Generating non-jumping numbers recursively

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Abstract

Let \( r \geq 2 \) be an integer. A real number \( z \in [0, 1) \) is a jump for \( r \) if there is a constant \( c > 0 \) such that for any \( \epsilon > 0 \) and any integer \( m \) where \( m \geq r \), there exists an integer \( n_0 \) such that any \( r \)-uniform graph with \( n > n_0 \) vertices and density \( \geq z + \epsilon \) contains a subgraph with \( m \) vertices and density \( \geq z + c \). It follows from a fundamental theorem of Erdős and Stone that every \( z \in [0, 1) \) is a jump for \( r = 2 \). Erdős asked whether the same is true for \( r \geq 3 \). Frankl and Rödl gave a negative answer by showing some non-jumping numbers for every \( r \geq 3 \). In this paper, we provide a recursive formula to generate more non-jumping numbers for every \( r \geq 3 \) based on the current known non-jumping numbers.

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1. Introduction

For a finite set \( V \) and a positive integer \( r \) we denote by \( \binom{V}{r} \) the family of all \( r \)-subsets of \( V \). An \( r \)-uniform graph \( G \) consists of a set \( V(G) \) of vertices and a set \( E(G) \subseteq \binom{V(G)}{r} \) of edges. The density of \( G \) is defined by \( d(G) = |E(G)|/|\binom{V(G)}{r}| \). An \( r \)-uniform graph \( H \) is called a subgraph of an \( r \)-uniform graph \( G \) if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). We write \( H \leq G \) if \( H \) is a subgraph of \( G \). A subgraph \( H \) of \( G \) is called induced if \( E(H) = E(G) \cap \binom{V(H)}{r} \).

It is easy to show that for an \( r \)-uniform graph \( G \), the average of densities of all its induced subgraphs with \( m \geq r \) vertices is \( d(G) \) (cf. [6]). Therefore, there exists a subgraph with \( m \) vertices and density \( \geq d(G) \). A natural question is whether there exists a subgraph of any given order with density \( \geq d(G) + c \), where \( c > 0 \) is a constant? To be precise, the concept of ‘jump’ is given below.

Definition 1.1. A real number \( z \in [0, 1) \) is a jump for \( r \) if there is a constant \( c > 0 \) such that for any \( \epsilon > 0 \) and any integer \( m \) where \( m \geq r \), there exists an integer \( n_0(\epsilon, m) \) such that any \( r \)-uniform graph with \( n > n_0(\epsilon, m) \) vertices and density \( \geq z + \epsilon \) contains a subgraph with \( m \) vertices and density \( \geq z + c \).

It follows from a fundamental theorem of Erdős and Stone [2] that every \( z \in [0, 1) \) is a jump for \( r = 2 \). For \( r \geq 3 \), Erdős [1] proved that every \( z \in [0, r! / r^r) \) is a jump. Furthermore, Erdős proposed the jumping constant conjecture: Every \( z \in [0, 1) \) is a jump for every integer \( r \geq 2 \). In [5], Frankl and Rödl disproved the Conjecture by showing that...
Theorem 1.1 (Frankl and Rödl [5]). \(1 - (1/ll^{-1})\) is not a jump for \(r\) if \(r \geq 3\) and \(l > 2r\).

Following a similar approach in [5], some other non-jumping numbers are given in [4,8–11]. But there are still a number of important unsolved questions regarding jumps for hypergraphs. For example, a well-known question of Erdős is to determine whether or not \(l!/(rr')\) is a jump and what is the smallest non-jumping number for \(r \geq 3\) (recall that every \(x \in [0, r!/r']\) is a jump for \(r \geq 3\)). At this moment, the current smallest known non-jumping number for \(r \geq 3\) is \(\frac{1}{2}r!/r'\) given in [4]. Another question raised in [4] is whether there is an interval of non-jumping numbers for \(r \geq 3\). From the definition of ‘jump’, we see that, if a number \(x\) is a jump, then there exists a constant \(c > 0\) such that every number in \([x, x + c]\) is a jump. Consequently, if there is a set of non-jumping numbers whose limits form an interval (number \(a\) is a limit of a set \(A\) if there is a sequence \(\{a_n\}_{n=1}^{\infty}, a_n \in A\) such that \(\lim_{n \to \infty} a_n = a\)), then every number in this interval is not a jump. We do not know whether or not such a ‘dense enough’ set of non-jumping numbers exists and it seems to be hard to answer this question. We strongly feel that there might exist a number \(a\) such that every number in \([a, 1]\) is a non-jumping number for some \(r \geq 3\). If this is true, more non-jumping numbers should be found in addition to the current known non-jumping numbers. So it is somehow interesting to produce more non-jumping numbers in a general way. In this paper we provide a recursive formula to generate more non-jumping numbers based on the known non-jumping numbers given in [4,5,8–11].

