**Theorem 1:** Consider the hybrid impulsive system (3). Given two scalars \( \lambda_0 \in \mathbb{R}_+ \) and \( \mu > 1 \), if there exist a...
positive definite matrix $P \in \mathbb{S}_{n+nc}^+$, a rectangular matrix $\Pi \in \mathbb{R}^{(n+nc) \times nw}$ and a positive scalar $\gamma \in \mathbb{R}^+$ such that the following conditions hold.

$$
A_{cl} \Pi - \Pi A_{cl} + A_{cl,w} w = 0 \quad (5)
$$

$$
C_{cl} \Pi + C_{cl,w} w = 0 \quad (6)
$$

$$
\begin{bmatrix}
B e \{ P A_{cl} \} + \lambda_0 P & * \\
B \Pi - C_{cl}^T \Pi^T P & -\gamma I & * \\
C_{cl} & D_{cl} & -\gamma I
\end{bmatrix} < 0 \quad (7)
$$

$$
\begin{bmatrix}
\mu P & * \\
\mu P - P A_{cl} & (\mu - 1) P & * \\
P A_{cl} & 0 & P
\end{bmatrix} \geq 0 \quad (8)
$$

Then, the hybrid impulsive system (3) is GUAS with a weighted $\dot{L}_2$ gain $\gamma$, and achieves almost asymptotic regulation for every impulsive time sequence $\{ t_k \}$ with average dwell time $\tau_k \geq \ln(\mu)/\lambda_0$. \hfill (9)

**Proof:** Apply the coordinate transformation $\tilde{x}_{cl} = x_{cl} - \Pi w$ to the system state in (3), we obtain the following hybrid impulsive system

$$
\begin{cases}
\dot{\tilde{x}}_{cl}(t) = A_{cl} \tilde{x}_{cl}(t) + (A_{cl} \Pi - \Pi A_{cl} + A_{cl,w} w) u(t) + (B_{cl} - \Pi Q) d(t), & t \neq t_k, \ k \in \{1, 2, \ldots\} \\
\tilde{x}_{cl}(t) = A_{cl} \tilde{x}_{cl}(t) - \Pi w(t), & t = t_k, \ k \in \{1, 2, \ldots\} \\
e(t) = C_{cl} \tilde{x}_{cl}(t) + (C_{cl} \Pi + C_{cl,w} w) u(t) + D_{cl} d(t)
\end{cases} \quad (10)
$$

Note that different from the original impulsive system (3), the discrete dynamics governing the system state jumps in the transformed system (10) also depend on the exosystem state $w(t)$, in addition to the system state itself $\tilde{x}_{cl}(t)$. Furthermore, with conditions (5)–(6), the transformed system (10) reduces to

$$
\begin{cases}
\dot{\tilde{x}}_{cl}(t) = A_{cl} \tilde{x}_{cl}(t) + (A_{cl} \Pi - \Pi A_{cl} + A_{cl,w} w) u(t) + (B_{cl} - \Pi Q) d(t), & t \neq t_k, \ k \in \{1, 2, \ldots\} \\
\tilde{x}_{cl}(t) = A_{cl} \tilde{x}_{cl}(t) - \Pi w(t), & t = t_k, \ k \in \{1, 2, \ldots\} \\
e(t) = C_{cl} \tilde{x}_{cl}(t) + (C_{cl} \Pi + C_{cl,w} w) u(t) + D_{cl} d(t)
\end{cases} \quad (11)
$$

Since $e(t)$ does not explicitly relate to the exosystem state $w(t)$ and $D_{cl}$ is a constant matrix, according to the Definition 3, to obtain almost asymptotic regulation, it suffices to prove that the above transformed system (11) is GUAS.

