On the $q$-differences of the generalized $q$-factorials

Ch.A. Charalambides
Department of Mathematics, University of Athens, Panepistemiopolis, 157 84 Athens, Greece

Abstract

The $q$-differences of the generalized $q$-factorial of $t$ of order $n$ and increment $h$, at $t = 0$, are examined. These $q$-numbers are the coefficients of the expansion of the generalized $q$-factorial of $t$ of order $n$ and increment $h$ into $q$-factorials of $t$ with unit increment. A combinatorial interpretation of these coefficients as $q$-rook numbers of a constant jump Ferrers board is provided. Further, explicit expressions, recurrence relations, limiting expressions, orthogonality relation and other properties of these $q$-numbers are derived.

AMS classification: Primary: 05A30; secondary: 05A15

Keywords: Ferrers boards; Inversion statistic; $q$-factorials; $q$-Stirling numbers; Reciprocal $q$-factorials; $q$-derivative operator

1. Introduction

A triangular sequence of numbers $C(n,k,s)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$, with $s$ a real (or more generally a complex) parameter, which may be defined by

$$(st)^n = \sum_{k=0}^{n} C(n,k,s)t^{(k)}, \quad n = 0, 1, \ldots,$$

where

$u^{(n)} = u(u-1) \cdots (u-n+1), \quad n = 1, 2, \ldots, u^{(0)} = 1,$

with $u$ a real (or more generally a complex) number, is the usual falling (descending) factorial of $u$ of order $n$, was systematically investigated by Charalambides (1976, 1977, 1979). Since the generalized factorial of $t$ of order $n$ and increment $h$,

$t^{(n,h)} = t(t-h) \cdots (t-hn+h), \quad n = 1, 2, \ldots, t^{(0,h)} = 1,$

may be expressed as $t^{(n,h)} = h^n(st)^{(n)}, \quad s = 1/h$, it follows that

$t^{(n,h)} = h^n \sum_{k=0}^{n} C(n,k,s)t^{(k)}, \quad s = 1/h, \quad n = 0, 1, \ldots$
and the number $C(n,k,s)$ is called the coefficient of the generalized factorial or simply C-number. Notice that these numbers for $s = -1$ reduce to the Lah numbers $L(n,k)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$. Further, they are polynomials in $s$ of degree $n$,

$$C(n,k,s) = \sum_{r=k}^{n} s(n,r)S(r,k)s^r,$$

with $s(n,r)$ and $S(r,k)$ the Stirling numbers of the first and second kind, respectively. In this form the C-numbers were first appeared in Jordan (1933) (cf. Charalambides and Singh (1988) and the references therein).

A $q$-analogue of the singless (absolute) Lah numbers was introduced by Hahn (1949) and further studied by Garsia and Remmel (1980) under the name $q$-Laguerre numbers.

In the present paper the $k$th $q$-difference of the generalized $q$-factorial of $t$ of order $n$ and increment $h$ at $t = 0$, which for $h = -1$ reduces to the $q$-Lah number, is investigated. A combinatorial interpretation of these $q$-differences when $h$ is a negative integer is given in Section 2. Two explicit expressions, a triangular and a vertical recurrence relations are derived in Section 3. Further, in Section 4, an expression connecting these numbers with the $q$-Stirling numbers of the first and second kind is obtained. It is also shown that in the limit when $h \to \infty$ and when $h \to 0$ these $q$-differences equal the $q$-Stirling numbers of first and second kind, respectively. In Section 5, it is shown that these numbers satisfy a quasi-orthogonal relation and that they are the coefficients in the expansion of the reciprocal generalized $q$-factorial into a series of reciprocal generalized $q$-factorials with different increments. Finally, the operator $(t^{h+1}D_q)^n$, where $D_q$ is the $q$-derivative operator, is expressed in terms of the operator $D_q^k$ and vice versa.

### 2. Definitions and notation

Let $0 < q < 1$, $t$ a real number and $n$ a positive integer. The number

$$[t]_q = (1 - qt)/(1 - q)$$

is called $q$-real number and particularly the number $[n]_q$ is called $q$-positive integer.