2. Preliminaries and the main result

We first give the definition of the Lagrangian of an \(r\)-uniform graph, a helpful tool in our approach. More studies of Lagrangians can be found in [3,5,7,12].

Definition 2.1. For an \(r\)-uniform graph \(G\) with vertex set \(\{1, 2, \ldots, m\}\), edge set \(E(G)\) and a vector \(\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m\), define
\[
\hat{\lambda}(G, \vec{x}) = \sum_{\{i_1, \ldots, i_r\} \subseteq E(G)} x_{i_1}x_{i_2}\ldots x_{i_r},
\]
where \(x_i\) is called the weight of the vertex \(i\).

Definition 2.2. Let \(S = \{\vec{x} = (x_1, x_2, \ldots, x_m) : \sum_{i=1}^{m} x_i = 1, x_i \geq 0\text{ for } i = 1, 2, \ldots, m\}\). The Lagrangian of \(G\), denoted by \(\lambda(G)\), is defined as
\[
\lambda(G) = \max\{\hat{\lambda}(G, \vec{x}) : \vec{x} \in S\}.
\]

A vector \(\vec{y} \in S\) is called an optimal vector for \(\lambda(G)\) if \(\hat{\lambda}(G, \vec{y}) = \lambda(G)\).

The following fact is easily implied by the definition of the Lagrangian.

Fact 2.1. If \(G_1 \subseteq G_2\), then
\[
\hat{\lambda}(G_1) \leq \hat{\lambda}(G_2).
\]

We introduce the blow-up of an \(r\)-uniform graph \(G\) allowing us to construct \(r\)-uniform graphs with arbitrary large number of vertices and densities close to \(r!\hat{\lambda}(G)\).

Definition 2.3. Let \(G\) be an \(r\)-uniform graph with \(V(G) = \{1, 2, \ldots, m\}\) and \(\vec{n} = (n_1, \ldots, n_m)\) be a positive integer vector. Define the \(\vec{n}\) blow-up of \(G\), \(\vec{n} \otimes G\) as an \(m\)-partite \(r\)-uniform graph with vertex set \(V_1 \cup \cdots \cup V_m\), \(|V_i| = n_i\), \(1 \leq i \leq m\), and edge set \(E(\vec{n} \otimes G) = \{(v_{i_1}, v_{j_2}, \ldots, v_{i_r}) : \{v_{i_1}, v_{j_2}, \ldots, v_{i_r}\} \in E(G)\}\) and \(v_{ij} \in V_i\) for \(1 \leq i \leq r\).

Remark 2.2 (Frankl and Rödl [5]). Let \(G\) be an \(r\)-uniform graph with \(m\) vertices and \(\vec{y} = (y_1, \ldots, y_m)\) be an optimal vector for \(\lambda(G)\). Then for any \(\varepsilon > 0\), there exists an integer \(n_1(\varepsilon)\), such that for any integer \(n \geq n_1(\varepsilon)\),
\[
d((\lceil ny_1 \rceil, \lceil ny_2 \rceil, \ldots, \lceil ny_m \rceil) \otimes G) \geq r!\lambda(G) - \varepsilon.
\]

Let us also state a fact which follows directly from the definition of the Lagrangian.
Fact 2.3 (Frankl and Rödl [5]). For every r-uniform graph G and every integer n, \( \lambda((n, n, \ldots, n) \otimes G) = \lambda(G) \) holds.

Lemma 2.4 proved in [5] gives a necessary and sufficient condition for a number \( \alpha \) to be a jump. This lemma establishes the connection between a jump and Lagrangians of some related uniform graphs. Before describing it, we need the following definition.

Definition 2.4. For \( \alpha \in [0, 1) \) and a family \( \mathcal{F} \) of r-uniform graphs, we say that \( \alpha \) is a threshold for \( \mathcal{F} \) if for any \( \varepsilon > 0 \) there exists an \( n_0 = n_0(\varepsilon) \) such that any r-uniform graph \( G \) with \( d(G) \geq \alpha + \varepsilon \) and \( |V(G)| \geq n_0 \) contains some member of \( \mathcal{F} \) as a subgraph. We denote this fact by \( \alpha \rightarrow \mathcal{F} \).

Lemma 2.4 (Frankl and Rödl [5]). The following two properties are equivalent.

1. \( \alpha \) is a jump for \( r \).
2. \( \alpha \rightarrow \mathcal{F} \) for some finite family \( \mathcal{F} \) of r-uniform graphs satisfying \( \lambda(F) > \alpha / r! \) for all \( F \in \mathcal{F} \).

We also need the following lemma from [5] in the proof of our main result.

Lemma 2.5 (Frankl and Rödl [5]). For any \( \sigma \geq 0 \) and any integer \( k \geq r \), there exists \( t_0(k, \sigma) \) such that for every \( t \geq t_0(k, \sigma) \), there exists an r-uniform graph \( A = A(k, \sigma, t) \) satisfying:

1. \( |V(A)| = t \),
2. \( |E(A)| = \sigma t^{r-1} \),
3. for all \( V_0 \subset V(A) \), \( r \leq |V_0| \leq k \), we have \( |E(A) \cap (V_0 \times V_0)| \leq |V_0| - r + 1 \).