Consider the Lyapunov-like function $V(\tilde{x}_{cl}) = \tilde{x}_{cl}^T P \tilde{x}_{cl}$ for the system (11) with $d(t) \equiv 0$. From the (1,1) element of the matrix in condition (7), we have

$$
V(\tilde{x}_{cl}) = \tilde{x}_{cl}^T (A_{cl}^T P + P A_{cl}) \tilde{x}_{cl} = -\lambda_0 V(\tilde{x}_{cl}) \quad (12)
$$

for all $\tilde{x}_{cl} \in \mathbb{R}^{n+nc}$. On the other hand, by Schur complement, condition (8) is equivalent to

$$
\begin{bmatrix}
\mu P & -A_{cl}^T P A_{cl} \\
\mu P - P A_{cl} & \mu P - P
\end{bmatrix} \geq 0 \quad (13)
$$

which implies that

$$
\begin{bmatrix}
x_{cl} \\
-\Pi w
\end{bmatrix}^T
\begin{bmatrix}
\mu P & P & P \\
P & P & P \\
-\Pi w & -A_{cl}^T P A_{cl} & A_{cl}^T P
\end{bmatrix}
\begin{bmatrix}
x_{cl} \\
-\Pi w
\end{bmatrix} \geq 0,
$$

for all $x_{cl} \in \mathbb{R}^{n+nc}$ and $w \in \mathbb{R}^{nw}$. \hfill (14)

Condition (12) with (14) yields that for any $t \in [t_k, t_{k+1})$, $k \in \{1, 2, \ldots\}$,

$$
\begin{align*}
V(\tilde{x}_{cl}(t)) &< e^{-\lambda_0(t-t_k)} V(\tilde{x}_{cl}(t_k)) \leq e^{-\lambda_0(t-t_k)} V(\tilde{x}_{cl}(t_k)) \\
&< \mu^2 e^{-\lambda_0(t-t_{k-1})} V(\tilde{x}_{cl}(t_{k-1})) < \cdots \\
&< \mu^{N(t,t_0)} e^{-\lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0)) \\
&= e^{\frac{1}{\tau_k} \ln(\mu) - \lambda_0(t-t_0)} V(\tilde{x}_{cl}(t_0))
\end{align*}
$$

Thus, if the average dwell time $\tau_k$ satisfies constraint (9), we conclude that $V(\tilde{x}_{cl})$ converges to zero as $t \to \infty$. Thereby, system (11) is GUAS since $P$ is a constant positive definite matrix, and $e(t)$ is almost asymptotically regulatable.

Regarding the weighted $\dot{L}_2$ performance from $d(t)$ to $e(t)$, we obtain from condition (7) that

$$
V(\tilde{x}_{cl}) < -\lambda_0 V(\tilde{x}_{cl}) - \frac{1}{\gamma} e^{T} e + \gamma d^T d \quad (15)
$$

for all $\tilde{x}_{cl} \in \mathbb{R}^{n+nc}$ and $d \in \mathbb{R}^{nu}$. Based on the conditions (15) and (14), the proof can be completed by following a similar line of the proof of Theorem 1 presented in [17] for switched linear systems, readers are referred to this reference for more details.

Conditions (5)–(6) are two linear matrix equations which essentially correspond to the output regulation conditions for LTI systems [1], while condition (8) is termed as the boundary condition that constrains the possible increase of the system energy caused by the abrupt state jumps at each impulse time.

### IV. HYBRID SYNTHESIS AND CONTROLLER CONSTRUCTION

Without loss of generality, we parameterize as below the coefficient matrices associated with the continuous-time dynamics of the hybrid controller (2), i.e.,

$$
\begin{bmatrix}
A_c & B_c \\
C_c & D_c
\end{bmatrix} = \begin{bmatrix}
A_{cl} & -D_{cl} (C_{cl} w + C_{cl} \Pi w) & C_{cl} & D_{cl} \\
B_{cl} & C_{cl} & -B_{cl} (C_{cl} w + C_{cl} \Pi w) & A_{cl} \\
C_{cl} & D_{cl} & \frac{-D_{cl} C_{cl} w - B_{cl} C_{cl} \Pi w}{\Pi w} & C_{cl} \\
C_{cl} & D_{cl} & \frac{-D_{cl} C_{cl} w - B_{cl} C_{cl} \Pi w}{\Pi w} & C_{cl}
\end{bmatrix}
$$

where $A_{cl} \in \mathbb{R}^{(n+nc) \times (n+nc)}$, $B_{cl} \in \mathbb{R}^{(n+nc) \times nw}$, $C_{cl} \in \mathbb{R}^{n+nc \times (n+nc)}$, $D_{cl} \in \mathbb{R}^{n+nc \times nw}$, which together with $\Pi w \in \mathbb{R}^{nw}$, $\Gamma \in \mathbb{R}^{nu \times nw}$ and the matrix $J_c \in \mathbb{R}^{n+nc \times n+nc}$ in a general form corresponding to the discrete-time dynamics, are free variables subject to design. The adoption of this type of controller structure is motivated by the internal model principle for LTI systems [18].

