Young’s inequality and related results on time scales

Fu-Hsiang Wong\textsuperscript{a,}\textsuperscript{*}, Cheh-Chih Yeh\textsuperscript{b}, Shiueh-Ling Yu\textsuperscript{c}, Chen-Huang Hong\textsuperscript{d}

\textsuperscript{a}Department of Mathematics, National Taipei Teacher’s College, 134, Ho-Ping E. Rd, Sec2, Taipei 10659, Taiwan, ROC
\textsuperscript{b}Department of Information Management, Lunghua University of Science and Technology, Keishan Taoyuan, 33306 Taiwan, ROC
\textsuperscript{c}Holistic Education Center, St. John’s and St. Mary’s Institute of Technology, Tamsui, Taipei, Taiwan, ROC
\textsuperscript{d}Department of Mathematics, National Central University, Chung-Li 32054, Taiwan, ROC

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Abstract

We establish the classical Young inequality on time scales as follows:

\[ ab \leq \int_{0}^{a} g^{\sigma}(x) \Delta x + \int_{0}^{b} (g^{-1})^{\sigma}(y) \Delta y \]

if \( g \in C_{\sigma}([0, c], \mathbb{R}) \) is strictly increasing with \( c > 0 \) and \( g(0) = 0, a \in [0, c], b \in [0, g(c)] \). Using this inequality, we can extend Hölder’s inequality and Minkowski’s inequality on time scales.

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1. Introduction

The Young inequality \([1–4]\) is not only interesting in itself but also very useful. The purpose of this work is to establish this renowned inequality on time scales. We first briefly introduce the time scales theory.

\textsuperscript{*}Corresponding author.
E-mail addresses: wong@tea.ntptc.edu.tw (F.-H. Wong), ccyeh@mail.lhu.edu.tw (C.-C. Yeh), slyu@mail.sjsmit.edu.tw (S.-L. Yu), hongch@wangwei.math.ncu.edu.tw (C.-H. Hong).

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By a time scale \( T \) we mean any closed subset of \( \mathbb{R} \) with order and topological structure present in a canonical way. Since a time scale \( T \) may or may not be connected, we need the concept of jump operators.

**Definition.** Let \( t \in T \), where \( T \) is a time scale; then two mappings
\[
\sigma, \rho : T \to \mathbb{R}
\]
satisfying
\[
\sigma(t) = \inf\{s \in T | s \rangle t\}, \rho(t) = \sup\{s \in T | s \rangle t\}
\]
are called the jump operators.

If \( \sigma(t) > t, t \in T \), we say \( t \) is right-scattered. If \( \rho(t) < t, t \in T \), we say \( t \) is left-scattered. If \( \sigma(t) = t, t \in T \), we say \( t \) is right-dense. If \( \rho(t) = t, t \in T \), we say \( t \) is left-dense.

**Definition.** A mapping \( f : T \to \mathbb{R} \) is called \( \text{rd-continuous} \) if

(a) \( f \) is continuous at each right-dense point or maximal point of \( T \);
(b) at each left-dense point \( t \in T \),
\[
\lim_{s \to t^-} g(s) = g(t^-)
\]
exists.

The set of all \( \text{rd-continuous} \) functions from \( T \to \mathbb{R} \) is denoted by \( C_{\text{rd}}[T, \mathbb{R}] \).

Let
\[
T^k = \begin{cases} T - \{m\}, & \text{if } T \text{ has a left-scattered maximal point } m. \\ T, & \text{otherwise}. \end{cases}
\]

If \( f : T \to \mathbb{R} \) is a function, then we define the function \( f^\sigma : T \to \mathbb{R} \) by
\[
f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in T,
\]
i.e., \( f^\sigma = f \circ \sigma \).

**Definition.** Assume that \( f : T \to \mathbb{R} \) and \( t \in T^k \); then we define \( f^\Delta(t) \) to be the number (if it exists) with the property that for any given \( \epsilon > 0 \), there exists a neighborhood \( U \) of \( t \) such that
\[
|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|
\]
for all \( s \in U \). In this case \( f^\Delta(t) \) is called the \( \Delta \)-derivative of \( f(t) \) at \( t \). If \( f \) is differentiable at each \( t \in T \), then \( f \) is called \( \Delta \)-differentiable on \( T \).