The following theorem is our main result. For positive integers \( x \) and \( y \), denote \([x]_y = x(x-1)(x-2) \cdots (x-y+1)\) if \( x \geq y \) and \([x]_y = 0\) if \( x < y \).

Theorem 2.6. Let \( l_i, i \geq 1 \) be a sequence of positive integers. Let \( N^{(i)}, i \geq 1 \) be a sequence of numbers defined inductively as below:

\[
N^{(1)} = 1 - \frac{1}{(l_1)^{r-1}}, \quad \text{where} \quad l_1 \geq 2;
\]

\[
N^{(i+1)} = \frac{[l_{i+1} - 1]_{r-1}}{(l_{i+1})^{r-1}} + \frac{N^{(i)}}{(l_{i+1})^{r-1}} \quad \text{for} \quad i \geq 1.
\]

(2)

Then \( N^{(i)} \)s are non-jumping numbers for \( r \geq 3 \) if

\[
l_{i+1} \text{ satisfies } \frac{[l_{i+1} - 1]_{r-1}}{(l_{i+1})^{r-1}} - 1 \geq N^{(i)}.
\]

(3)

Remark 2.7. \( N^{(1)} \) in Theorem 2.6 can be replaced by other non-jumping numbers given in [4,8–11].

Let us give two examples to illustrate Theorem 2.6:

1. Let \( N^{(1)} = \frac{5}{9} \) and \( r = 3 \) in this theorem (it was proved that \( \frac{5}{9} \) is a non-jumping number for \( r = 3 \) in [4]), then we have: \( N^{(2)} = 1 - \frac{3}{l_2} + \frac{23}{9(l_2)^2} \) is a non-jumping number for \( r = 3 \) if integer \( l_2 \geq 6 \).

2. Let \( N^{(1)} = 1 - \frac{1}{(l_2)^2} \) and \( r = 3 \) in this theorem, then \( N^{(2)} = 1 - \frac{3}{l_2} + \frac{3}{(l_2)^2} - \frac{1}{(l_2)^3} \) are non-jumping numbers for \( r = 3 \) if integers \( l_1 \) and \( l_2 \) satisfy \( l_1 \geq 2 \) and \( l_2 \geq 3(l_1)^2 - 1 \).

Remark 2.8. 1. Condition (3) guarantees that

\[N^{(i+1)} \geq N^{(i)}.\]

2. Condition (3) implies that \( l_{i+1} \geq r \).
The approach in proving Theorem 2.6 is sketched as follows: Assuming that \( N(i) \) is a jump, we will derive a contradiction in two steps.

**Step 1:** Construct an \( r \)-uniform graph with the Lagrangian slightly smaller than \( N(i)/r! \), then use Lemma 2.5 to add an \( r \)-uniform graph with a large enough number of edges but sparse enough (guaranteed by properties 2 and 3 in Lemma 2.5) and obtain an \( r \)-uniform graph with the Lagrangian slightly larger than \( N(i)/r! \). Then we ‘blow up’ it to an \( r \)-uniform graph, say \( H(i) \) with a large enough number of vertices and density \( \geq N(i) + \varepsilon \) for some \( \varepsilon > 0 \) (see Remark 2.2). If \( N(i) \) is a jump, then by Lemma 2.4, \( N(i) \) is a threshold for some finite family \( \mathcal{F}(i) \) of \( r \)-uniform graphs with Lagrangians \( > N(i)/r! \). So \( H(i) \) must contain some member of \( \mathcal{F}(i) \) as a subgraph.

**Step 2:** Show that any subgraph of \( H(i) \) with the number of vertices not greater than \( \max\{|V(F(i))|; F(i) \in \mathcal{F}(i)\} \) has the Lagrangian \( \leq N(i)/r! \) and derive a contradiction.

We would like to point out that it is certainly nontrivial to construct an \( r \)-uniform graph satisfying the properties in both Steps 1 and 2. Generally, whenever we find such a construction, we can obtain a corresponding non-jumping number. This method was first developed by Frankl and Rödl in [5], then it was used in [4,8–11] to find more non-jumping numbers by giving this type of construction. The critical and technical part in the proof of the main theorem in this paper is to show that the construction satisfies the property in Step 2 (Lemma 3.2 in Section 3).

### 3. Proof of Theorem 2.6

Let \( l_1 \) and \( N(i), i \geq 0 \) be given as in Theorem 2.6. We will show that \( N(i) \)'s are non-jumping numbers as described in two steps from last section.