With this parametrization, the resulting closed-loop system (3) can be rewritten by partitioning $x_{cl} = [x_{cl,w} \ x_{cl,c}]^T$, where $x_{cl,w} \in \mathbb{R}^{nw}$ and $x_{cl,c} \in \mathbb{R}^{n+nc}$. Accordingly, the system matrices become

$$
\begin{bmatrix}
A_{cl} & 0 \\
B_{cl} & 0
\end{bmatrix} = \begin{bmatrix}
A_{cl} & 0 \\
B_{cl} & 0
\end{bmatrix} \begin{bmatrix}
D_{cl} & C_{cl} \\
B_{cl} & C_{cl}
\end{bmatrix},
$$

$$
B_{cl} = \begin{bmatrix}
B_{cl,w} & 0 \\
0 & D_{cl}
\end{bmatrix}
$$
\[ C_{cl} = \begin{bmatrix} \begin{bmatrix} C_{p1} & D_{p12} \Gamma \end{bmatrix} + \begin{bmatrix} D_{p12} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{D}}_{c1} & \bar{\bar{D}}_{c2} \end{bmatrix} \times \begin{bmatrix} C_{p2} & -(C_{w2} + C_{p2} \Pi_{p}) \end{bmatrix} \end{bmatrix} \end{bmatrix}. \]

\[ D_{cl} = D_{p11} + [D_{p12} \quad 0] \begin{bmatrix} \bar{\bar{D}}_{c1} & \bar{\bar{D}}_{c2} \end{bmatrix} J_{p12}, \]

\[ A_t = \begin{bmatrix} J_p & 0 & 0 \\ 0 & J_{c11} & J_{c12} \\ 0 & J_{c21} & J_{c22} \end{bmatrix}. \]

where \( J_{c11} \in \mathbb{R}^{n_u \times n_w}, J_{c12} \in \mathbb{R}^{n_u \times (n_c-n_w)}, J_{c21} \in \mathbb{R}^{(n_c-n_u) \times n_w}, J_{c22} \in \mathbb{R}^{(n_c-n_u) \times (n_c-n_w)}. \)

Then, we have the hybrid synthesis conditions, together with the hybrid controller construction algorithms, summarized in the following theorem.

**Theorem 2:** Consider the hybrid impulsive closed-loop system (3). Given two scalars \( \lambda_0 \in \mathbb{R}_+ \) and \( \mu > 1 \), if there exist positive definite matrices \( R \in \mathbb{S}^{n_u+n_w}, \bar{S} \in \mathbb{S}^{n_c}, \) rectangular matrices \( \bar{A}_c \in \mathbb{R}^{(n_u+n_w) \times (n_u+n_w)}, \bar{B}_c \in \mathbb{R}^{(n_u+n_w) \times n_y}, \bar{D}_c \in \mathbb{R}^{(n_u+n_w) \times n_z}, J_{c11} \in \mathbb{R}^{n_u \times n_w}, J_{c12} \in \mathbb{R}^{n_u \times (n_c-n_w)}, J_{c21} \in \mathbb{R}^{(n_c-n_u) \times n_w}, J_{c22} \in \mathbb{R}^{(n_c-n_u) \times (n_c-n_w)}, \) and a positive scalar \( \gamma \in \mathbb{R}_+ \) such that the conditions (18)–(22) hold.

\[ \Pi_{A_c} A_c - A_c \Pi_{A_c} + B_{p2} R = \begin{bmatrix} R & \bar{S} \\ \bar{S} & I \end{bmatrix} \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix}^T \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix} > 0 \]