The (falling) factorial of the $q$-number $[t]_q$ (or the $q$-(falling) factorial of $t$) of order $n$ is defined by

$$[t]_q^{(n)} = [t]_q[t - 1]_q \cdots [t - n + 1]_q$$

$$= (1 - q^t)(1 - q^{t-1}) \cdots (1 - q^{t-n+1})/(1 - q)^n. \quad (2.2)$$
Particularly,

\[ [n]_q! = [1]_q[2]_q \cdots [n]_q = (1 - q)(1 - q^2) \cdots (1 - q^n)/(1 - q)^n. \]

The \( q \)-binomial coefficient (or \textit{Gaussian polynomial}) is defined by

\[ \binom{t}{n}_q = \frac{[t]_q [t]_q \cdots [t - n + 1]_q}{[n]_q!} = \frac{(1 - q)(1 - q^2) \cdots (1 - q^{n - 1})}{(1 - q)(1 - q^2) \cdots (1 - q^n)}. \] (2.3)

The \textit{rising factorial} of the \( q \)-number \( [t]_q \) (or the \textit{\( q \)-rising factorial of} \( t \)) of order \( n \) is defined by

\[ [t]_q^{[n]} = [t]_q [t + 1]_q \cdots [t + n - 1]_q = (1 - q)(1 - q^2) \cdots (1 - q^{n - 1})(1 - q^n). \] (2.4)

Notice that

\[ [t]_q^{[n]} = [t + n - 1]_q^{[n]} = [ - 1]_q^n [ - t]_q^{[n]} = (-1)^n q^{n} [ - t]_q^{[n]}. \] (2.5)

Further, the \textit{rising factorial} of the \( q \)-number \( [t]_q \) of order \( n \) may be expressed as polynomial of the falling factorials of the same number of orders \( k = 0, 1, \ldots, n \). Specifically,

\[ [-t]_q^{[n]} = q^n \sum_{k=0}^{n} L_q(n, k)[t]_q^{[k]}, \quad n = 0, 1, \ldots \] (2.6)

so that by (2.5),

\[ [t]_q^{[n]} = \sum_{k=0}^{n} [L_q(n, k)]_q [t]_q^{[k]}, \quad n = 0, 1, \ldots, \] (2.7)

where \( L_q(n, k), k = 0, 1, \ldots, n, n = 0, 1, \ldots \) are the \textit{\( q \)-Lah numbers} and

\[ [L_q(n, k)]_q = (-1)^n L_q(n, k), \quad k = 0, 1, \ldots, n, \quad n = 0, 1, \ldots \] (2.8)

are the \textit{signless (absolute) \( q \)-Lah numbers} (or \textit{\( q \)-Laguerre numbers} in the terminology of Hahn (1949) and Garsia and Remmel (1980)). These numbers for \( q = 1 \) reduce to the usual Lah and signless Lah numbers, respectively.

Consider now the \textit{generalized \( q \)-factorial} of \( t \) of order \( n \) and increment \( h \).

\[ [t]_q^{(n, h)} = [t]_q [t - h]_q \cdots [t - nh + n]_q, \quad n = 1, 2, \ldots, [t]_q^{(n, h)} = 1. \] (2.9)

where for \( h > 0 \) is the \textit{\( q \)-falling factorial} of \( t \) of order \( n \) and increment \( h \) and for \( h < 0 \) is the \textit{\( q \)-rising factorial} of \( t \) of order \( n \) and increment \( -h \). Notice that

\[ [t]_q^{(n, h)} = [h]_q^{[n]_q^{[h]_q^{[n]_q}}}. \]
Further, the generalized q-factorial of t of order n and increment h may be expressed as polynomial of q-factorials of t of orders k = 0, 1, ..., n. Specifically, a polynomial $f_n(t)$ in $q^t$ of degree less than or equal to n may be expressed as a polynomial of q-factorials of t by

$$f_n(t) = \sum_{k=0}^{n} \frac{1}{[k]_q!} A_q^k f_n(t) [t]_q^{(k)},$$

where $A_q^k$ is the q-difference operator of order k defined by

$$A_q^k = (E - 1)(E - q) \cdots (E - q^{k-1}), \quad (2.10)$$

with $E$ the usual shift operator, i.e., $Ef(t) = f(t + 1)$. Thus, with $f_n(t) = [t]_q^{(n,h)}$ and putting

$$R_q(n, k; h) = q^{h(n)-\binom{n}{k}} \sum_{k=0}^{n} q^{\binom{k}{2}} R_q(n, k; h) [t]_q^{(k)}, \quad n = 0, 1, \ldots, \quad (2.11)$$

it follows that

$$[t]_q^{(n,h)} = q^{-h(n)} \sum_{k=0}^{n} q^{\binom{k}{2}} R_q(n, k; h) [t]_q^{(k)}, \quad n = 0, 1, \ldots, \quad (2.12)$$