**Step 1:** Let \( t \) be a positive integer. We first define a sequence of \( r \)-uniform graphs \( G_t(i) \) inductively on \( z_i = \prod_{j=1}^i l_j \) pairwise disjoint sets \( V_1, V_2, \ldots, V_{z_i} \), each of cardinality \( t \) as below.

Define \( G_t(i) \) on \( l_1 \) pairwise disjoint sets \( V_1, V_2, \ldots, V_{l_1} \), where \( |V_1| = t \), and the edge set of \( G_t(i) \) is \( \left( \bigcup_{i=1}^{l_i} V_i \right) - \bigcup_{j=1}^{l_j} \left( V_j \right) \). (If \( N(i) \) is any other non-jumping number given in [4,8–11], then \( G_t(i) \) will be replaced by the corresponding construction given in those papers.)

Now suppose that \( G_t(i) \) has been defined on \( \bigcup_{j=1}^{z_{i+1}} V_j \). Define \( G_t(i+1) \) on \( z_{i+1} = z_i l_{i+1} \) pairwise disjoint sets \( V_j, 1 \leq j \leq z_{i+1} \), each of cardinality \( t \) as follows: for each \( p, 1 \leq p \leq l_{i+1} \), take a copy of \( G_t(i) \) on the vertex set \( \bigcup_{(p-1)z_i+1 \leq j \leq pz_i} V_j \), then add all crossing edges taking at most one vertex from each copy of \( G_t(i) \), i.e., the edges in the form of \( \{v_{j_1}, v_{j_2}, \ldots, v_{j_k}\} \), where \( 1 \leq j_1 < j_2 < \cdots < j_k \leq l_{i+1} \), and for each \( q, 1 \leq q \leq r, v_{j_q} \in \bigcup_{(q-1)z_i+1 \leq j \leq qz_i} V_j \).

For any integer \( k \geq r \) and \( \sigma > 0 \), let \( t_0(k, \sigma) \) be given as in Lemma 2.5. Take \( t > t_0(k, \sigma) \) and an \( r \)-uniform graph \( A(k, \sigma, t) \) satisfying the conditions in Lemma 2.5 with \( V(A(k, \sigma, t)) = V_1 \). The \( r \)-uniform graph \( H(i) (k, \sigma, t) \) is obtained by adding \( A(k, \sigma, t) \) to \( G_t(i) \).

Suppose that \( N(i) \) is a jump. In view of Lemma 2.4, there exists a finite collection \( \mathcal{F}(i) \) of \( r \)-uniform graphs satisfying the following:

(i) \( \lambda(F(i)) > \frac{1}{r} N(i) \) for all \( F(i) \in \mathcal{F}(i) \), and

(ii) \( N(i) \) is a threshold for \( \mathcal{F}(i) \).

Set \( k_i = \max\{|V(F(i))|; F(i) \in \mathcal{F}(i)\} \). Let \( \sigma_1 = (2|l'_j| + \left( \frac{1}{r} \right) (l_i'^{-1} - l_1))/r! \), and \( \sigma_i = \sigma_1 \prod_{j=2}^i l_j \) if \( i \geq 2 \). Let \( t_i \) be a fixed integer large enough to satisfy both Lemma 2.5 and Claim 3.1 as follows (i.e., \( t_i > \max\{t_0(k_i, \sigma_i, t_i)\} \)).

**Claim 3.1.** For sufficiently large \( t \),

\[
\frac{|E(G_t(i))| + \sigma t^{-1}}{(z_i t)^r} \geq \frac{1}{r!} \left( N(i) + \frac{1}{z_i^r t} \right)
\]

holds. Consequently,

\[
\lambda(H_t(i) (k_i, \sigma_i, t_i)) \geq \frac{1}{r!} \left( N(i) + \frac{1}{z_i^r t_i} \right).
\]
Proof of Claim 3.1. If (4) holds, corresponding to the $z_i t_i$ vertices of $H^{(i)}(k_i, \sigma_i, t_i)$, we take the vector $\bar{x} = (x_1, \ldots, x_{z_i t_i})$, where $x_j = 1/(z_i t_i)$ for each $j$, $1 \leq j \leq z_i t_i$, then

$$\lambda(H^{(i)}(k_i, \sigma_i, t_i)) \geq \lambda(H^{(i)}(k_i, \sigma_i, t_i), \bar{x})$$

$$= \frac{|E(H^{(i)}(k_i, \sigma_i, t_i))|}{(z_i t_i)^r} = \frac{|E(G^{(i)}_{t_i})| + \sigma_i t_i r^{r-1}}{(z_i t_i)^r}$$

$$\geq \frac{1}{r!} \left( N^{(i)} + \frac{1}{z_i t_i} \right),$$

Now we prove (4) by induction on $i$. If $i = 1$, then

$$|E(G^{(1)}_{t_1})| + \sigma_1 t_1 r^{r-1} = \frac{(l_1 t_1 - l_1 \left( \frac{t_1}{r} \right) + \sigma_1 t_1 r^{r-1}}{(l_1 t_1)^r}$$

$$= \frac{1}{r!} \left( 1 - \frac{1}{(l_1)^{r-1}} + \frac{2}{r} + o\left( \frac{1}{r} \right) \right)$$

$$\geq \frac{1}{r!} \left( N^{(1)} + \frac{1}{z_1 t_1} \right)$$

for $t$ large enough.