**Definition.** A function \( g : T \to \mathbb{R} \) is called an antiderivative of \( f : T \to \mathbb{R} \) if \( g^\Delta(t) = f(t) \) for all \( t \in T^k \), and in this case, we define the integral of \( f \) by
\[
\int_s^t f(u) \Delta u = g(t) - g(s)
\]
for all \( s, t \in T \), and we say that \( f \) is integrable on \( T \).

Throughout this work, we suppose that

(a) \( T \) is a time scale;
(b) \( a, b \in T \) with \( a < b \);
(c) an interval means the intersection of a real interval with the given time scale.

For further information concerning time scales, see [5, 6]. In order to establish our main results, we need the following lemma which is due to Bohner and Peterson [5].

**Lemma A.** Let \( v \in C_{rd}(T, \mathbb{R}) \) be strictly increasing and \( \hat{T} = v(T) \) be a time scale. If \( f \in C_{rd}(T, \mathbb{R}) \), then for \( a, b \in T \),

\[
\int_{a}^{b} f(x) v(x) \Delta x = \int_{\text{v}(a)}^{\text{v}(b)} f(v^{-1}(y)) \Delta y.
\]

2. Main results

We now can state and prove our main result as follows:

**Theorem 1.** Let \( g \in C_{rd}([0, c], \mathbb{R}) \) be a strictly increasing function with \( c > 0 \). If \( g(0) = 0 \), \( a \in [0, c] \) and \( b \in [0, g(c)] \), then

\[
ab \leq \int_{0}^{a} g(x) \Delta x + \int_{0}^{b} (g^{-1}(y)) \Delta y. \tag{R_1}
\]

**Proof.** Clearly,

\[
\int_{0}^{b} (g^{-1})^\sigma(x) \Delta x = \int_{0}^{b} g^{-1}(\sigma(x)) \Delta x \geq \int_{0}^{b} g^{-1}(x) \Delta x
\]

because \( \sigma(x) \geq x \) and \( g^{-1} \) is strictly increasing.

Letting \( v(x) = g(x) \) and \( f(x) = x \) in Lemma A, we see that

\[
\int_{0}^{g^{-1}(b)} g^\Delta(x) \Delta x = \int_{0}^{g(g^{-1}(b))} g^{-1}(y) \Delta y = \int_{0}^{b} g^{-1}(y) \Delta y. \tag{2}
\]

It follows from Theorem 1.77 in [5] that

\[
\int_{0}^{g^{-1}(b)} g^\Delta(x) \Delta x = (g(x)x)^{g^{-1}(b)}_{0} - \int_{0}^{g^{-1}(b)} g(\sigma(x)) \Delta x
\]

\[
= bg^{-1}(b) - \int_{0}^{g^{-1}(b)} g^\sigma(x) \Delta x.
\]

This and (1) and (2) imply

\[
\int_{0}^{a} g^\sigma(x) \Delta x + \int_{0}^{b} (g^{-1})^\sigma(y) \Delta y \geq bg^{-1}(b) + \int_{g^{-1}(b)}^{a} g^\sigma(x) \Delta x. \tag{3}
\]

Case (a). \( a \geq g^{-1}(b) \). It follows from the strictly increasing property of \( g \) that

\[
\int_{g^{-1}(b)}^{a} g^\sigma(x) \Delta x \geq \int_{g^{-1}(b)}^{a} g(\sigma(g^{-1}(b))) \Delta x \geq \int_{g^{-1}(b)}^{a} g(g^{-1}(b)) \Delta x
\]

\[
= b(a - g^{-1}(b)) = ab - bg^{-1}(b).
\]
This and (3) imply
\[ \int_0^a g^a(x) \Delta x + \int_0^b (g^{-1})^a(y) \Delta y \geq ab. \]