More generally,

$$[t]_q^{(n,a)} = q^{-a(n)} \sum_{k=0}^{n} q^{\binom{k}{2}} b^{n-k} R_q(n, k; a/b) [t]_q^{(k,h)}, \quad n = 0, 1, \ldots, \quad (2.13)$$

Notice that the coefficient $R_q(n, k; h)$ of the generalized q-factorial for $q = 1$ reduces to the coefficient $h^n C(n, k, 1/h)$ of the generalized factorial and for $h = -1$ to

$$R_q(n, k; -1) = q^{\binom{k}{2}+\binom{k}{1}} |L_q(n, k)|,$$

where $|L_q(n, k)|$ is the signless q-Lah number.

A combinatorial interpretation of the q-numbers $R_q(n, k; h)$ when $h$ is a negative integer may be given in terms of q-counting configurations as follows. Garsia and Remmel (1986) considered the q-rook number for a given Ferrers board $A$:

$$R_k(A, q) = \sum_{C \in C_k(A)} q^{\text{inv}(C)}, \quad (2.14)$$

where $C_k(A)$ denotes the collection of all configurations of k non-taking rooks in $A$ and $\text{inv}(C)$ is a generalization of the number of inversions of a permutation. Then for a board $A$ of width $n$ with $a_j$ squares in the $j$th column, $j = 1, 2, \ldots, n$, they derived the identity

$$\sum_{k=0}^{n} R_{n-k}(A, q) [t]_q^{(k)} = \prod_{j=1}^{n} [t + a_j - j + 1]_q.$$
Thus, by (2.12), for a positive integer \( m \),

\[
q^{m(n-1)}R_q(n, k; -m) = R_{n-k}(S_{n,m}, q),
\]

(2.15)

where \( S_{n,m} \) is a Ferrers board of width \( n \) with \((m + 1)(j - 1)\) squares in the \( j \)th column, \( j = 1, 2, \ldots, n \).

3. Explicit expressions and recurrence relations

**Theorem 3.1.** The coefficients of the generalized \( q \)-factorials are given by

\[
R_q(n, k; h) = \frac{q^{h_{(n-1)}}}{[k]_q!} \sum_{r=0}^{k} (-1)^{k-r} q^{(k-r)} \left[ \begin{array}{c} k \\ r \end{array} \right]_q [r]^q_{(n,h)}. 
\]

(3.1)

Also,

\[
R_q(n, k; h) = \frac{1}{(1-q)^{n-k}} \sum_{r=k}^{n} (-1)^{n-r} q^{h_{(n-1)}} \left[ \begin{array}{c} n \\ r \end{array} \right]_q \left[ \begin{array}{c} r \\ k \end{array} \right]_q. 
\]

(3.2)

**Proof.** The \( q \)-difference operator of order \( k \), \( \Delta_q^k \), by expanding (2.10) via the \( q \)-binomial theorem,

\[
\prod_{j=1}^{k} (x + yq^{j-1}) = \sum_{r=0}^{k} q^{(k-r)} \left[ \begin{array}{c} k \\ r \end{array} \right]_q x^r y^{k-r},
\]

may be expressed in terms of the usual shift operator \( E \) as

\[
\Delta_q^k = \sum_{r=0}^{k} (-1)^{k-r} q^{(k-r)} \left[ \begin{array}{c} k \\ r \end{array} \right]_q E^r.
\]

Performing this operator on \([t]_q^{(n,h)}\), (2.11) yields (3.1).