Now suppose that for integer $j \geq 1$,

$$\frac{|E(G^{(j)}_{t_j})| + \sigma_j t_j r^{r-1}}{(z_j t_j)^r} \geq \frac{1}{r!} \left( N^{(j)} + \frac{1}{z_j t_j} \right)$$

holds for sufficiently large $t$. Recall that $G^{(j+1)}_{t_j}$ consists of $l_{j+1}$ disjoint copies of $G^{(j)}_{t_j}$ and all crossing edges among these disjoint copies of $G^{(j)}_{t_j}$. Therefore,

$$\frac{|E(G^{(j+1)}_{t_{j+1}})| + \sigma_{j+1} t_{j+1} r^{r-1}}{(z_{j+1} t_{j+1})^r} = \frac{l_{j+1} \left| E(G^{(j)}_{t_j}) \right| + \left( \frac{l_{j+1}}{r} \right) z_j t_j r^{r-1} + l_{j+1} \sigma_j t_j r^{r-1}}{(z_{j+1} t_{j+1})^r}$$

$$= l_{j+1} \left( \frac{z_j t_j r^{r-1}}{l_{j+1}} + \frac{1}{z_j l_{j+1} t_{j+1}} \right)$$

$$\geq \frac{1}{r!} \left( N^{(j)} + \frac{1}{(z_{j+1})^r} \right).$$

This completes the proof of the Claim. □

Now assume that $\bar{y} = (y_1, y_2, \ldots, y_{z_i t_i})$ is an optimal vector for $\lambda(H^{(i)}(k_i, \sigma_i, t_i))$. Let $\epsilon_i = 1/(2(z_{j+1})^r t)$ and $n \geq n_1(\epsilon_i)$ be given as in Remark 2.2. Then $r$-uniform graph $S_n = ([ny_1], \ldots, [ny_{z_i t_i}]) \otimes H^{(i)}(k_i, \sigma_i, t_i)$ has density at least $N^{(i)} + \epsilon_i$.

On the other hand, in view of Lemma 2.4, some member $F^{(i)}$ of $\mathcal{F}^{(i)}$ is a subgraph of $S_n$ for $n \geq \max\{n_0(\epsilon_i), n_1(\epsilon_i)\}$. For such a member $F^{(i)} \in \mathcal{F}^{(i)}$, there exists a subgraph $M^{(i)}$ of $H^{(i)}(k_i, \sigma_i, t_i)$ with $|V(M^{(i)})| \leq k_i$ so that $F^{(i)} \subseteq (n, n, \ldots, n) \otimes M^{(i)}$. 
By Facts 2.1 and 2.3, we have
\[ \lambda(F(i)) \leq \lambda((n, n, \ldots, n) \otimes M(i)) = \lambda(M(i)). \quad (8) \]

Step 2: Theorem 2.6 will follow from Lemma 3.2, whose proof will be given in the next section.

**Lemma 3.2.** For any positive integer \( k \) and \( \sigma > 0 \), let \( t_0(k, \sigma) \) be given as in Lemma 2.5 and \( t > t_0(k, \sigma) \) be any integer. If \( M(i) \) is a subgraph of \( H(i)(k, \sigma, t) \) with \( |V(M(i))| \leq k \), then

\[ \lambda(M(i)) \leq \frac{1}{r!} N(i) \quad (9) \]

holds.

Now applying Lemma 3.2 to (8), we have
\[ \lambda(F(i)) \leq \frac{1}{r!} N(i) \]
which contradicts our choice of \( F(i) \), i.e., contradicts that \( \lambda(F(i)) > \frac{1}{r!} N(i) \) for all \( F(i) \in F(i) \). This completes the proof of Theorem 2.6. □

4. **Proof of Lemma 3.2**

Let \( U_j = V(M(i)) \cap V_j \) for \( 1 \leq j \leq z_i \) and \( \vec{z}^{(i)} \) be an optimal vector for \( \lambda(M(i)) \). Let \( a_j^{(i)} \) be the sum of the weights of \( \vec{z}^{(i)} \) in \( U_j \), \( 1 \leq j \leq z_i \) respectively. Notice that

\[ \begin{align*}
\sum_{j=1}^{z_i} a_j^{(i)} &= 1, \\
a_j^{(i)} &\geq 0, \quad 1 \leq j \leq z_i.
\end{align*} \quad (10) \]

Now we will prove Lemma 3.2 based on induction on \( i \). For \( i = 1 \), the following result is proved in [5].