Then, there exists a hybrid controller of the form (2) with order \( n_c = n + 2n_w \) that renders the closed-loop system (3) GUAS with a weighted \( L_2 \) gain \( \gamma \), and achieves almost asymptotic regulation for every impulse time sequence \( \{t_k\} \) with average dwell time \( \tau_d \geq \frac{\ln(\mu)}{\lambda_0} \). Moreover, the hybrid controller gain matrices are readily obtained through the following algorithm:

1. Let \( W = \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix} \) to obtain \( S = W^{-T} \bar{S} W^{-1} \). Solve \( N, M \) from the factorization problem \( I - R S = M N^T \).
2. Compute the associated matrices with
   \[
   \begin{bmatrix}
   \bar{A}_c & \bar{B}_c \\
   \bar{C}_c & \bar{D}_c
   \end{bmatrix} = \begin{bmatrix}
   W^T & W^T \bar{S} & \begin{bmatrix} B_{p2} & 0 \end{bmatrix} & 0 & I
   \end{bmatrix}^{-1}
   \]
   \[ \times \begin{bmatrix}
   \bar{A}_c - W^T S & \begin{bmatrix} A_p & B_{p2} \Gamma \end{bmatrix} R & \bar{B}_c \\
   \bar{C}_c & \begin{bmatrix} 0 & 0 \end{bmatrix} & \bar{D}_c
   \end{bmatrix}^{-1}
   \times \begin{bmatrix}
   C_{p2} - (C_{w2} + C_{p2} \Pi_{p}) & 0 & \begin{bmatrix} M^T \\ I \end{bmatrix} & I & 0
   \end{bmatrix} \]
   \[ \begin{bmatrix} J_{c22} & J_{c21} \\
   J_{c12} & J_{c11}
   \end{bmatrix} = \begin{bmatrix}
   W^T & W^T \bar{S} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & 0 & I
   \end{bmatrix}^{-1}
   \]
   \[ \times \begin{bmatrix}
   J_{c22} - W^T S & \begin{bmatrix} J_p & 0 \end{bmatrix} R & J_{c21} - W^T S & \begin{bmatrix} J_p \Pi_p \\ 0 \end{bmatrix} \\
   J_{c12} & J_{c11}
   \end{bmatrix} \times \begin{bmatrix}
   M^T & 0 & 0 \\ I & R & I
   \end{bmatrix}^{-1}
   \]
3. Obtain the realization of the hybrid impulsive controller in the form of (2) via (16).

**Proof:** Based on the results of Theorem 1, the regulation conditions (5)–(6) can be verified by using the controller structure (16) and specifying \( \Pi = \begin{bmatrix} \Pi_{\gamma} & I_{n_w} & 0 \end{bmatrix}^T \).

For conditions (7)–(8), we partition the Lyapunov matrix
\[ P = \begin{bmatrix} S & N \\ N^T & X \end{bmatrix}^{-1} \]
with \( S \in \mathbb{S}^{n_u+n_w}, X^{-1} \in \mathbb{S}^{n_c}, \) and let
\[ Z_1 = \begin{bmatrix} R & 0 \\ M^T & I \end{bmatrix}, Z_2 = \begin{bmatrix} I & S \\ 0 & N^T \end{bmatrix} \]
such that \( PZ_1 = Z_2, NM^T = I - SR \) and \( X^{-1} = -N^T RM^{-T} \). Then, we specify
\[ W = \begin{bmatrix} I & \Pi_p \\ 0 & I \end{bmatrix}, \quad \tilde{Z}_1 = \begin{bmatrix} R & W \\ M^T & I \end{bmatrix}, \quad \tilde{Z}_2 = \begin{bmatrix} I & SW \\ 0 & N^TW \end{bmatrix}. \]