The generalized \( q \)-factorial of \( t \) of order \( n \), on using the \( q \)-binomial formula and the identity

\[
(q^r)^t = \sum_{k=0}^{r} (-1)^k (1-q)^k q^{(r)} \left[ \begin{array}{c} r \\ k \end{array} \right]_q [t]_q^{(k)}
\]

may be expressed as a polynomial of \( q \)-factorials of \( t \) as

\[
q^{h_{(r)}} [t]_q^{(n,h)} = \frac{1}{(1-q)^{t}} \prod_{j=0}^{n-1} (q^j - q^t) = \frac{1}{(1-q)^{t}} \sum_{r=0}^{n} (-1)^r q^{h_{(r)}} \left[ \begin{array}{c} n \\ r \end{array} \right]_q (q^t)^r
\]

\[
= \frac{1}{(1-q)^{t}} \sum_{r=0}^{n} (-1)^r q^{h_{(r)}} \left[ \begin{array}{c} n \\ r \end{array} \right]_q \sum_{k=0}^{r} (-1)^k (1-q)^k q^{(r)} \left[ \begin{array}{c} r \\ k \end{array} \right]_q [t]_q^{(k)}
\]

\[
= \sum_{k=0}^{n} \left\{ \frac{1}{(1-q)^{t-k}} \sum_{r=k}^{n} (-1)^{k-r} q^{h_{(r)}} \left[ \begin{array}{c} n \\ r \end{array} \right]_q \left[ \begin{array}{c} r \\ k \end{array} \right]_q q^{(r)} \left[ t \right]_q^{(k)} \right\}
\]

yielding (3.2).
Theorem 3.2. The coefficients of the generalized q-factorials satisfy (a) the 'triangular' recurrence relation

\[ R_q(n + 1, k; h) = ([k]_q - [nh]_q)R_q(n, k; h) + R_q(n, k - 1; h), \quad (3.3) \]

\( k = 1, 2, \ldots, n + 1, n = 0, 1, \ldots \) and (b) the 'vertical' recurrence relation

\[ R_q(n + 1, k + 1; h) = q^{h(n+1)} \sum_{r=k}^{n} q^{h(r) - nh r - k} \left[ \begin{array}{c} n \\ r \end{array} \right]_q [1 - h]_q^{n-r; q} R_q(r, k; h), \quad (3.4) \]

\( k = 0, 1, \ldots, n, n = 0, 1, \ldots, \) with initial conditions

\[ R_q(0, 0; h) = 1, \quad R_q(0, k; h) = 0, \quad k \neq 0, \quad R_q(n, 0; h) = 0, \quad n \neq 0. \]

Proof. (a) Expanding both members of the recurrence relation

\[ [t]_q^{(n+1; h)} = [t - nh]_q [t]_q^{(n; h)} \]

into q-factorials of \( t \) by the aid of (2.12) and since

\[ [t - nh]_q = q^{-nh}([t]_q - [nh]_q), \]

we get the relation

\[ \sum_{k=0}^{n+1} q^{(t)} R_q(n + 1, k; h)[t]_q^{(k)} = \sum_{r=0}^{n} q^{(r)} R_q(n, r; h)[t]_q^{(r)} - \sum_{k=0}^{n} q^{(t)} [nh]_q R_q(n, k; h)[t]_q^{(k)}, \]

which, on using the expression

\[ [t]_q^{(r)} = q^r [t]_q^{(r+1)} + [r]_q [t]_q^{(r)}, \]

yields

\[ \sum_{k=0}^{n+1} q^{(t)} R_q(n + 1, k; h)[t]_q^{(k)} = \sum_{r=0}^{n} q^{(r+1)} R_q(n, r; h)[t]_q^{(r+1)} + \sum_{r=0}^{n} q^{(r)} [r]_q R_q(n, r; h)[t]_q^{(r)} - \sum_{k=0}^{n} q^{(t)} [nh]_q R_q(n, k; h)[t]_q^{(k)}. \]

Equating the coefficients of \([t]_q^{(k)}\) in both sides of the last relation, the recurrence relation (3.3) is deduced.