**Lemma 4.1 (cf. Frankl and Rödl [5]).**

\[ \lambda(M(1)) \leq \frac{1}{r!} N(1). \quad (11) \]

Assuming that
\[ \lambda(M(i)) \leq \frac{1}{r!} N(i), \quad (12) \]
we are going to show that
\[ \lambda(M(i+1)) \leq \frac{1}{r!} N(i+1). \quad (13) \]

Recall that \( H(i+1)(k, \sigma, t) \) is obtained by adding \( A(k, \sigma, t) \) to \( G_i^{(i+1)} \) which consists of \( l_{i+1} \) disjoint copies of \( G_i^{(i)} \) and all crossing edges among these disjoint copies of \( G_i^{(i)} \). Therefore, \( H(i+1)(k, \sigma, t) \) consists of a copy of \( H(i)(k, \sigma, t) \), \( l_{i+1} - 1 \) copies of \( G_i^{(i)} \) and all crossing edges among these disjoint copies of \( H(i)(k, \sigma, t) \) and \( G_i^{(i)} \). Then by the assumption (12), the summation of the terms in \( \lambda(M(i+1)) \) restricted on \( \bigcup_{j=1}^{z_i} U_j^{(i+1)} \) is no more than \( \frac{1}{r!} N(i) (\sum_{j=1}^{z_i} a_j^{(i+1)})^r \). Similarly, for each \( p \), \( 2 \leq p \leq l_{i+1} \), the summation of the terms in \( \lambda(M(i+1)) \) restricted on vertex set \( \bigcup_{j=(p-1)z_i+1}^{pz_i} U_j^{(i+1)} \) is no more than \( N(i) (\sum_{j=(p-1)z_i+1}^{pz_i} a_j^{(i+1)})^r / r! \). For each \( p \), \( 1 \leq p \leq l_{i+1} \), let
\[ b_p = \frac{\sum_{j=(p-1)z_i+1}^{pz_i} a_j^{(i+1)}}{z_i}. \]
Then
\[
\lambda(M^{(i+1)}) \leq \frac{1}{r!} N^{(i)}(z_{i+1}) \sum_{p=1}^{l_{i+1}} (z_i b_p)^r + \sum_{1 \leq i_1 < \cdots < i_r \leq l_{i+1}} z_i^r b_{i_1} b_{i_2} \cdots b_{i_r}
\]
\[= f(b_1, b_2, \ldots, b_{l_{i+1}}). \quad (14)\]

Using a direct calculation,
\[
f \left( \frac{1}{z_{i+1}}, \frac{1}{z_{i+1}}, \ldots, \frac{1}{z_{i+1}} \right) = \frac{1}{r!} N^{(i+1)}.
\]

Therefore, we need to verify the following claim.

**Claim 4.2.**

\[
f(b_1, b_2, \ldots, b_{l_{i+1}}) \leq f \left( \frac{1}{z_{i+1}}, \frac{1}{z_{i+1}}, \ldots, \frac{1}{z_{i+1}} \right)
\]
holds under the constraints
\[
\begin{align*}
\sum_{j=1}^{l_{i+1}} b_j & = \frac{1}{z_i}, \\
b_j \geq 0, & \quad 1 \leq j \leq l_{i+1}.
\end{align*}
\]

The proof of Claim 4.2 follows from the following conclusion: if function \( f \) reaches the maximum at \( (b_1, b_2, \ldots, b_{l_{i+1}}) \), then \( b_1 = b_2 = \cdots = b_{l_{i+1}} = 1/z_{i+1} \). We proceed to prove this conclusion in two claims.

**Claim 4.3.** Let \( p, q, 1 \leq p < q \leq l_{i+1} \), be a pair of integers and \( \varepsilon \) be a real number. Let \( c_p = b_p + \varepsilon, c_q = b_q - \varepsilon \), and \( c_s = b_s \) for \( s \neq p, q \). Let \( (b_q - b_p)A(b_1, b_2, \ldots, b_{l_{i+1}}) \) and \( B(b_1, b_2, \ldots, b_{l_{i+1}}) \) be the coefficients of \( \varepsilon \) and \( \varepsilon^2 \) in \( f(c_1, c_2, \ldots, c_{l_{i+1}}) - f(b_1, b_2, \ldots, b_{l_{i+1}}) \) respectively, i.e.,
\[
f(c_1, c_2, \ldots, c_{l_{i+1}}) - f(b_1, b_2, \ldots, b_{l_{i+1}}) = (b_q - b_p)A(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon + B(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon^2 + o(\varepsilon^2).
\]

If \( b_p \neq b_q \), then
\[
A(b_1, b_2, \ldots, b_{l_{i+1}}) + B(b_1, b_2, \ldots, b_{l_{i+1}}) > 0.
\]