Perform congruence transformation on condition (7) with matrices \( diag(\tilde{Z}_1, I, I) \), we have
\[ \tilde{Z}_1^T P \tilde{Z}_1 = \begin{bmatrix} R & W \\ W^T & \bar{S} \end{bmatrix}, \]
\[ \tilde{Z}_1^T P A_c \tilde{Z}_1 = \tilde{Z}_2^T A_c \tilde{Z}_1 \]
\[ = \begin{bmatrix}
   A_p & 0 & B_{p2} R_p \\
   0 & A_e & 0 & B_{p2} \Gamma \\
   0 & 0 & A_e & 0 & B_{p2} \Gamma \\
   J_{c22} - W^T S & \begin{bmatrix} J_p & 0 \end{bmatrix} R & J_{c21} - W^T S & \begin{bmatrix} J_p \Pi_p \\ 0 \end{bmatrix} & \begin{bmatrix} M^T & 0 \\ I & R \end{bmatrix}
   \end{bmatrix}^{-1} \]

(17)

Thus, condition (20) confirms that \( P \) is positive definite, and the results obtained above verify the equivalence between conditions (7) and (21).

To verify the boundary condition (8), we perform again the congruence transformation on (8) with another matrix
\( \begin{bmatrix} H \hat{c} \left\{ \begin{array}{ll} A_p & 0 \\ 0 & A_c \end{array} \right\} R + \begin{bmatrix} B_{p2} \\ 0 \end{bmatrix} \hat{c} c + \lambda_0 R \\ \hat{A}_c + \begin{bmatrix} A_p & A_p \pi_p + B_{p2} \Gamma \end{bmatrix}^T \\ + \begin{bmatrix} C_{p1}^T \\ -C_{p2}^T \end{bmatrix} \hat{D}_I \begin{bmatrix} B_{p2} \\ 0 \end{bmatrix}^T \right\} + \lambda_0 R \right) ^\ast + \left( \begin{array}{ll} ^\ast \end{array} \right) \left\{ \begin{array}{ll} 0 \\ 0 \end{array} \right\} \right\} 
\end{align*}

\( \begin{align*}
\mu R & \begin{bmatrix} \mu I - J_p & \mu \pi_p - J_p \pi_p \\ \mu I & \mu \pi_p \end{bmatrix}^T \\
\mu S & \begin{bmatrix} \mu I - J_p & \mu I - J_c \pi_p \\ \mu I & \mu I \end{bmatrix}^T \\
\hat{J}_{c22} & \begin{bmatrix} J_p & J_p \pi_p \\ 0 & \hat{J}_{c11} \end{bmatrix} \end{align*} 
\)

\begin{align*}
J_{c11} & = J_{c11}, \\
J_{c22} & = J_{c22} M^T + J_{c11} [0 \ I] R, \\
J_{c21} & = W^T S [J_p \pi_p + W^T S [0 \ I] J_{c11} + W^T N J_{c21}, \\
J_{c22} & = W^T S [J_p 0 + W^T S [0 \ I] J_{c11} [0 \ I] ] R \end{align*}

\begin{align*}
+ W^T N J_{c21} [0 \ I] R + W^T S [0 \ I] J_{c12} M^T \\
+ W^T N J_{c22} M^T. 
\end{align*}

With this, condition (22) can be deduced from condition (8). Furthermore, the hybrid controller formulas (23) can also be established by converting the relations (24) and (25).

Note that all the design variables appear linearly in the synthesis conditions (18)–(22). The results provided in Theorem 2 can be used to pose an optimization problem of minimizing the weighted \( L_2 \) gain \( \gamma \), and thus to provide a solution to the hybrid synthesis problem of jointly designing the hybrid controller gain matrices with respect to both continuous-time and discrete-time dynamics. Given the average dwell time parameters \( \lambda_0, \mu \), the hybrid control synthesis can be formulated as the following LMIs optimization problem

\[
\begin{align*}
\min_{\gamma, R, S, \hat{A}_c, \hat{D}_c, \hat{C}_c, \hat{J}_{c11}, \hat{J}_{c21}, \hat{J}_{c22}, \pi_p, \Gamma} & \gamma, \\
\text{s.t.} & \ (18)–(22).
\end{align*}
\]

\[\text{V. SIMULATION RESULTS}\]