(b) From (2.12) and since \( R_q(n + 1, 0; h) = 0, n = 0, 1, \ldots \) we have

\[ q^{-h{n+1 \choose 2}} \sum_{k=0}^{n} q^{h{k+1 \choose 2}} R_q(n + 1, k + 1; h) [t + 1]_q^{(k+1)} = [t + 1]_q^{(n+1; h)} \]

so that

\[ q^{-h{n+1 \choose 2}} \sum_{k=0}^{n} q^{h{k+1 \choose 2}} R_q(n + 1, k + 1; h) [t]_q^{(k)} = [t + 1 - h]_q^{(n; h)}. \]
Since, by the $q$-Vandermonde formula,
\[
\left[ t + 1 - h \right]_{q}^{n} = \sum_{r=0}^{n} q^{2^{h(n)} - h r n + r} \left[ \frac{n}{r} \right]_{q}^{h} \left[ 1 - h \right]_{q}^{n-r} \left[ t \right]_{q}^{r},
\]
it follows that
\[
\sum_{k=0}^{n} q^{k+1} R_{q}(n+1, k+1; h) \left[ t \right]_{q}^{(k)}
= q^{h(n)} \sum_{r=0}^{n} q^{2^{h(n)} - h r n + r} \left[ \frac{n}{r} \right]_{q}^{h} \left[ 1 - h \right]_{q}^{n-r} \left[ t \right]_{q}^{r}.
\]
Expanding the $q$-factorial of $t$ with increment $h$ into $q$-factorials of $t$ with unit increment by the aid of (2.12) we get the identity
\[
\sum_{k=0}^{n} q^{k+1} R_{q}(n+1, k+1; h) \left[ t \right]_{q}^{(k)}
= q^{h(n)} \sum_{r=0}^{n} q^{h(n)} - h r n + r \left[ \frac{n}{r} \right]_{q}^{h} \left[ 1 - h \right]_{q}^{n-r} \sum_{k=0}^{r} q^{h(r)} R_{q}(r, k; h) \left[ t \right]_{q}^{(k)}
= q^{h(n)} \sum_{k=0}^{n} q^{k+1} \left\{ \sum_{r=k}^{n} q^{h(n)} - h r n - k \left[ \frac{n}{r} \right]_{q}^{h} \left[ 1 - h \right]_{q}^{n-r} R_{q}(r, k; h) \right\} \left[ t \right]_{q}^{(k)},
\]
which implies (3.4).

4. Connection with $q$-Stirling numbers

The $q$-Stirling numbers of the first and second kind denoted by $s_{q}(n, k)$ and $S_{q}(n, k)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$, respectively, may be defined by (cf. Carlitz, 1948; Gould, 1961)
\[
\left[ t \right]_{q}^{n} = q^{-\left( \frac{n}{2} \right)} \sum_{k=0}^{n} s_{q}(n, k) \left[ t \right]_{q}^{k}
\]
and
\[
\left[ t \right]_{q}^{n} = \sum_{k=0}^{n} q^{\left( \frac{n}{2} \right)} S_{q}(n, k) \left[ t \right]_{q}^{(k)}.
\]

**Theorem 4.1.** The coefficients of the generalized $q$-factorials (a) are connected with the $q$-Stirling numbers of the first and second kind $s_{q}(n, k)$ and $S_{q}(n, k)$, $k = 0, 1, \ldots, n$, $n = 0, 1, \ldots$ by
\[
R_{q}(n, k; h) = \sum_{r=0}^{n} s_{q}(n, r) S_{q}(r, k) \left[ h \right]_{q}^{n-r}
\]
and (b) satisfy the limiting expressions
\[
\lim_{h \to \infty} R_{q}(n, k; h)/\left[ h \right]_{q}^{n-k} = s_{q}(n, k).
\]
\[
R_{q}(n, k; 0) = S_{q}(n, k).
\]
Proof. (a) Expanding \([t/h]_q^{(n)}\) into powers of \([t/h]_q = [t]_q/[h]_q\) by the aid of (4.1) and then expanding the powers of \([t]_q\) into factorials by the aid of (4.2) we get the relation

\[
[t]_q^{(n,h)} = [h]_q^n [t/h]_q^{(n)} = q^{-h/[2]} \sum_{r=0}^{n} s_q(n, r) [h]_q^{n-r} [t]_q^r
\]

\[
= q^{-h/[2]} \sum_{r=0}^{n} s_q(n, r) [h]_q^{n-r} \sum_{k=0}^{r} q^{(3)} S_q(r, k) [t]_q^k
\]

\[
= q^{-h/[2]} \sum_{k=0}^{n} q^{(3)} \left\{ \sum_{r=k}^{n} s_q(n, r) S_q(r, k) [h]_q^{n-r} \right\} [t]_q^k
\]

which, by virtue of (2.12), implies (4.3).