**Proof of Claim 4.3.** Without loss of generality, take \( p = 1 \) and \( q = 2 \). In view of the definition of the function \( f(b_1, b_2, \ldots, b_{l_{i+1}}) \), we have
\[
f(b_1 + \varepsilon, b_2 - \varepsilon, b_3, \ldots, b_{l_{i+1}}) - f(b_1, b_2, b_3, \ldots, b_{l_{i+1}})
\]
\[= \frac{1}{r!} N^{(i)}(z_i) \sum_{(i_4 \leq \cdots \leq i_r \leq l_{i+1})} (b_{i_1} + \varepsilon)^r (b_{i_2} - \varepsilon)^r - b_{i_1}^r - b_{i_2}^r
\]
\[+ \sum_{3 \leq i_3 < i_4 < \cdots < i_r \leq l_{i+1}} (b_{i_1} + \varepsilon)(b_{i_2} - \varepsilon) - b_1 b_2]
\]
\[= f_1(b_1, b_2, \ldots, b_{l_{i+1}})(\varepsilon) + f_2(b_1, b_2, \ldots, b_{l_{i+1}})(\varepsilon)\]
\[= (b_1 + \varepsilon)^r + (b_2 - \varepsilon)^r - b_1^r - b_2^r.
\]

Let
\[
f_1(b_1, b_2, \ldots, b_{l_{i+1}})(\varepsilon) = (b_1 + \varepsilon)^r + (b_2 - \varepsilon)^r - b_1^r - b_2^r,
\]
\[
f_2(b_1, b_2, \ldots, b_{l_{i+1}})(\varepsilon) = (b_1 + \varepsilon)(b_2 - \varepsilon) - b_1 b_2.
\]

Let \( (b_2 - b_1)A_j(b_1, b_2, \ldots, b_{l_{i+1}}) \) and \( B_j(b_1, b_2, \ldots, b_{l_{i+1}}) \) be the coefficients of \( \varepsilon \) and \( \varepsilon^2 \) in \( f_j(b_1, b_2, \ldots, b_{l_{i+1}})(\varepsilon) \) for each \( j, 1 \leq j \leq 2 \). We estimate \( A_j(b_1, \ldots, b_{l_{i+1}}) + B_j(b_1, \ldots, b_{l_{i+1}}) \) for each \( j \). By direct calculation,
\[
A_2(b_1, \ldots, b_{l_{i+1}}) + B_2(b_1, \ldots, b_{l_{i+1}}) = 1 - 1 = 0
\]
\[= (b_q - b_p)A(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon + B(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon^2 + o(\varepsilon^2).
\]
and 
\[ B_1(b_1, \ldots, b_{l_{i+1}}) + A_1(b_1, \ldots, b_{l_{i+1}}) \]
\[ = \left( \frac{r}{2} \right) b_1^{r-2} + \left( \frac{r}{2} \right) b_2^{r-2} - r(b_1^{r-2} + b_1^{r-3}b_2 + b_1^{r-4}b_2^2 + \cdots + b_1b_2^{r-3} + b_2^{r-2}) \]
\[ = \frac{r}{2}[(r-1)(b_1^{r-2} + b_2^{r-2}) - 2(b_1^{r-2} + b_1^{r-3}b_2 + b_1^{r-4}b_2^2 + \cdots + b_1b_2^{r-3} + b_2^{r-2})] \]
\[ = \frac{r}{2} \sum_{j=1}^{r-3} (b_1^{r-j} + b_2^{r-j} - b_1^{r-j}b_2 - b_2^{r-j}b_1) \]
\[ > 0 \] (18)
if \( b_1 \neq b_2 \). Claim 4.3 follows from (16) to (18). \( \square \)

**Claim 4.4.** Let \( p, q, 1 \leq p < q \leq l_{i+1} \) be a pair of integers. Let \( A(b_1, b_2, \ldots, b_{l_{i+1}}) \) and \( B(b_1, b_2, \ldots, b_{l_{i+1}}) \) be given as in Claim 4.3. One of the following two cases holds:

1. (Case 1) If \( A(b_1, b_2, \ldots, b_{l_{i+1}}) > 0 \) then \( b_p = b_q \).
2. (Case 2) If \( A(b_1, b_2, \ldots, b_{l_{i+1}}) \leq 0 \), then \( b_p = b_q \) or \( \min\{b_p, b_q\} = 0 \).

**Proof of Claim 4.4.** Without loss of generality, take \( p = 1 \) and \( q = 2 \). If \( b_1 = b_2 \), then we are done. Otherwise, without loss of generality, assume that \( b_1 < b_2 \).

**Case 1:** If \( A(b_1, b_2, \ldots, b_{l_{i+1}}) > 0 \), then
\[ f(b_1 + \varepsilon, b_2 - \varepsilon, b_3, \ldots, b_{l_{i+1}}) - f(b_1, b_2, b_3, \ldots, b_{l_{i+1}}) = (b_2 - b_1)A(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon + B(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon^2 + o(\varepsilon^2) > 0 \] (19)
holds for small enough \( \varepsilon > 0 \) and this contradicts to the assumption that \( f \) reaches the maximum at \( (b_1, b_2, \ldots, b_{l_{i+1}}) \).