In this section, we will demonstrate the proposed hybrid output regulation approach for linear impulsive systems using the following example

\[
\begin{align*}
\dot{x}_p(t) &= \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} x_p(t) + A_w w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\
& \quad t \neq t_k, k \in \{1, 2, \ldots\} \\
x_p^+(t_k) &= \begin{bmatrix} 1 & -1 \\ 0 & 1.5 \end{bmatrix} x_p(t_k). \\
e(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_p(t), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_p(t) + 0.1 d(t)
\end{align*}
\]

with

\[
A_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_c = \begin{bmatrix} 0 & -3 & 0 & -1 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}
\]

The controlled plant is a second-order linear impulsive system with only the second state undergoing a jump at each impulse time. The perturbation \( d(t) \) provides a small disturbance to both the exosystem and the plant as a sensor noise. The exosystem generates a command signal \( w(t) \) with multiple frequencies at 2.80 rad/sec and 1.07 rad/sec.

For simulation purpose, suppose that the impulsive plant (27) is required to operate over the time interval \( t \in [0, 50] \) sec with the following impulse time sequence \( \{5, 12, 20, 22, 25, 30, 40\} \) sec. The impulse time sequence is deliberately chosen to impose more dense impulses at the interval \( [20, 30] \) sec, so as to illustrate the effect of the average dwell time \( \tau_a \). Its associated average dwell time index can be calculated by \( \tau_a = 50/7 = 7.1429 \) sec,
where $7$ is the number of impulses occurred during the time interval $[0, 50]$ sec. As such, we specify the constant parameters $\lambda_0 = 0.1$ and $\mu = 2$ in the optimization problem (26) for hybrid controller synthesis, such that the average dwell time constraint (9) is satisfied, i.e., $\tau_a^* = \frac{\ln(\mu)}{\ln(\lambda_0)} = 6.9315 < \tau_a$. By solving the optimization problem (26) and using the controller construction algorithm in Theorem 2, one can obtain the hybrid regulator in the form of (2). Note that in order to overcome the numerical issue associated with ill-conditioning matrices, we have also constrained the eigenvalues of the closed-loop system matrix $Acl$ within a circle $|c + 10| \leq 10$ by using the method of [19]. Due to the space limitation, we have only listed the solutions of the regulation equations and the associated optimized weighted $L_2$ gain as $\Pi_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $\Gamma = \begin{bmatrix} 0 & 0 & 2 & 3 \end{bmatrix}$ and $\gamma = 0.0397$.

With initial conditions $x_p(0) = 0$, $x_e(0) = 0$, $w(0) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ and a vanishing perturbation input $d(t) = e^{-0.1t} \sin(t)$, we then carry out the time-domain simulation using the synthesized hybrid regulator. The simulation results for almost asymptotic regulation are plotted in Fig. 1. As can be seen, in spite of the effects of the system state jumps, the persistent exosignal $w(t)$ as well as the unknown perturbation $d(t)$, the hybrid impulsive regulator is capable to render the overall impulsive system stable and suppress these effects from the error output $e(t)$. Furthermore, when the perturbation signal $d(t)$ is vanished, the exosignal $w(t)$ with multiple frequency components is completely rejected in the error output $e(t)$, and exact output regulation is achieved. On the other hand, as expected, due to the impulse effects, both error output $e(t)$ and control input signal $u(t)$ exhibit abrupt jumps at each impulse time instants. Moreover, larger control efforts are enforced at the time interval $[20, 30]$ sec due to the impulses of higher frequency occurred.

VI. CONCLUSIONS

In this paper, an important problem of hybrid output regulation for linear impulsive systems has been addressed by using an average dwell time technique. The regulation problem under consideration was generalized for a class of exosystems subject to unknown perturbations, and the hybrid regulator design was treated by considering both objectives of almost asymptotic regulation and weighted $L_2$ performance. The hybrid regulator was constructed as a linear impulsive system that undergoes synchronous impulses with the controlled plant, and its synthesis conditions were formulated in terms of two matrix equations plus a set of LMIs. The proposed hybrid regulation scheme provides a systematic and unified framework for the joint design problem of hybrid regulator gains, whose effectiveness has been demonstrated via a numerical example.

REFERENCES


Fig. 1: Simulation results of hybrid output regulation.