(b) Letting \(h \to \infty\) in

\[
[t]_q^{(n)} = q^{-5} \sum_{k=0}^{n} q^{(1/h)^{(5)}} R_q^{(n)}(n, k; h) [h]_q^{-n+k} [t]_q^{(1/h)},
\]

we get the expression

\[
[t]_q^{(n)} = q^{-5} \sum_{k=0}^{n} \left\{ \lim_{h \to \infty} R_q^{(n)}(n, k; h) [h]_q^{-n+k} \right\} [t]_q^k
\]

which, by virtue of (4.1), implies (4.4). The expansion (2.12) for \(h = 0\) reduces to

\[
[t]_q^n = \sum_{k=0}^{n} q^{(3)} R_q(n, k; 0) [t]_q^k
\]

which, by virtue of (4.2), yields (4.5).

Remark 4.1. The expression (3.2) of the coefficients of the generalized q-factorials, by virtue of (4.4) and (4.5), yields the following known expressions of the q-Stirling numbers (Gould, 1961; Carlitz, 1933)

\[ s_q(n, k) = \frac{1}{(1 - q)^{n-k}} \sum_{r=k}^{n} (-1)^{r-k} q^{r(r+1)/2} \binom{n}{r} \binom{r}{k} \]

and

\[ S_q(n, k) = \frac{1}{(1 - q)^{n-k}} \sum_{r=k}^{n} (-1)^{r-k} \binom{n}{k} \binom{r}{k} \]

5. Orthogonality relation and other properties

Theorem 5.1. The coefficients of the generalized q-factorials satisfy the quasi-orthogonal relation

\[
\sum_{r=k}^{n} R_q(n, r; h) R_q(r, k; 1/h) [h]_q^{-r-k} = \delta_{n,k}.
\]
Proof. Using successively (2.13) we find the relation
\[
[t]_{q}^{(n,n,h)} = q^{-h(b/2)} \sum_{r=0}^{n} q^{h[r]} R_{q}(n,r;h) [t]_{q}^{(r)}
\]
\[
= q^{-h(b/2)} \sum_{r=0}^{n} R_{q}(n,r;h) \sum_{k=0}^{r} q^{h[k]} R_{q}(r,k;1/h) [h]_{q}^{-k} [t]_{q}^{(k,h)}
\]
\[
= q^{-h(b/2)} \sum_{k=0}^{n} q^{h[k]} \left\{ \sum_{r=k}^{n} R_{q}(n,r;h) R_{q}(r,k;1/h) [h]_{q}^{-k} \right\} [t]_{q}^{(k,h)}
\]
which implies (5.1).

The reciprocal generalized q-factorial of t of order n and increment h is defined by
\[
[t]_{q}^{(-n,h)} = \frac{1}{[t + nh]_{q}^{(n,h)}}, \quad n \text{ positive integer}, \quad (5.2)
\]
so that
\[
[t]_{q}^{(n,h)}[t - nh]_{q}^{(m,h)} = [t]_{q}^{(n+m,h)}
\]
for all integers n and m. An extension of the expansion (2.12) and more generally of the expansion (2.15) for negative integer values of n is given in the next theorem.

**Theorem 5.2.** The expansion of the reciprocal generalized q-factorial \([t]_{q}^{(-n,n)}\), \(n = 1, 2, \ldots\) into a formal series of reciprocal generalized q-factorials \([t]_{q}^{(-k,h)}\), \(k = n, n+1, \ldots, b \neq a\), is given by
\[
[t]_{q}^{(-n,a)} = q^{-a(b+1)} \sum_{k=n}^{\infty} q^{b(k+1)} [t]_{q}^{(-k,n)} R_{q}(k,n; b/a) [t]_{q}^{(-k,b)}
\]
\[
(5.3)
\]
and in particular
\[
[t]_{q}^{(-n,h)} = q^{-h(b+1)} \sum_{k=n}^{\infty} q^{b(k+1)} [t]_{q}^{(-k,n)} R_{q}(k,n;1/h) [t]_{q}^{(-k,b)}.
\]
\[
(5.4)
\]