Case (b). \( a \leq g^{-1}(b) \). Let \( h = g^{-1} \). Then \( a \leq h(b) \), that is, \( h^{-1}(a) \leq b \). Applying case (a),
\[ ab \leq \int_0^b h^a(x) \Delta x + \int_0^a (h^{-1})^a(y) \Delta y = \int_0^b (g^{-1})^a(x) \Delta x + \int_0^a g^a(y) \Delta y. \]

Combining cases (a) and (b), we obtain the desired result \( (R_1) \). \( \square \)

As an application of Theorem 1, we have the following:

**Corollary 2.** Let \( p > 1 \) and \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( a \geq 0 \) and \( b \geq 0 \), then
\[ ab \leq \int_0^a (\sigma(x))^{p-1} \Delta x + \int_0^b (\sigma(y))^{q-1} \Delta y. \] \( (R_2) \)

**Proof.** Taking \( g(x) = x^{p-1} \) on \([0, \infty)\), \( g^{-1}(y) = y^{q-1} \) on \([0, \infty)\). Thus, \( (R_2) \) follows from Theorem 1. \( \square \)

**Corollary 3.** Let \( p \) and \( q \) be defined as in **Corollary 2**. If \( a \geq 0 \), \( b \geq 0 \), and \( \sigma(t) - t \) is constant on \([0, \infty)\), then
\[ ab \leq \frac{(\sigma(a))^p}{p} + \frac{(\sigma(b))^q}{q} - \frac{(\sigma(0))^p}{p} - \frac{(\sigma(0))^q}{q}. \]

**Proof.** Let \( f(t) = \frac{t^p}{p} \). Then it follows from Theorem 1.24 and exercise 1.35 in [5] that
\[ (f^\sigma(t))^\Delta = [f^\Delta(t)]^\sigma = \left( \frac{t^p}{p} \right)^\Delta \bigg|_{t=\sigma(t)} = \sum_{k=0}^{p-1} (\sigma(t))^k t^{p-1-k} \frac{\sigma(\sigma(t)))^k (\sigma(t))^{p-1-k}}{p}. \]

Thus
\[ ab \leq \int_0^a (\sigma(t))^{p-1} \Delta t + \int_0^b (\sigma(t))^{q-1} \Delta t \]
\[ \leq \sum_{k=0}^{p-1} \frac{\sigma(\sigma(t)))^k (\sigma(t))^{p-1-k}}{p} \Delta t + \sum_{k=0}^{q-1} \frac{\sigma(\sigma(t)))^k (\sigma(t))^{q-1-k}}{q} \Delta t \]
\[ = \frac{(\sigma(a))^p}{p} + \frac{(\sigma(b))^q}{q} - \frac{(\sigma(0))^p}{p} - \frac{(\sigma(0))^q}{q}. \] \( \square \)
Remark 4. Letting $\mathbb{T} = \mathbb{R}$ in Corollary 3,
\[ ab \leq \frac{a^p + b^q}{p + q}. \]  
(R3)

Theorem 5 (Hölder’s Inequality I). Let $f, g, h \in C_{rd}([a, b], \mathbb{R})$ and \( \frac{1}{p} + \frac{1}{q} = 1 \) with $p > 1$; then
\[ \left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \geq \int_a^b |h(x)||f(x)g(x)| \Delta x. \]  
(R4)

Proof. Setting
\[ A(t) = \left( \frac{|h(t)|^\frac{1}{p}|f(t)|}{\left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^\frac{1}{p}} \right) \quad \text{and} \quad B(t) = \left( \frac{|h(t)|^\frac{1}{q}|g(t)|}{\left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^\frac{1}{q}} \right) \quad \text{on} \quad [a, b]. \]
Following from (R3), we see that
\[ \frac{1}{p} \int_a^b A(t)B(t) \Delta t \leq \frac{1}{p} \int_a^b [A^p(t) + B^q(t)] \Delta t \]
\[ = \frac{1}{p} \int_a^b \frac{|h(t)||f(t)|^p}{\int_a^b |h(x)||f(x)|^p \Delta x} \Delta t + \frac{1}{q} \int_a^b \frac{|h(t)||g(t)|^q}{\int_a^b |h(x)||g(x)|^q \Delta x} \Delta t \]
\[ = \left( \frac{1}{p} \right) + \left( \frac{1}{q} \right) = 1. \]
Therefore, we obtain the desired result. \( \square \)