**Case 2:** If \( A(b_1, b_2, \ldots, b_{l_{i+1}}) \leq 0 \), we claim that \( b_1 = 0 \). Otherwise, take \( \varepsilon < 0 \) with \( |\varepsilon| \) small enough, then in view of the assumption that \( A(b_1, b_2, \ldots, b_{l_{i+1}}) \leq 0 \) and \( b_1 < b_2 \), we have \( (b_2 - b_1)A(b_1, b_2, \ldots, b_{l_{i+1}})\varepsilon \geq 0 \). By the assumption \( A(b_1, b_2, \ldots, b_{l_{i+1}}) \leq 0 \) and Claim 4.3, we have \( B(b_1, b_2, \ldots, b_{l_{i+1}}) > 0 \). Applying this to (19), we obtain that
\[ f(b_1 + \varepsilon, b_2 - \varepsilon, b_3, \ldots, b_{l_{i+1}}) - f(b_1, b_2, b_3, \ldots, b_{l_{i+1}}) > 0 \]
for \( \varepsilon < 0 \) with \( |\varepsilon| \) small enough, and get a contradiction to the assumption that \( f \) reaches the maximum at \( (b_1, b_2, \ldots, b_{l_{i+1}}) \). This completes the proof of Claim 4.4. \( \square \)

Now let us turn to the proof of Claim 4.2. By Claim 4.4, either \( b_1 = b_2 = \cdots = b_{l_{i+1}} = (1/z_{i+1}) \) or for some \( q < l_{i+1} \), \( b_{i_1} = b_{i_2} = \cdots = b_{i_q} = (1/q z_i) \) and other \( b_j = 0 \).

Now we compare \( f\left(1/z_{i+1}, 1/z_{i+1}, \ldots, 1/z_{i+1}\right) = \frac{N(i+1)}{r!} \) to \( f\left(1/q z_i, \ldots, 1/q z_i, 0, \ldots, 0\right) \).

**Case a:** If \( q \leq r - 1 \), then
\[ f\left(1/q z_i, \ldots, 1/q z_i, 0, \ldots, 0\right) \leq \frac{N(i+1)}{r!}. \]

Condition (3) guarantees that \( N(i+1) \geq N(i) \). Therefore,
\[ f\left(1/z_{i+1}, 1/z_{i+1}, \ldots, 1/z_{i+1}\right) \geq f\left(1/q z_i, \ldots, 1/q z_i, 0, \ldots, 0\right). \]
Case b: If \( q \geq r \), then
\[
f \left( \frac{1}{qz_i}, \ldots, \frac{1}{qz_i}, 0, \ldots, 0 \right) = \frac{N(i)}{r!q^{r-1}} + \frac{(q - 1)(q - 2) \cdots (q - r + 1)}{r!q^{r-1}}.
\]

It is sufficient to show that
\[
g(x) = \frac{(x - 1)(x - 2) \cdots (x - r + 1)}{x^{r-1}} + \frac{N(i)}{x^{r-1}}
\]
increases when \( x \geq r \). In fact, by a direct calculation,
\[
g'(x) = \frac{x^{r-1}(x - 1) \cdots (x - r + 1) \left( \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-r+1} \right) - (r - 1)x^{r-2}(x - 1) \cdots (x - r + 1)}{(x^{r-1})^2} - \frac{(r - 1)N(i)}{x^r}
\]
\[
= \frac{x(x - 1) \cdots (x - r + 1) \left( \frac{x}{x-1} + \frac{x}{x-2} + \cdots + \frac{x}{x-r+1} \right) - (r - 1)(x - 1)(x - 2) \cdots (x - r + 1)}{x^r} - \frac{(r - 1)N(i)}{x^r}
\]
\[
= \frac{(x - 1) \cdots (x - r + 1) \left[ \frac{x}{x-1} + \frac{x}{x-2} + \cdots + \frac{x}{x-r+1} - (r - 1) \right] - (r - 1)N(i)}{x^r}
\]
\[
= \frac{(x - 1) \cdots (x - r + 1) \left[ \left( \frac{x}{x-1} - 1 \right) + \left( \frac{x}{x-2} - 1 \right) + \cdots + \left( \frac{x}{x-r+1} - 1 \right) \right] - (r - 1)N(i)}{x^r}
\]
\[
= \frac{(x - 1) \cdots (x - r + 1) \left( \frac{1}{x-1} + \frac{2}{x-2} + \cdots + \frac{r-1}{x-r+1} \right) - (r - 1)N(i)}{x^r}
\]
\[
\geq \frac{(r - 1) - (r - 1)N(i)}{x^r} \geq 0
\]

if \( x \geq r \). This completes the proof of Claim 4.2. \( \square \)

Acknowledgments

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References