Proof. Multiplying both sides of the recurrence relation \(R_{q}(k,n; b/a) = ([n]_{q} - [(k - 1)b/a])_{q} R_{q}(k - 1, n; b/a) + R_{q}(k - 1, n - 1; b/a)\), \(k = n, n+1, \ldots\) by
\[
q^{b(k)} [t]_{q}^{(-k,n)} R_{q}(k,n; b/a) + q^{b(k-1)} [t]_{q}^{(-k+1,n)} R_{q}(k-1, n; b/a),
\]
using in the left-hand side the recurrence relation
\[
[t]_{q}^{(-k+1,n)} = [t + kb]_{q} [t]_{q}^{(-k,n)} = q^{bk} ([t]_{q} - [-kb]_{q}) [t]_{q}^{(-k,b)}
\]
and in the right-hand side the expressions
\[
[n]_{q} = [-na]_{q}/[-a]_{q}, \quad [(k - 1)b/a]_{q} = [-(k - 1)b]_{q}/[-a]_{q}.
\]
it follows that
\[ q^{b(t+1)} \left[ -a \right]_q^{k-n} \left[ \left[ t \right]_q - \left[ -kb \right]_q \right] R_q \cdot (k, n; b/a) \left[ t \right]_q^{(-k,b)} = q^{b(t)} \left[ -a \right]_q^{k-n-1} \left[ \left[ na \right]_q - \left[ -(k-1)b \right]_q \right] R_q \cdot (k-1, n; b/a) \left[ t \right]_q^{(-k+1,b)} + q^{b(t)} \left[ -a \right]_q^{k-n} R_q \cdot (k-1, n-1; b/a) \left[ t \right]_q^{(-k+1,b)}, \quad k = n, n+1, \ldots . \]

Summing this relation for \( k = n, n + 1, \ldots \) we get for the series
\[ R_n(t) = \sum_{k=n}^{\infty} q^{b(t+1)} \left[ -a \right]_q^{k-n} R_q \cdot (k, n; b/a) \left[ t \right]_q^{(-k,b)}, \]
the recurrence relation
\[ R_n(t) = \frac{1}{\left[ t \right]_q - \left[ -na \right]_q} R_{n-1}(t) = \frac{q^{na}}{\left[ t + na \right]_q} R_{n-1}(t), \quad n = 1, 2, \ldots, \quad R_0(t) = 1, \]
which yields
\[ R_n(t) = \frac{q^{a(n+1)}}{\left[ t + na \right]_q^{(n,a)}} = q^{a(t+1)} \left[ t \right]_q^{(-n,a)}, \]
as required by (5.3).

Putting successively in (5.3) \( a = 0 \) and \( b = 0 \) by virtue of (4.4) and (4.5) the following corollary is deduced.

**Corollary 5.1.** The expansions of a reciprocal \( q \)-power into a formal series of reciprocal generalized \( q \)-factorials and vice versa are given by
\[ \left[ t \right]_q^{-n} = \sum_{k=n}^{\infty} q^{b(t+1)} \left[ -b \right]_q^{k-n} S_q \cdot (k, n) \left[ t \right]_q^{(-k,b)} \tag{5.5} \]
and
\[ \left[ t \right]_q^{-(n,a)} = q^{-a(t+1)} \sum_{k=n}^{\infty} \left[ -a \right]_q^{k-n} S_q \cdot (k, n) \left[ t \right]_q^{-k} \tag{5.6} \]
respectively.

**Remark 5.1.** The expansion (5.3), on using the expression
\[ \left[ t \right]_q^{(-n,h)} = \frac{1}{\left[ t + nh \right]_q^{(n,h)}} = \frac{1}{\left[ -1 \right]_q^{n} \left[ -t - h \right]_q^{(n,h)}} = \frac{1}{\left[ -1 \right]_q^{n} \left[ -t \right]_q^{(-n+1,h)}} \]
and changing \( n + 1, k + 1, q^{-1} \) and \( -t \) to \( n, k, q \) and \( t \), respectively, leads to the expansion
\[ \frac{1}{\left[ t \right]_q^{(n,a)}} = q^{a(3)} \sum_{k=n}^{\infty} q^{-b(3)} \left[ a \right]_q^{k-n} R_q \cdot (k - 1, n - 1; b/a) \frac{1}{\left[ t \right]_q^{(k,b)}}. \tag{5.7} \]
Notice that this expansion by putting successively \( a = 0 \) and \( b = 0 \) and using (4.4) and (4.5) reduces to