Using Theorem 5, we can prove the following:

Theorem 6 (Hölder’s Inequality II). Let $f, g, h \in C_{rd}([a, b], \mathbb{R})$ and \( \frac{1}{p} + \frac{1}{q} = 1 \) with $p < 0$ or $q < 0$; then
\[ \left( \int_a^b |h(x)||f(x)|^p \Delta x \right)^{\frac{1}{p}} \left( \int_a^b |h(x)||g(x)|^q \Delta x \right)^{\frac{1}{q}} \leq \int_a^b |h(x)||f(x)g(x)| \Delta x. \]

Proof. Without loss of generality, we may assume that $p < 0$. Set
\[ P = -\frac{p}{q}, \quad Q = \frac{1}{q}. \]
Then \( \frac{1}{p} + \frac{1}{Q} = 1 \) with $P > 1$ and $Q > 0$. Letting $f(x) = F(x)$ and $g(x) = G(x)$ in (R4), we have
\[ \left( \int_a^b |h(x)||F(x)|^P \Delta x \right)^{\frac{1}{P}} \left( \int_a^b |h(x)||G(x)|^Q \Delta x \right)^{\frac{1}{Q}} \geq \int_a^b |h(x)||F(x)G(x)| \Delta x. \]
Taking $F(x) = f^{-q}(x)$ and $G(x) = f^q(x)g^q(x)$ in the above inequality, we obtain the desired result. \( \square \)

Using our Hölder’s inequalities, we can show Minkowski’s inequality on time scales as follows:
**Theorem 7** (Minkowski’s Inequality). Let \( f, g, h \in C_{rd}([a, b], \mathbb{R}) \) and \( p > 1 \). Then

\[
\left( \int_a^b |h(x)||f(x) + g(x)|^p \, \Delta x \right)^{\frac{1}{p}} \leq \left( \int_a^b |h(x)||f(x)|^p \, \Delta x \right)^{\frac{1}{p}} + \left( \int_a^b |h(x)||g(x)|^p \, \Delta x \right)^{\frac{1}{p}}.
\]

**Proof.** It follows from Theorem 5 that

\[
\int_a^b |h(x)||f(x) + g(x)|^p \, \Delta x
\]

\[
= \int_a^b |h(x)||f(x) + g(x)|^{p-1}(|f(x) + g(x)|) \, \Delta x
\]

\[
\leq \int_a^b |h(x)||f(x) + g(x)|^{p-1}(|f(x)| + |g(x)|) \, \Delta x
\]

\[
= \int_a^b |h(x)||f(x) + g(x)|^{p-1}|f(x)| \, \Delta x + \int_a^b |h(x)||f(x) + g(x)|^{p-1}|g(x)| \, \Delta x
\]

\[
\leq \left\{ \int_a^b |h(x)||f(x) + g(x)|^{p-1}y \, \Delta x \right\}^{\frac{1}{q}} \left( \int_a^b |h(x)||f(x)|^p \, \Delta x \right)^{\frac{1}{p}}
\]

\[
+ \left\{ \int_a^b |h(x)||f(x) + g(x)|^{p-1}y \, \Delta x \right\}^{\frac{1}{q}} \left( \int_a^b |h(x)||g(x)|^p \, \Delta x \right)^{\frac{1}{p}}
\]

\[
= \left\{ \int_a^b |h(x)||f(x) + g(x)|^p \, \Delta x \right\}^{\frac{1}{q}}
\]

\[
\cdot \left\{ \left( \int_a^b |h(x)||f(x)|^p \, \Delta x \right)^{\frac{1}{p}} + \left( \int_a^b |h(x)||g(x)|^p \, \Delta x \right)^{\frac{1}{p}} \right\}.
\]

Therefore, we obtain the desired result. \( \square \)

**References**