\[
\frac{1}{[t]_q^a} = \sum_{k=n}^{\infty} q^{h_b} \left[ b \right]_q^{k-n} s_q(k-1,n-1) \frac{1}{[t]_q^{k,b}}
\]

and

\[
\frac{1}{[t]_q^{n,a}} = q^{a+t_b} \sum_{k=n}^{\infty} S_q(k-1,n-1) \frac{1}{[t]_q^k},
\]

respectively.

The \( q \)-derivative operator \( D_q \) is defined by

\[
D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t},
\]

so that \( D_q 1 = 0 \) and

\[
D_q t^m = [m]_q t^{m-1}, \quad m \neq 0.
\]

Note that \( D_q \) for \( q = 1 \) reduces to the ordinary derivative operator \( D \). The operator

\[
\Theta_q = tD_q
\]

is the \( q \)-operator of the well-known operator \( \Theta = tD \).

\[
\Theta_q \left[ \Theta \right]_q = (1 - q^\Theta)/(1 - q),
\]

so that \( \Theta_q 1 = 0 \) and

\[
\Theta_q t^m = [m]_q t^m, \quad m \neq 0.
\]

The operator \( \Theta_q^n = (tD_q)^n \) and more generally the operator \( (t^{h+1} D_q)^n \) can be expressed in terms of the operator \( D_q^n \) and vice versa (cf. Comtet, 1973 for \( q = 1 \)).

**Theorem 5.3.**

\[
(t^{h+1} D_q)^n = \sum_{k=0}^{n} q^{h(t)+h^k} R_q(n,k; -h)t^{hn+k} D_q^k,
\]

(5.8)

\[
D_q^n = \sum_{k=0}^{n} q^{-h(t)} R_q(n,k; -1/h)[ -h]_q^{n-k} t^{-n-hk} (t^{h+1} D_q)^k.
\]

(5.9)

**Proof.** The operator \( t^{h+1} D_q \) repeated, on using the rule

\[
D_q f(t) g(t) = g(t) D_q f(t) + f(qt) D_q g(t),
\]
gives
\[(t^{h+1}D_q)^2 = \left[ (h+1)_q t^{2h+1}D_q + q^{h+1} t^{2h+2}D_q^2 \right],\]
\[(t^{h+1}D_q)^3 = \left[ (h+1)_q [2h+1]_q t^{3h+1}D_q + q^{h+1} [h+1]_q \right. \]
\[+ \left. [2h+2]_q t^{3h+2}D_q^2 + q^{3h+3} t^{3h+3}D_q^3 \right].\]

and, in general,
\[(t^{h+1}D_q)^n = \sum_{k=0}^{n} C_{n,k} t^{hn+k}D_q^k.\]

The coefficients \(C_{n,k}, k = 0, 1, \ldots, n\) are independent of the function on which the operator is performed. Therefore, for their determination the most convenient function can be chosen. Let \(f(t) = t^z\). Then
\[(t^{h+1}D_q)^n t^z = \left[ z \right]_{q}^{(n-h)} t^{zh+n}, \quad D_q^n t^z = \left[ z \right]_{q}^{(k)} t^{z-k}\]
and so
\[\left[ z \right]_{q}^{(n-h)} = \sum_{k=0}^{n} C_{n,k} \left[ z \right]_{q}^{(k)}\]
Thus, by virtue of (2.12),
\[C_{n,k} = q \binom{h}{(3)} + \binom{1}{(3)} R_q(n, k; -h)\]
as required by (5.8). Inverting (5.8), by the aid of (5.1), (5.9) is obtained.

Putting in (5.8) and (5.9) \(h = 0\), by virtue of (4.4) and (4.5), the following corollary is deduced.

**Corollary 5.2.**
\[
\Theta_q^n = \sum_{k=0}^{n} q^{(3)} S_q(n, k) t^k D_q^k, \quad (5.10)
\]
\[
D_q^n = q^{-(3)} t^{-n} \sum_{k=0}^{n} S_q(n, k) \Theta_q^k, \quad (5.11)
\]

**References**


